Bob Chen
Math 142B
HW #4
3/5/13

8.3.2 Let $f(x) = \frac{1}{x}$ for $x \in (0, 2)$.

(a) Find the nth Taylor polynomial for f.

(b) Show that for every n,

$$f(x) - p_n(x) = \frac{(1-x)^{n+1}}{x}.$$

(c) Use part (b) to prove that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

when $|x - 1| < 1$.

(a) The derivatives are

$$f^{(n)}(x) = (-1)^n n! x^{-n-1}$$

so that

$$f^{(n)}(1) = (-1)^n n!.$$

Then

$$p_n(x) = \sum_{k=0}^{n} (-1)^n (x-1)^n = \sum_{k=0}^{n} (1-x)^n.$$

(b) Since $x \in (0, 2)$, we have $(x - 1) < 1$. Hence

$$f(x) - p_n(x) = \frac{1}{x} - \frac{1-(1-x)^{n+1}}{1-(1-x)} = \frac{(1-x)^{n+1}}{x}.$$

(c) Part (b) shows that

$$f(x) - p_n(x) \to 0,$$

so that

$$f(x) = \lim p_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1)}{k!}(x-1)^k$$

for $x \in (0, 2)$.

■

8.4.1 Prove that for each natural number n, blah blah blah

The solution is in the proof of Theorem 8.16. To get an error bound of 10^{-4}, pick $n = 3$. Then we have

$$|\ln(1+x) - p_2(x)| \leq \frac{x^{n+1}}{n+1} \leq (.1)^4.$$

So, we just need to compute $p_3(0.1)$. ■

8.6.2 Let $f(x) = e^{-\frac{1}{x^2}}$. Show that there is no positive number M so that for each n,

$$|f^{(n)}(x)| \leq M^n$$

for all x.

Suppose there exists an M so that

$$|f^{(n)}(x)| \leq M^n$$

for $x \in [0,1]$. Then by Theorem 8.14, $p_n(x) \to f(x)$ on $[0,1]$. But this contradicts Theorem 8.22, which shows that f is not analytic. ∎

8.6.4 **NB:** This is not a starred problem, but I included the solution because I think it is rather tricky. Suppose g has derivatives of all orders and that for each n there exists c_n, δ_n so that

$$|g(x)| < c_n|x|^n$$

whenever $|x| < \delta_n$. Prove that $g^{(n)}(0) = 0$ for all n.

It is easy to show that $g(0) = 0$ since $g(0) \leq c_1|0|$. We'll do the rest by induction on n.

By LRT (at $x_0 = 0$), we have

$$g(x) = g(0) + \cdots + \frac{g^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{g^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

for some $c \in [0,x]$. (since by the induction hypothesis, all the derivatives up to $g^{(n)}(0) = 0$). Now divide by x^{n+1} and take the limit as $x \to 0$; the RHS tends to

$$\frac{g^{(n+1)}(0)}{(n+1)!}.$$

The LHS is bounded by

$$|g(x)x^{-n-1}| \leq |c_{n+2}x|,$$

which vanishes as $x \to 0$. Hence the RHS must be 0 also, so $g^{(n+1)}(0) = 0$. ∎

8.7.5 Show that e^x cannot be uniformly approximated by a polynomial on $\mathbb{R}$.

Suppose to the contrary, that we have $p(x)$ a polynomial for which

$$|p(x) - e^x| < 1$$

for every $x \in \mathbb{R}$. Then

$$\frac{e^x - 1}{e^x} \leq \frac{p(x)}{e^x} \leq \frac{e^x + 1}{e^x}.$$

Use the squeeze theorem as $x \to \infty$; this shows that

$$p(x)e^{-x} \to 1$$

as $x \to \infty$. But this is a contradiction (l'Hôpital's Rule shows that the limit must actually be 0). ∎