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Innovative Applications of O.R.

Copula-based Black–Litterman portfolio optimization

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ABSTRACT

We extend the Black-Litterman (BL) approach to incorporate tail dependency in portfolio optimization and estimate the posterior joint distribution of returns using vine copulas. Our novel copula-based BL (CBL) model leads to flexibility in modeling returns symmetric and asymmetric multivariate distribution from a range of copula families. Based on a sample of the Eurostoxx 50 constituents (also for S&P 100 as robustness check), we evaluate the performance of the suggested CBL approach and portfolio optimization technique using out-of-sample back-testing. Our empirical analysis and robustness check indicate better performance for the CBL portfolios in terms of lower tail risk and higher risk-adjusted returns, compared to the benchmark strategies.

1. Introduction

The mean-variance approach suggested by Markowitz (1952) plays an integral role in modern asset allocation and has been vastly studied in the finance literature. Using statistical measures of mean and variance, Markowitz developed the modern portfolio theory that provides investors with the optimal asset allocation considering their risk-return trade-off. Although the mean-variance optimization has been widely used in practice, it is sensitive to input parameters causing a number of shortcomings including extreme weights, corner solutions, and highly changing weights over time (Bera & Park, 2008; Best & Grauer, 1991; Chopra & Ziemba, 1993; Grauer & Shen, 2000; Green & Hollifield, 1992; Jobson & Korkie, 1980; Michaud, 1989). Several approaches e.g., Bayesian framework, robust portfolio optimization, higher moments, tail risk optimization and weight constraints have been developed to alleviate the effects of estimation error (see Kolm, Tütüncü, & Fabozzi, 2014, and reference therein). Other extensions include constrained portfolio optimization (Clarke, De Silva, & Thorley, 2002), transaction costs (e.g., De Roon, Nijman, & Werker, 2001), risk-based constraints (Levy & Levy, 2014), multi-period portfolios (Cui, Gao, Shi, & Zhu, 2019; Merton, 1969; Östermark, 1991) and new risk measures (e.g., Ahmadi-Javid & Fallah-Tafti, 2019; Fulga, 2016; Huang, Zhu, Fabozzi, & Fukushima, 2010; Rockafellar & Uryasev, 2000).

One of the challenges faced by fund managers is to combine specific views on underlying assets with allocation techniques. In particular, when using a risk model representing the market equilibrium, fund managers might be interested in applying some quantifiable views or opinions in constructing diversified portfolios. The Black–Litterman (BL) model is a widely used approach to incorporate investors' views into Markowitz-type allocation problems (Black & Litterman, 1991; 1992). The BL framework belongs to the Bayesian and shrinkage approaches that are also used to reduce the sensitivity of the Markowitz's mean-variance optimization concerning input parameters (Kolm et al., 2014). In the BL model expected returns are estimated based on the market equilibrium and the Bayesian method is used to obtain the posterior distribution of the mean and covariance matrix, which include both subjective relative and absolute views of returns (He & Litterman, 1999).

Although the BL approach, in its original form, improves the Markowitz's portfolio optimization by reducing estimation errors, it enforces normality and constant conditional distribution for asset returns. As regards to the normality assumption, several studies have extended the original BL model to capture non-normal distributions (Fabozzi, Focardi, & Kolm, 2006; Giacometti, Bertocchi, Rachev, & Fabozzi, 2007). For the constant distribution assumption, volatility modeling, both univariate and multivariate, has been incorporated into the BL approach (Beach & Orlov, 2007; Deng, 2018; Harris, Stoja, & Tan, 2017; Palomba, 2008). Although these studies extend the BL framework to non-Gaussian distributions, mostly Student-*t*, their suggested models fail to capture asymmetries in the return distribution. For this reason, copula pooling techniques

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have been developed, where an implied (mixture) distribution is obtained by blending investors' views with market-based returns (Meucci, 2006; 2008). As discussed in Palczewski & Palczewski (2019), a disadvantage of the distribution pooling methods is a lack of statistical interpretation. Kolm & Ritter (2017) provide a generalized BL approach allowing views on asset pricing models' parameters, e.g., risk premia in the Arbitrage Pricing Theory. Using Verbal Decision Analysis, Silva, Pinheiro, & Poggi (2017) address the difficulty of quantifying subjective views and suggest an alternative approach to incorporate investor's profile and perspective. Pang & Karan (2018) suggest a closed-form solution for the classical BL portfolio optimization problem using conditional value-at-risk (CVaR) as the risk measure.

In this paper, we extend the BL approach to incorporate both symmetric and asymmetric dependence in return distribution by using copula modeling. The copula models can capture symmetric or asymmetric tail dependence, which is an important aspect of financial risk modeling and portfolio optimization based on downside risk (Embrechts, Höing, & Juri, 2003; Patton, 2004). Several studies have shown the importance and advantages of asymmetric dependence modeling in asset allocation using Archimedean copulas (see Al Janabi, Hernandez, Berger, & Nguyen, 2017; Boubaker & Sghaier, 2013; Low, Alcock, Faff, & Brailsford, 2013; Okimoto, 2014). Furthermore, vine copulas are more flexible for estimating a pairwise dependency structure (Czado, 2019; Joe, 2014).

We combine the regular vine (Rvine) copula model with the BL approach. This leads to a generalized version of the BL approach in which the covariance matrix is estimated from the joint distribution obtained from copula modeling. In our vine copula-based BL (CBL) approach, directional investors' views are constructed by applying cointegration rank tests and the vector error correction model (VECM) to asset prices. We evaluate the CBL approach in a portfolio optimization setting. To further reduce the sensitivity of our CBL approach to input parameters, we impose tail constraints in reward/risk maximization as suggested in Alexander, Baptista, & Yan (2007). An empirical analysis of CBL portfolio optimization reveals that there are gains from incorporating copula modeling into the BL approach to maximize the investor's utility function, particularly for the Min CVaR and Max Sharpe ratio (SR) portfolios. Most of the tail constrained optimal portfolios result in lower risk (both in terms of volatility and downside risk). In general, we find that the CBL models outperform the benchmark portfolios for most out-of-sample measures.

This study makes the following contributions. First, we show how to incorporate the vine copula models into the BL approach. The estimation of tail dependency for the posterior conditional distribution of returns enhances the CBL approach with the flexibility to model downside risk. This leads to a generalized version of the BL approach in which the covariance matrix is estimated from the joint distribution obtained from copula modeling. Second, we contribute to the literature on the BL approach with non-Gaussian equilibrium returns by modeling the multivariate distribution from asymmetric copula models. Third, we extend the risk-adjusted approach in Giacometti et al. (2007) to a copula-based equilibrium model. Our novel CBL model leads to (i) more flexibility in modeling returns symmetric and asymmetric multivariate distribution from a range of copula families, (ii) a simulation-based approach that can be applied not only to downside risk modeling but also for portfolio optimization, and (iii) scenario-based optimization and stochastic programming.

The remainder of the paper is structured as follows. Section 2 presents the methodology including the CBL approach, the capital asset pricing model (CAPM)-based and risk-adjusted equilibrium, investors' views, truncated vine copulas, and optimal portfolios with tail constraints. The data set used for the empirical investigation is described in Section 3. Section 4 provides the empirical results including out-of-sample portfolio back-testing. In Section 5, we present robustness checks. Finally, concluding remarks are provided in Section 6.

2. Methodology

In the CBL model, the posterior distribution of returns is estimated using (i) the prior distribution, (ii) investors' views, and (iii) the dependency structure between assets. To estimate the prior distribution, an equilibrium model is used to obtain the prior mean (also known as equilibrium returns). Furthermore, a prior covariance matrix is required to estimate the equilibrium returns. Therefore, copula models are applied to obtain both the prior covariance and the dependency structure, which is one of the novelties of the current paper. Using the directional views obtained from the VECM/VAR (vector autoregressive) model, we estimate the CBL posterior mean and covariance matrix. We first describe the CBL approach using the CAPM-based and risk-adjusted equilibrium models. Thereafter, we present the truncated Rvine copula model and portfolio optimization methods. Finally, we discuss the steps involved in estimating the copula-based optimal portfolios.

2.1. BL approach

Black & Litterman (1991, 1992) show how to incorporate investors' views when constructing portfolio strategies. While the BL approach imposes consistency between the expected returns and market equilibrium, it allows investors' views to cause deviations from the market equilibrium. The BL approach also allows us to include as many views as investors require. Based on a Bayesian framework, the BL model combines investors' views with expected returns that are consistent with the market equilibrium (known as equilibrium returns). In this model, inputs include a prior distribution estimated from the equilibrium model, investors' views and their confidence levels, and a scaling parameter indicating the deviation from the market equilibrium.

Let d be the total number of assets, $\mathbf{r}_t = (r_{1t}, r_{2t}, ..., r_{dt})$, $r_{jt} = [\frac{p_{jt}-p_{j,t-1}}{p_{j,t-1}}] - r_{ft}$ be the excess returns computed based on the observed adjusted prices p_{jt} and risk-free rate of return r_{ft} at time t, with a $d \times d$ $\boldsymbol{\Sigma}$ covariance matrix and a $d \times 1$ vector of expected returns $\boldsymbol{\mu}$ such that $\mathbf{r}_t = \boldsymbol{\mu} + \boldsymbol{\epsilon}_t$, $\boldsymbol{\epsilon}_t \sim \mathcal{D}(0, \boldsymbol{\Sigma})$, where $\mathcal{D}(.,.)$ is a location-scale distribution. Denoting $\boldsymbol{\pi}$ as equilibrium returns (prior mean) and imposing the scaling parameter τ, the prior distribution for $\boldsymbol{\mu}$ is defined as:

$$\boldsymbol{\mu}_{\text{prior}} \sim \mathcal{D}(\boldsymbol{\pi}, \tau\boldsymbol{\Sigma}), \tag{1}$$

where $\tau\boldsymbol{\Sigma}$ is the prior covariance for $\boldsymbol{\mu}$. Let $\mathbf{q}$ be a vector of K views with a $K \times K$ matrix, $\boldsymbol{\Lambda}$, representing the confidence in investors' views. Given a $K \times d$ pick matrix, $\mathbf{P}$, the posterior distribution for $\boldsymbol{\mu}$ is:

$$\begin{aligned} &\boldsymbol{\mu}_{\text{posterior}}|\mathbf{q}; \boldsymbol{\Lambda} \sim \mathcal{D}(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{\boldsymbol{\mu}_{BL}}), \\ &\boldsymbol{\mu}_{BL} = [(\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}^\intercal\boldsymbol{\Lambda}^{-1}\mathbf{P}]^{-1}[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \mathbf{P}^\intercal\boldsymbol{\Lambda}^{-1}\mathbf{q}], \\ &\boldsymbol{\Sigma}_{\boldsymbol{\mu}_{BL}} = [(\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}^\intercal\boldsymbol{\Lambda}^{-1}\mathbf{P}]^{-1}, \end{aligned} \tag{2}$$

where $\boldsymbol{\mu}_{BL}$ and $\boldsymbol{\Sigma}_{\boldsymbol{\mu}_{BL}}$ denote the posterior mean and covariance for $\boldsymbol{\mu}$.[1] To construct confidence in investors' views, we follow Meucci (2009, 2010) and set $\boldsymbol{\Lambda} = \frac{1}{\kappa}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\intercal$, where $\kappa \in (0, \infty)$ is the general confidence level. The pick matrix $\mathbf{P}$ is used to represent the K views on d returns. Each row in $\mathbf{P}$ includes weights for the returns, expressed as a views vector.[2]

[1] See Meucci (2010) and He & Litterman (1999) for more details on the derivation of the posterior distribution.

[2] We set $\kappa = 1$; hence, the confidence in investors' views $\boldsymbol{\Lambda}$ is a diagonal matrix containing conditional variances.

Rewriting the posterior covariance as $\boldsymbol{\Sigma}_{BL} = \boldsymbol{\Sigma}_{\boldsymbol{\mu}_{BL}} + \boldsymbol{\Sigma}$, the posterior distribution for $\mathbf{r}$ is given as:

$$\begin{aligned} &\mathbf{r}_{BL}|\mathbf{q}; \boldsymbol{\Lambda} \sim \mathcal{D}(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL}), \\ &\boldsymbol{\Sigma}_{BL} = (1+\tau)\boldsymbol{\Sigma} - \tau^2 \boldsymbol{\Sigma}\mathbf{P}^\intercal(\tau\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\intercal + \boldsymbol{\Lambda})^{-1}\mathbf{P}\boldsymbol{\Sigma}. \end{aligned} \tag{3}$$

2.2. CAPM-based equilibrium

In the original BL model, a CAPM-based market equilibrium is used to estimate the expectations for implied returns. Considering the investor's general utility function maximization as:

$$\underset{\mathbf{w}}{\text{maximize}} \qquad \mathbf{w}^\intercal \boldsymbol{\pi} - \tfrac{\delta}{2}\mathbf{w}^\intercal \boldsymbol{\Sigma}\mathbf{w}, \tag{4}$$

and applying reverse optimization, given the optimal portfolio weights $\mathbf{w}$, we can obtain the equilibrium returns as:

$$\boldsymbol{\pi} = \delta\boldsymbol{\Sigma}\mathbf{w}, \tag{5}$$

where δ is the risk aversion coefficient, which can be estimated as the ratio of the market risk premium (expected excess returns) to market variance. Assuming the CAPM holds, setting $\mathbf{w}$ to the market capitalization weights leads to a CAPM-based equilibrium model. In this conventional approach, the historical covariance matrix is used to create the equilibrium returns. Furthermore, no actual optimization is conducted at this stage; instead, the weight vector $\mathbf{w}$ is directly based on market capitalization.

Considering the BL model in Eq. (3), we make three main assumptions: (1) normally distributed returns, (2) a constant conditional distribution, and (3) no tail dependency (either symmetric or asymmetric). Relaxing the normality assumption is addressed by Meucci (2006, 2008). Regarding the constant conditional distribution assumption, Harris et al. (2017) use the time-varying CAPM suggested by Bollerslev, Engle, & Wooldridge (1988), which allows modeling the time-varying conditional mean and covariance matrix for the equilibrium returns.

In the dynamic conditional correlation (DCC) based BL approach suggested by Harris et al. (2017), the marginal distribution is considered as normal or Student-t, which cannot capture an asymmetric tail distribution. Therefore, we suggest using copula modeling to estimate returns' joint distribution. As mentioned above, copula modeling can also be used to estimate the covariance matrix between assets. To do so, first, GARCH models can be used to estimate the univariate conditional volatilities and marginal distributions:

$$\begin{aligned} &r_{jt} = \mu_j + \epsilon_{jt} \\ &\epsilon_{jt} = \sqrt{h_{jt}} z_{jt} \\ &z_{jt} \approx (iid), \forall j \in \{1, 2, ..., d\} \\ &h_{jt} = \omega_j + \alpha_j \epsilon^2_{j,t-1} + \beta_j h_{j,t-1}, \forall j \in \{1, 2, ..., d\}. \end{aligned} \tag{6}$$

Then, the pseudo observations from the inverse of the marginal distributions are added into the copula model. Having estimated the copula parameters, asset returns $\hat{\mathbf{r}}$ can be simulated from the joint distribution.[3] We use these simulations to create a copula-based prior covariance matrix $\boldsymbol{\Sigma} = cov(\hat{\mathbf{r}})$. Inserting this covariance matrix into Eq. (5), we obtain a copula-based equilibrium model.

2.3. Risk-adjusted equilibrium model

In the original BL approach, equilibrium returns are modeled using the reverse optimization of an investor's utility function. In this utility function, portfolio variance is considered as risk measure. Giacometti et al. (2007) suggest an adjustment that incorporates tail risk measures, both VaR and CVaR, to estimate equilibrium returns. Furthermore, market capitalization weights are considered as an optimal portfolio when the CAPM holds. However, the market capitalization portfolio does not account for an asset's tail risk. Instead, we suggest using the CVaR-adjusted equilibrium model suggested by Giacometti et al. (2007) with the weights for a portfolio constructed from copula models. In this case, the risk-adjusted equilibrium returns become:

$$\boldsymbol{\pi} = \frac{\delta}{2}\left(\mathrm{CVaR}_\alpha \frac{\boldsymbol{\Sigma}\hat{\mathbf{w}}}{\sqrt{\hat{\mathbf{w}}^\intercal\boldsymbol{\Sigma}\hat{\mathbf{w}}}} - E(\mathbf{r})\right), \tag{7}$$

where $\hat{\mathbf{w}}$ is a vector of the optimal weights from the mean-CVaR optimization for the Rvine copula model. CVaR_α is the $\alpha-$level CVaR, defined in Eq. (22), and $E(\mathbf{r})$ are the expected returns for the optimal portfolio.

2.4. Investors' Views

The original BL model allows incorporating subjective views in the portfolio optimization, and therefore, provides an efficient approach that does not require the utilization of historical data. This could be considered as one of the main motivations for applying the BL approach. However, the complex nature of financial returns and advancements in econometric time-series modeling, inspired several studies to extend the BL approach and to apply data-generating processes for constructing investors' views. For instance, Zhou (2009) suggests a Bayesian learning approach to include historical data, market equilibrium, and investors' views. Geyer & Lucivjanská (2016) use predictive regressions, with price-to-earnings and dividend-to-price ratios as regressors, to construct investors' views. Beach & Orlov (2007) suggest using the GARCH process in defining views vector and its uncertainty, while Deng (2018) incorporates investors' views using the VECM model augmented with DCC. All these studies aim at improving the BL approach by incorporating objective views that consider the distributional properties of financial data. The VECM, developed in the seminal work of Engle & Granger (1987), takes advantage of the cointegration relations between asset prices in a multivariate setting and has been applied to both the investigation of the interdependence of financial assets (e.g., Östermark, 2001) and the forecasting of stock prices (e.g., Cheung, Cheung, & Wan, 2009; Kuo, 2016).

To construct investors' views, we follow Pfaff (2016) and use the VECM to create expectations of price changes. The VECM allows modeling possible cointegration between asset prices that might lead to better estimation. Let $\mathbf{p}_t = (p_{1t}, p_{2t}, ..., p_{dt})$ be the observed asset prices at time t. Then, the VAR process is given by:

$$\mathbf{p}_t = A_1\mathbf{p}_{t-1} + \cdots + A_\rho\mathbf{p}_{t-\rho} + \mathbf{e}_t. \tag{8}$$

The corresponding VECM, with a transitory form, is given as:

$$\begin{aligned} &\Delta\mathbf{p}_t = \boldsymbol{\alpha}\boldsymbol{\beta}^\intercal\mathbf{p}_{t-1} + \Gamma_1\Delta\mathbf{p}_{t-1} + \cdots + \Gamma_{\rho-1}\mathbf{p}_{t-\rho+1} + \mathbf{e}_t, \\ &\Gamma_i = -(A_{i+1} + \ldots + A_\rho), \quad i = 1, \ldots, \rho-1, \end{aligned} \tag{9}$$

where Γ_i represents the cumulative transitory effects, $\boldsymbol{\alpha}\boldsymbol{\beta}^\intercal$ is of reduced rank in the case of cointegration, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the $d \times k$ loading and long-run coefficient matrices with k cointegration rank, respectively (see Pfaff, 2008). The VECM can be estimated using maximum likelihood estimation as suggested by Johansen (1995). To find the best VAR lag order ρ, we use the Schwartz information criterion (see supplementary material, Fig. S1). To select an optimal cointegration relation k, we use the maximum likelihood cointegration test (see Section 3).

Converting the VECM into a VAR process, we obtain the one-step ahead price trajectories with 10% confidence intervals ($\hat{p}^L_{j,t+1}$ and $\hat{p}^U_{j,t+1}$). Finally, comparing the last adjusted price p_{jt} with the forecasts, the return views $\mathbf{q}_t = (q_{1t}, q_{2t}, ..., q_{dt})$ are computed as:

$$q_{jt} = \begin{cases} \frac{\hat{p}^L_{j,t+1} - p_{jt}}{p_{jt}}, & p_{jt} < \hat{p}^L_{j,t+1} \\ 0, & \hat{p}^L_{j,t+1} < p_{jt} < \hat{p}^U_{j,t+1} \\ \frac{\hat{p}^U_{j,t+1} - p_{jt}}{p_{jt}}, & \hat{p}^U_{j,t+1} < p_{jt}. \end{cases} \tag{10}$$

[3] For more information and the steps involved in GARCH-copula models, see Sahamkhadam, Stephan, & Östermark (2018).

Since we estimate the views vector $\mathbf{q}_t$ using Eq. (10) with $K = d$, the pick matrix is set to identity $\mathbf{P} = \mathbf{I}_d$ such that each row in $\mathbf{P}$ captures the expected return of one asset.[4]

2.5. CBL posterior distributions

In our CBL approach, we suggest estimating returns' posterior distribution using a prior covariance matrix $\boldsymbol{\Sigma}$ from the copula model. To do so, the copula-based posterior mean $\hat{\boldsymbol{\mu}}_{CBL}$ and covariance $\hat{\boldsymbol{\Sigma}}_{CBL}$ are estimated using Eqs. (2) and (3). Given the residuals $\hat{\boldsymbol{\eta}}$ simulated from copula modeling, the CBL posterior distribution is:

$$\hat{\mathbf{r}}_{CBL} = \hat{\boldsymbol{\mu}}_{CBL} + [\hat{\boldsymbol{\Sigma}}_{CBL}]^{\frac{1}{2}}\hat{\boldsymbol{\eta}}, \tag{11}$$

where $\hat{\boldsymbol{\eta}} \sim i.i.d.$ is the vector of the covariance-standardized residuals, which can be simulated from a joint distribution estimated with the copula density function. In the CBL approach, both the posterior covariance matrix and the simulated residuals preserve the dependency structure between assets.

2.6. Rvine copula

Sklar (1959) showed how to obtain the d-dimensional copula C which for the distribution function F with marginal distributions $F_1, ..., F_d$ is:

$$\begin{aligned}\forall \mathbf{z} \in \Re^d : F(z_1, z_2, \dots, z_d) &= C(F_1(z_1), F_2(z_2), \dots, F_d(z_d)) \\ &= C(u_1, u_2, \dots, u_d),\end{aligned} \tag{12}$$

where $z_j = F_j^{-1}(u_j)$, $u_j \sim U[0,1]^d, \forall j \in \{1, 2, \dots, d\}$. If all the marginals are continuous, then C is unique; if C is a d-dimensional copula, then F is a multivariate distribution function with the marginals $F_1, F_2, ..., F_d$ and therefore:

$$C(u_1, u_2, ..., u_d) = F(F_1^{-1}(u_1), F_2^{-1}(u_2), ..., F_d^{-1}(u_d)) = F(z_1, z_2, ..., z_d). \tag{13}$$

Let $\boldsymbol{\Omega}$ be the parameter set in the copula multivariate distribution function $C(u_1, u_2, \dots, u_d|\boldsymbol{\Omega})$ and f_j the derivative of the univariate marginal distribution F_j. Therefore, the density function for the d-dimensional joint distribution is given as:

$$\begin{aligned}f(z_1, z_2, \dots, z_d) &= \frac{\partial^d C(F_1(z_1), F_2(z_2), \dots, F_d(z_d)|\boldsymbol{\Omega})}{\partial z_1, \partial z_2, \dots, \partial z_d} \\ &= c(F_1(z_1), F_2(z_2), \dots, F_d(z_d)|\boldsymbol{\Omega}) \times \prod_{j=1}^{d} f_j(z_j).\end{aligned} \tag{14}$$

In Eqs. (12)–(14), only one copula function C is used to construct the joint distribution, which means only one copula family is considered for the whole set of marginal uniforms $u_1, u_2, \dots, u_d$. Drawing upon the idea of Joe (1996) to decompose the joint density function to several pair-copula densities, Bedford & Cooke (2001) and Bedford, Cooke et al. (2002) derive a graphical representation of the pair-copula construction (PCC) in the form of nested trees including regular (Rvine), drawable, and canonical structures. Aas, Czado, Frigessi, & Bakken (2009) show how to derive the latter two from Rvine structures. More properties and statistical inference of vine copulas are developed in Joe (2014) and Czado (2019).

For a d-dimensional set of continuous random variables, there exist $d(d-1)/2$ pair-copulas, and the copula density c can be decomposed into a product of these pair-copulas' densities. Using a sequence of $i = 1, 2, \dots, d-1$ linked trees, the decomposition can be presented in a graphical PCC, known as the regular vine. Let $e \in E_i$ be the edge between two nodes. n_e and k_e represent a pair-copula $c_{n_e,k_e;D_e}$ conditioned on D_e, with the copula parameter(s) $\boldsymbol{\Omega}_{n_e,k_e|D_e}$. Let $\boldsymbol{u}_{D_e} = \{u_i | i \in D_e\}$ be the variables in the conditioning set D_e, and $C_{n_e|D_e}$ is the conditional distribution of $U_{ne}|\boldsymbol{U}_{D_e}$. When the number of trees increases, the conditioning set D_e also grows and it is common to consider only the dependence of $c_{n_e,k_e;D_e}$ on the indices in D_e, ignoring the impact of $\boldsymbol{u}_{D_e}$. This is the so-called simplifying assumption (see Acar, Genest, & NešLehová, 2012; Haff et al., 2013). Several studies have investigated the quality of this assumption. For instance, Haff, Aas, & Frigessi (2010) show that the simplified PCC provides an acceptable approximation of the dependence structure. Killiches, Kraus, & Czado (2017) also conclude that the simplified PCC results in a smooth fit and is preferred over the non-simplified vine copulas for practical applications with high-dimensional data.

The copula density for a simplified Rvine copula is:

$$c(u|\boldsymbol{\Omega}) = \prod_{i=1}^{d-1} \prod_{e \in E_i} c_{n_e,k_e;D_e}\left(C_{n_e|D_e}(u_{n_e}|\boldsymbol{u}_{D_e}), C_{k_e|D_e}(u_{k_e}|\boldsymbol{u}_{D_e})|\boldsymbol{\Omega}_{n_e,k_e|D_e}\right), \tag{15}$$

with the corresponding log-likelihood function:

$$\mathscr{L}(\boldsymbol{\Omega}|u) = \sum_{j=1}^{d}\sum_{i=1}^{d-1}\sum_{e \in E_i} \ln[c_{n_e,k_e;D_e}(C_{n_e|D_e}(u_{j,n_e}|u_{j,D_e}), C_{k_e|D_e}(u_{j,k_e}|u_{j,D_e})|\boldsymbol{\Omega}_{n_e,k_e|D_e})]. \tag{16}$$

Although vine copulas exhibit flexibility in estimating tail dependency, there is a tradeoff between higher flexibility and increased computational load in high-dimensional settings. For this reason, truncated and simplified vine structures that allow for high-dimensional dependence estimation have been developed (see e.g., Brechmann, Czado, & Aas, 2012; Brechmann & Joe, 2015; Heinen, Valdesogo et al., 2009; Kurowicka, 2011). In particular, Brechmann et al. (2012) show how to truncate the Rvine structure. This truncation is applied to the number of trees in the vine by setting an independence copula at each edge from a specific tree $I \in \{1, 2, \dots, d-1\}$ to the final tree. The I-level truncated Rvine has a density of:

$$c^{Truncated}(u) = \prod_{i=1}^{I} \prod_{e \in E_i} c_{n_e,k_e|D_e}\left(C_{n_e|D_e}(u_{n_e}|u_{D_e}), C_{k_e|D_e}\left(u_{k_e}|u_{D_e}\right)|\boldsymbol{\Omega}_{n_e,k_e|D_e}\right). \tag{17}$$

Nagler, Bumann, & Czado (2019) suggest a modified version of the Bayesian Information Criteria (BIC) for vine copulas and show advantages in its performance for portfolio downside risk forecasting. This selection criteria can be used to identify both the truncation level I and the copula families in a mixed vine copula structure. Following Nagler et al. (2019), the mBICV is defined as:

$$\text{mBICV}(\hat{\boldsymbol{\Omega}}) = -2\mathscr{L}(\hat{\boldsymbol{\Omega}}) + \hat{\imath}\ln d - 2\sum_{i=1}^{d-1}\left[\hat{\varrho}_i \ln\varphi_0^i + \left(d - i - \hat{\varrho}_i\right)\ln\left(1 - \varphi_0^i\right)\right], \tag{18}$$

where $\hat{\varrho}_i$ represents the number of independent copulas in the i:th tree, $\hat{\imath}$ is the number non-zero parameters in the set $\hat{\boldsymbol{\Omega}}$ respectively. $\mathscr{L}(\hat{\boldsymbol{\Omega}}) = \sum_{j=1}^{d} c^{\hat{\boldsymbol{\Omega}}}(\mathbf{u}_j)$ is the log-likelihood and φ_0^i is the probability of not having an independence copula in tree i.[5]

[4] Regarding Eq. (10), the investor's wealth or market evolution (e.g., bull/bear conditions) requiring different investment strategies in different times and contingent on wealth as a dynamic concept could be incorporated (see Östermark, 2017).

[5] See Nagler et al. (2019) for the derivation of the mBICV.

2.7. Portfolio Optimization Methods

According to Markowitz (1952), for a d-dimensional portfolio with asset returns $\hat{\mathbf{r}}_t = (\hat{r}_{1t}, \hat{r}_{2t}, ..., \hat{r}_{dt})$, asset weights $\hat{\mathbf{w}}_t = (\hat{w}_{1t}, \hat{w}_{2t}, ..., \hat{w}_{dt})$, a $d \times d$ positive-definite covariance matrix $\hat{\boldsymbol{\Sigma}}_t$, and a $d \times 1$ vector of asset means $\hat{\boldsymbol{\mu}}_t = (\hat{\mu}_{1t}, \hat{\mu}_{2t}, ..., \hat{\mu}_{dt})$ at time (out-of-sample iteration) t, the portfolio's expected return and variance are $\hat{\mathbf{w}}_t^{\intercal}\hat{\boldsymbol{\mu}}_t$ and $\hat{\mathbf{w}}_t^{\intercal}\hat{\boldsymbol{\Sigma}}_t\hat{\mathbf{w}}_t$, respectively. Markowitz (1952) suggests that investors should maximize the mean-variance utility function. Another possibility is to maximize the Sharpe ratio (SR) which is defined as a ratio of the expected portfolio return to the standard deviation (c.f., Sharpe, 1966; 1994). SR maximization aims to find the optimal portfolio with the highest SR from Markowitz's mean-variance efficient frontier (see Stoyanov, Rachev, & Fabozzi, 2007, for a discussion on reward/risk portfolios). Both reward (portfolio expected return) and risk (portfolio variance) measures are positive homogeneous. Furthermore, the reward and risk measures are concave and convex functions, respectively. Therefore, using the fractional programming suggested by Charnes & Cooper (1962) and Dinkelbach (1967), Max SR optimization reduces to convex linearly constrained quadratic programming and can be posed as:

$$\begin{array}{lll}
\underset{\tilde{\mathbf{w}}_t,\nu}{\text{minimize}} & \tilde{\mathbf{w}}_t^{\intercal}\hat{\boldsymbol{\Sigma}}_t\tilde{\mathbf{w}}_t & \text{portfolio risk} \\
\text{subject to} & \tilde{\mathbf{w}}_t^{\intercal}\hat{\boldsymbol{\mu}}_t \geq \mu_0 & \text{portfolio return} \\
 & \tilde{\mathbf{w}}_t^{\intercal}\mathbf{1} = \nu & \text{full investment} \\
 & 0 \leq \tilde{w}_{jt} \leq \nu, \forall j \in \{1, 2, .., d\} & \text{long positions only} \\
 & \nu > 0, &
\end{array} \tag{19}$$

where μ_0 is the target portfolio return, ν is an auxiliary scaling variable (scalar) and $\tilde{\mathbf{w}}_t$ is a vector of the unconstrained weights; the final optimal weights are obtained as $\hat{w}_{jt} = \frac{\tilde{w}_{jt}}{\nu}$. An important assumption in this optimization problem is the limitation of positive reward and risk measures.

Downside risk is another risk measure that has been considered in many studies of optimal portfolio allocation. For instance, in combination with the classical Markowitz mean-variance portfolio optimization, VaR has been employed as the risk measure leading to an optimal mean-VaR portfolio (see e.g., Baixauli-Soler, Alfaro-Cid, & Fernandez-Blanco, 2011; Consigli, 2002; Lwin, Qu, & MacCarthy, 2017).

Rockafellar & Uryasev (2000) propose linearizing CVaR and using linear programming to solve the minimum CVaR optimization problem. Let $\mathfrak{f}(\hat{\mathbf{w}}_t, \hat{\mathbf{r}}_t)$ be the loss function from the simulated out-of-sample returns, $\alpha \in (0, 1)$ be the confidence level, and $p(\hat{\mathbf{r}}_t)$ be the probability density of $\hat{\mathbf{r}}_t$. The probability that $\mathfrak{f}(\hat{\mathbf{w}}_t, \hat{\mathbf{r}}_t)$ does not exceed a threshold value $l \in \Re$ is:

$$\Psi(\hat{\mathbf{w}}_t, l) = \int_{\mathfrak{f}(\hat{\mathbf{w}}_t, \hat{\mathbf{r}}_t) \leq l} p(\hat{\mathbf{r}}_t) d(\hat{\mathbf{r}}_t), \tag{20}$$

where $\Psi(\hat{\mathbf{w}}_t, l)$ is assumed to be non-decreasing and continuous w.r.t. l; therefore, it is used to derive $\alpha-$level VaR as:

$$\text{VaR}_\alpha(\hat{\mathbf{w}}_t) = \min\{l | \Psi(\hat{\mathbf{w}}_t, l) \geq \alpha\}. \tag{21}$$

CVaR is an alternative when the objective is to minimize losses beyond the VaR threshold:

$$\text{CVaR}_\alpha(\hat{\mathbf{w}}_t) = \frac{1}{1-\alpha} \int_{\mathfrak{f}(\hat{\mathbf{w}}_t, \hat{\mathbf{r}}_t) \geq \text{VaR}_\alpha(\hat{\mathbf{w}}_t)} \mathfrak{f}(\hat{\mathbf{w}}_t, \hat{\mathbf{r}}_t) p(\hat{\mathbf{r}}_t) d\hat{\mathbf{r}}_t, \tag{22}$$

where $\mathbb{P}[\mathfrak{f}(\hat{\mathbf{w}}_t, \hat{\mathbf{r}}_t) \geq \text{VaR}_\alpha(\hat{\mathbf{w}}_t)] = 1 - \alpha$ and $\Psi(\hat{\mathbf{w}}_t, l) = \alpha$.

Rockafellar & Uryasev (2000) suggest using a convex and continuously differentiable function $\mathscr{F}_\alpha(\hat{\mathbf{w}}_t, l)$ as:

$$\mathscr{F}_\alpha(\hat{\mathbf{w}}_t, l) = l + \frac{1}{1-\alpha} \int_{\hat{\mathbf{r}}_t \in \Re^d} [\mathfrak{f}(\hat{\mathbf{w}}_t, \hat{\mathbf{r}}_t) - l]^+ p(\hat{\mathbf{r}}_t) d\hat{\mathbf{r}}_t, \tag{23}$$

where $[\mathfrak{f}(\hat{\mathbf{w}}_t, \hat{\mathbf{r}}_t) - l]^+ = \max\{[\mathfrak{f}(\hat{\mathbf{w}}_t, \hat{\mathbf{r}}_t) - l], 0\}$, and:

$$\text{VaR}_\alpha(\hat{\mathbf{w}}_t) = \underset{l \in \Re}{\text{argmin}}\, \mathscr{F}_\alpha(\hat{\mathbf{w}}_t, l), \;\; \text{CVaR}_\alpha(\hat{\mathbf{w}}_t) = \mathscr{F}_\alpha\left(\hat{\mathbf{w}}_t, \text{VaR}_\alpha(\hat{\mathbf{w}}_t)\right). \tag{24}$$

To approximate $\mathscr{F}_\alpha(\hat{\mathbf{w}}_t, l)$ and express it as piecewise linear w.r.t. l, simulations can be drawn from the probability distribution of $\hat{\mathbf{r}}_t$ according to $p(\hat{\mathbf{r}}_t)$, such that $\hat{\mathbf{r}}_t = \{\hat{\mathbf{r}}_{mt},\ m = 1, ..., M\}$. In this case, we have:

$$\hat{\mathscr{F}}_\alpha(\hat{\mathbf{w}}_t, l) = l + \frac{1}{M(1-\alpha)} \sum_{m=1}^{M} \left[\mathfrak{f}(\hat{\mathbf{w}}_t, \hat{\mathbf{r}}_{mt}) - l\right]^+. \tag{25}$$

Rockafellar & Uryasev (2000) suggest using the approximation presented in Eq. (25) to minimize CVaR. This can be achieved by setting $\mathfrak{f}(\hat{\mathbf{w}}_t, \hat{\mathbf{r}}_{mt}) = -\hat{\mathbf{w}}_t^{\intercal}\hat{\mathbf{r}}_{mt}$ and $l = \text{VaR}_\alpha(\hat{\mathbf{w}}_t)$ and introducing an auxiliary variable $\boldsymbol{\upsilon}_t = (\upsilon_{1t}, \upsilon_{2t}, ..., \upsilon_{Mt})$ such that $\upsilon_{mt} = \left[-\hat{\mathbf{w}}_t^{\intercal}\hat{\mathbf{r}}_{mt} - \text{VaR}_\alpha(\hat{\mathbf{w}}_t)\right]^+$. Therefore, the CVaR minimization problem can be solved using linear programming:

$$\begin{array}{lll}
\underset{\hat{\mathbf{w}}_t,\text{VaR}_\alpha,\boldsymbol{\upsilon}_t}{\text{minimize}} & \text{VaR}_\alpha + \frac{1}{M(1-\alpha)} \sum_{m=1}^{M} \upsilon_{mt} & \text{portfolio risk} \\
\text{subject to} & \hat{\mathbf{w}}_t^{\intercal}\hat{\mathbf{r}}_{mt} + \text{VaR}_\alpha + \upsilon_{mt} \geq 0, & \\
 & \forall m \in \{1, 2, .., M\} & \\
 & \hat{\mathbf{w}}_t^{\intercal}\mathbf{1} = 1 & \text{full investment} \\
 & 0 \leq \hat{w}_{jt} \leq 1, \forall j \in \{1, 2, .., d\} & \text{long positions only} \\
 & \upsilon_{mt} \geq 0, &
\end{array} \tag{26}$$

The stable tail-adjusted return ratio, known as the STARR or mean/CVaR ratio, is given by $\frac{\hat{\mathbf{w}}_t^{\intercal}\hat{\boldsymbol{\mu}}_t}{\text{CVaR}_\alpha(\hat{\mathbf{w}}_t)}$, and can, similar to SR, also be maximized by applying fractional programming. In this case, the fractional objective function is reduced to a linear programming formulation:

$$\begin{array}{lll}
\underset{\tilde{\mathbf{w}}_t,\text{VaR}_\alpha,\boldsymbol{\upsilon}_t,\nu}{\text{minimize}} & \text{VaR}_\alpha + \frac{1}{M(1-\alpha)} \sum_{m=1}^{M} \upsilon_{mt} & \text{portfolio risk} \\
\text{subject to} & \tilde{\mathbf{w}}_t^{\intercal}\hat{\boldsymbol{\mu}}_t \geq \mu_0 & \text{portfolio return} \\
 & \tilde{\mathbf{w}}_t^{\intercal}\hat{\mathbf{r}}_{mt} + \text{VaR}_\alpha + \upsilon_{mt} \geq 0, & \\
 & \forall m \in \{1, 2, .., M\} & \\
 & \tilde{\mathbf{w}}_t^{\intercal}\mathbf{1} = \nu & \text{full investment} \\
 & 0 \leq \tilde{w}_{jt} \leq \nu, \forall j \in \{1, 2, .., d\} & \text{long positions only} \\
 & \upsilon_{mt} \geq 0 & \\
 & \nu > 0. &
\end{array} \tag{27}$$

As the CVaR problem can be expressed in linear terms, it is possible to add it as a constraint into the portfolio optimization problem.

As shown in Alexander et al. (2007), the CVaR constraint better reduces large losses in the mean-variance framework, compared to the VaR constraint. The choice of equality or inequality constraint depends on the extent to which a risk-neutral investor's preference differs from that of a risk-averse investor. In other words, assuming that the risk-neutral investor seeks to maximize the reward/risk ratio, the CVaR constraint limits downside risk to a value close to the risk-averse Min CVaR optimization. This difference can be imposed on the optimization problem by introducing an upper boundary for downside risk, which could be considered as a confidence level for CVaR estimated from the Min CVaR optimization. This boundary provides the CVaR level with flexibility when maximizing the reward/risk ratio. In the first step, we estimate the Min CVaR and Max SR (or STARR) portfolio weights and obtain the minimum level of CVaR for the corresponding portfolios. Then, we take the average of the CVaR values and use it as the upper boundary.[6]

[6] We use this method to obtain a plausible upper boundary for CVaR. Any other logical values can be imposed as long as there is a solution for the optimization.

In this case, the constrained Max SR portfolio optimization is given as:

$$\begin{aligned}
&\underset{\tilde{\mathbf{w}}_t,\mathrm{VaR}_\alpha,\boldsymbol{\upsilon}_t,\nu}{\text{minimize}} && \tilde{\mathbf{w}}_t^\intercal \hat{\boldsymbol{\Sigma}}_t \tilde{\mathbf{w}}_t && \text{portfolio risk}\\
&\text{subject to} && \tilde{\mathbf{w}}_t^\intercal \hat{\boldsymbol{\mu}}_t \geq \mu_0 && \text{portfolio return}\\
& && \mathrm{VaR}_\alpha + \frac{1}{M(1-\alpha)}\sum_{m=1}^{M}\upsilon_{mt} \leq \nu \mathrm{U}_{\mathrm{CVaR}} && \\
& && \tilde{\mathbf{w}}_t^\intercal \hat{\mathbf{r}}_{mt} + \mathrm{VaR}_\alpha + \upsilon_{mt} \geq 0,\ \forall m \in \{1,2,..,M\} && \\
& && \tilde{\mathbf{w}}_t^\intercal \mathbf{1} = \nu && \text{full investment}\\
& && 0 \leq \tilde{w}_{jt} \leq \nu, \forall j \in \{1,2,..,d\} && \text{long positions only}\\
& && \upsilon_{mt} \geq 0 && \\
& && \nu > 0, &&
\end{aligned} \tag{28}$$

where $\mathrm{U}_{\mathrm{CVaR}}$ is the maximum acceptable value of the downside risk of the portfolio. For the constrained Max STARR optimization, we define:

$$\begin{aligned}
&\underset{\tilde{\mathbf{w}}_t,\mathrm{VaR}_\alpha,\boldsymbol{\upsilon}_t,\nu}{\text{minimize}} && \mathrm{VaR}_\alpha + \frac{1}{M(1-\alpha)}\sum_{m=1}^{M}\upsilon_{mt} && \text{portfolio risk}\\
&\text{subject to} && \tilde{\mathbf{w}}_t^\intercal \hat{\boldsymbol{\mu}}_t \geq \mu_0 && \text{portfolio return}\\
& && \mathrm{VaR}_\alpha + \frac{1}{M(1-\alpha)}\sum_{m=1}^{M}\upsilon_{mt} \leq \nu \mathrm{U}_{\mathrm{CVaR}} && \\
& && \tilde{\mathbf{w}}_t^\intercal \hat{\mathbf{r}}_{mt} + \mathrm{VaR}_\alpha + \upsilon_{mt} \geq 0,\ \forall m \in \{1,2,..,M\} && \\
& && \tilde{\mathbf{w}}_t^\intercal \mathbf{1} = \nu && \text{full investment}\\
& && 0 \leq \tilde{w}_{jt} \leq \nu, \forall j \in \{1,2,..,d\} && \text{long positions only}\\
& && \upsilon_{mt} \geq 0 && \\
& && \nu > 0. &&
\end{aligned} \tag{29}$$

2.8. Steps

In this section, the steps involved in constructing the CBL-based portfolio strategies are presented. Let $T_t = O_t + H_t$ be the number of time points T_t in the observation interval O_t and the out-of-sample holdout interval H_t at time t, $O_t \cap H_t = \emptyset, \forall t$ (see Appendix A for more details). Repeat steps 1–4 for a specified set of copula, equilibrium, and portfolio optimization models.

Step 1: Estimate the return's prior distribution and investors' views for Step. For this, generate the parameters $\{\mathbf{u}, \boldsymbol{\Sigma}, \boldsymbol{\pi}, \mathbf{q}, \mathbf{P}, \boldsymbol{\Lambda}, \tau\}$:

(a) $\mathbf{u}$: Extract the standardized residuals $\mathbf{z}$ from the estimated univariate GARCH models in Eq. (6) for the chosen set of stocks using the observed returns $\mathbf{r}$ over O_t. Convert the standardized residuals $\mathbf{z}$ into pseudo-observations $\mathbf{u}$ using the probability marginal distribution.

(b) Copula cumulative distribution function: Estimate the conditional multivariate cumulative distribution function of the returns using the chosen copula model on the pseudo-observations from Step 1.a.

(c) $\boldsymbol{\Sigma}$: Simulate the returns from the cumulative distribution function estimated in Step (1.b) a sufficient number of times to compute the prior covariance matrix $\boldsymbol{\Sigma}$.

(d) $\boldsymbol{\pi}$: Apply the prior covariance matrix $\boldsymbol{\Sigma}$ from Step (1.c) to obtain the prior mean from the chosen equilibrium model in Eq. (5) or (7).

(e) $\mathbf{q}$: Carry out the VECM/VAR estimation in Eqs. (8) and (9) using the observed prices $\mathbf{p}$ over O_t to generate the price trajectories and estimate the directional views $\mathbf{q}$.

(f) Define $\mathbf{P} = \mathbf{I}_d$, $\boldsymbol{\Lambda} = \frac{1}{\kappa}\mathbf{P}\boldsymbol{\Sigma}\mathbf{P}^\intercal$, $\kappa \in (0, \infty)$ and τ. In this study we set $\kappa = 1$ and $\tau = 0.5$ (see Section 4 for more details on τ).

Step 2: Estimate the return's posterior distribution. Insert the estimated parameters $\{\boldsymbol{\Sigma}, \boldsymbol{\pi}, \mathbf{q}, \mathbf{P}, \boldsymbol{\Lambda}, \tau\}$ from Step 1 into Eqs. (2) and (3) and obtain the posterior mean $\hat{\boldsymbol{\mu}}_{CBL}$ and covariance $\hat{\boldsymbol{\Sigma}}_{CBL}$. Use the M simulated standardized residuals $\hat{\boldsymbol{\eta}}$ from the chosen distribution in Step 1.b, as well as the posterior mean and covariance in Eq. (11), and generate the M CBL-return vectors $\hat{\mathbf{r}}_t = \{\hat{\mathbf{r}}_{mt}, m = 1, ..., M\}$.

Step 3: Solve the chosen portfolio problem for the optimal scaled composition vector $\hat{\mathbf{w}}_t$ using the active optimization model discussed in Section 2.7. Register the performance of the portfolio $P_t = \hat{\mathbf{w}}_t^\intercal \mathbf{p}_t$ using the observed prices, with comparisons to the representative benchmark portfolios constructed and held over H_t.

Step 4: Let $t = t + \Delta t$ and repeat from Step 1 for $t \in [t_1, t_2]$.

3. Data

To test the suggested model, we focus on European stocks. Our data set includes daily adjusted prices for constituents of the Eurostoxx 50 index. The sample period runs from February 1998 to December 2020, resulting in 5833 trading days. The data including adjusted prices, historical constituent list, and market capitalization are obtained from Eikon Thompson Reuters' database. We use 3-month government bonds for Germany (until December 2006) and 3-month government AAA-rated spot rates for the Eurozone (from January 2007) as the risk-free rates.[7] Table S1 (see supplementary material) reports the descriptive statistics of the stock returns for constituents of the Eurostoxx 50 index. Except for AIB Group (AIBG.I) and Pharol SGPS SA (PHRA.LS), all series have positive average returns, with Adyen NV (ADYEN.AS) having the highest. The lowest volatility, 1.36%, is observed for Electrabel SA (ELCBt.BR.G07). The minimum and maximum returns are reported for Ageas SA (AGES.BR) and Volkswagen AG (VOWG.DE), respectively. While skewness differs by series, all have positive kurtosis along with the results of Jarque–Bera's normality test, indicating that these return time series are non-normally distributed. The results of the ARCH test (Engle, 1982) with one lag indicate volatility clustering and autocorrelation in the squared residuals for all the series, except for Volkswagen AG (VOWG.DE). The test statistics for the Ljung–Box test with 10 lags suggest the presence of serial correlation for most of the series.

To construct portfolio strategies, we consider the historical constituent list and only include 50 stocks that are the components of Eurostoxx 50 at each trading day. In the BL approach, we use the VECM model to construct investor's views. To investigate whether this model is suitable and there is cointegration rank among the stock price series, we perform the maximum likelihood test suggested by Johansen (1995). We test the null hypothesis of rank k against $k + 1$, and select the first rank when the test does not reject the null hypothesis. Fig. 1 plots the selected cointegration ranks and their p-values using a rolling window of 500 days. As one can see for most out-of-sample intervals, there are cointegration relationships between stock prices. We use these selected ranks when constructing investor's views $\mathbf{q}$ in the CBL approach.

As a robustness check, we also apply the suggested portfolio models to U.S. stocks (i.e., the constituents of the S&P 100 index). The sample for the U.S. market expands from August 1998 to December 2020, resulting in 5631 trading days. We use 3-month U.S. treasury bills as the risk-free rate.[8]

[7] AAA-rated refers to the highest rating a bond issuer can receive from credit-rating agencies.

[8] The historical constituent list and descriptive statistics of the U.S. stock returns are available upon request.

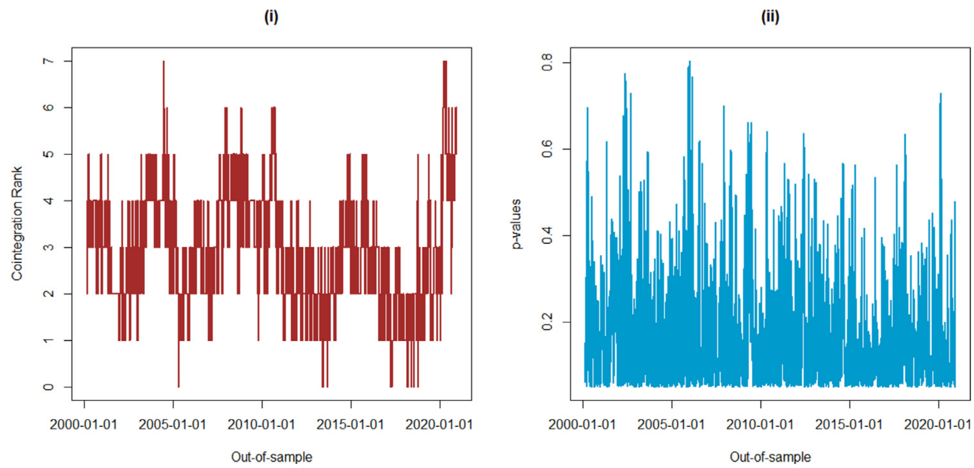

Fig. 1. This figure illustrates (i) selected cointegration ranks, and (ii) *p*-values of the cointegration rank test for the constituents of Eurostoxx 50 index over the out-of-sample using a rolling window of 500 days.

Table 1
Out-of-sample performance of the CBL Portfolios with the CAPM equilibrium.

Portfolio Strategy	Ave. Return	Std. Devation	Sortino Ratio	Sharpe Ratio	CVaR	VaR	STARR Ratio	Mean /VaR	Ave. Turnover	Portfolio Wealth
Panel A: Benchmark Portfolios										
(i) EQW	0.024	1.52	0.021	0.016	6.00	4.50	0.004	0.005	**0.010**	193.36
(ii) Historical-based										
Min CVaR	0.014	**1.17**	0.016	0.012	**4.45**	**3.26**	0.003	0.004	**0.032**	143.17
Max SR	0.010	1.45	0.009	0.007	5.65	3.88	0.002	0.003	**0.126**	91.53
Constrained Max SR	0.008	1.27	0.009	0.007	**4.75**	3.64	0.002	0.002	**0.127**	95.26
Max STARR	0.004	1.50	0.004	0.003	5.83	4.30	0.001	0.001	**0.176**	62.50
Constrained Max STARR	0.004	1.33	0.004	0.003	5.10	3.81	0.001	0.001	0.195	70.42
Panel B: BL Portfolios										
Min CVaR	0.029	**1.25**	0.032	0.023	**4.79**	**3.55**	0.006	0.008	1.39	146.54
Max SR	**0.057**	1.60	**0.048**	**0.036**	6.37	4.62	**0.009**	**0.012**	1.47	**492.46**
Constrained Max SR	**0.049**	1.46	**0.046**	**0.034**	5.75	4.07	**0.009**	**0.012**	1.41	**374.06**
Max STARR	**0.055**	1.58	**0.048**	**0.035**	6.13	4.21	**0.009**	**0.013**	1.65	**410.77**
Constrained Max STARR	**0.048**	1.47	**0.045**	**0.032**	5.64	4.11	**0.008**	**0.012**	1.58	**308.98**
Panel C: Mixed CBL Portfolios										
Min CVaR	0.036	**1.25**	0.038	0.029	4.91	**3.41**	0.007	0.011	1.20	241.59
Max SR	0.044	1.49	0.039	0.030	6.00	4.16	0.007	0.011	1.40	281.44
Constrained Max SR	0.044	1.36	**0.042**	**0.032**	5.41	3.81	**0.008**	**0.012**	1.31	**315.79**
Max STARR	0.017	1.68	0.011	0.010	7.29	4.19	0.002	0.004	1.47	51.39
Constrained Max STARR	0.011	1.59	0.007	0.007	7.06	3.95	0.002	0.003	1.40	40.87
Panel D: Student-*t* CBL Portfolios										
Min CVaR	0.031	**1.21**	0.033	0.025	**4.77**	**3.31**	0.006	0.009	1.28	174.14
Max SR	**0.046**	1.47	0.041	0.031	5.81	4.18	**0.008**	0.011	1.40	306.89
Constrained Max SR	0.042	1.34	0.041	**0.032**	5.31	3.79	**0.008**	0.011	1.32	293.85
Max STARR	0.023	1.64	0.015	0.014	7.05	4.09	0.003	0.006	1.59	67.17
Constrained Max STARR	0.019	1.55	0.013	0.012	6.72	3.82	0.003	0.005	1.52	62.22
Panel E: Clayton CBL Portfolios										
Min CVaR	0.028	**1.26**	0.030	0.023	**4.91**	**3.27**	0.006	0.009	1.08	167.42
Max SR	0.044	1.45	0.041	0.030	5.68	4.14	**0.008**	0.011	1.35	292.75
Constrained Max SR	0.041	1.34	0.040	0.030	5.27	3.70	**0.008**	0.011	1.25	278.99
Max STARR	0.012	1.74	0.008	0.007	7.44	4.58	0.002	0.003	1.34	38.85
Constrained Max STARR	0.011	1.63	0.007	0.007	7.15	4.18	0.002	0.003	1.27	42.23

Notes: The portfolios consist of 50 components of the Eurostoxx 50 index. The out-of-sample period runs from February 2000 to December 2020, consisting of 5333 trading days. The results are obtained by applying rolling window estimation with a training sample size of 500 days. Panel A reports the results for the benchmark model including the EQW and historical-based portfolios. Panel B reports the results for the BL approach. Panels C–E report the results for the CBL model with mixed, Student-*t*, and Clayton copula. VaR and CVaR are estimated empirically at the 1% level. Economic measures are average turnover and final wealth for the portfolio at the end of the sample, assuming a € 100 initial investment with the proportional transaction cost set to 1 basis point (bp). Bold values show the five best portfolios for each measure.

4. Empirical analysis

To evaluate the suggested BL model combined with copula modeling, we perform out-of-sample back-testing and examine the optimal portfolio strategies with a tail constraint. To do so, we use a training sample of 500 days and roll this window over the rest of the targeted sample, which yields 5333 out-of-sample iterations. We use the CBL model and estimate the one-step ahead conditional distribution. In the CBL model, we apply the truncated copula models in which the truncation level is selected based on the mBICV criterion. Estimating the one-step ahead conditional distribution and tail dependence structure, we forecast and sim-

Table 2
Out-of-sample performance of the CBL portfolios with the risk-adjusted equilibrium.

Portfolio Strategy	Ave. Return	Std. Devation	Sortino Ratio	Sharpe Ratio	CVaR	VaR	STARR Ratio	Mean /VaR	Ave. Turnover	Portfolio Wealth
Panel A: BL Portfolios										
Min CVaR	0.031	1.19	0.035	0.026	4.59	3.29	0.007	0.009	1.10	196.52
Max SR	0.042	1.35	0.042	0.031	5.42	3.76	0.008	0.011	**0.841**	376.07
Constrained Max SR	0.039	1.27	0.041	0.031	5.03	3.40	0.008	0.011	**0.820**	335.80
Max STARR	0.034	1.37	0.033	0.025	5.67	3.90	0.006	0.009	1.21	190.60
Constrained Max STARR	0.034	1.32	0.035	0.026	5.29	3.55	0.006	0.010	1.15	212.32
Panel B: Mixed CBL Portfolios										
Min CVaR	0.037	**1.12**	0.046	0.033	**4.11**	**2.97**	0.009	0.013	**1.10**	289.22
Max SR	**0.050**	1.21	**0.057**	**0.041**	4.45	3.24	**0.011**	**0.015**	1.19	**506.40**
Constrained Max SR	**0.047**	**1.16**	**0.056**	**0.040**	**4.18**	**3.07**	**0.011**	**0.015**	1.12	**465.27**
Max STARR	**0.046**	1.25	0.051	0.037	4.59	3.32	0.010	0.014	1.32	373.44
Constrained Max STARR	0.042	1.21	0.048	0.034	4.40	3.12	0.009	0.013	1.25	319.99
Panel C: Student-t CBL Portfolios										
Min CVaR	0.040	**1.13**	0.047	0.035	**4.23**	**2.89**	0.009	0.014	1.17	314.22
Max SR	**0.051**	1.20	**0.058**	**0.043**	4.49	3.20	**0.011**	**0.016**	1.21	**537.79**
Constrained Max SR	**0.046**	**1.15**	**0.054**	**0.040**	**4.30**	**3.04**	**0.011**	**0.015**	1.16	433.45
Max STARR	**0.048**	1.23	0.053	0.039	4.64	3.25	0.010	**0.015**	1.42	403.03
Constrained Max STARR	0.045	1.19	0.051	0.037	4.50	3.07	0.010	0.014	1.33	361.40
Panel D: Clayton CBL Portfolios										
Min CVaR	0.036	**1.15**	0.043	0.031	**4.33**	**3.06**	0.008	0.012	**1.00**	280.61
Max SR	**0.048**	1.22	**0.054**	**0.040**	4.52	3.29	**0.011**	**0.015**	1.13	**480.84**
Constrained Max SR	**0.046**	1.18	0.053	0.039	4.34	3.18	**0.011**	0.014	**1.07**	**445.32**
Max STARR	0.040	1.29	0.043	0.031	4.77	3.52	0.008	0.011	1.20	279.50
Constrained Max STARR	0.034	1.26	0.037	0.027	4.81	3.44	0.007	0.010	1.11	220.77

Notes: The portfolios consist of 50 components of the Eurostoxx 50 index. The out-of-sample period runs from February 2000 to December 2020, consisting of 5333 trading days. The results are obtained by applying rolling window estimation with a training sample size of 500 days. Panel A reports the results for the BL approach. Panels B–D report the results for the mixed, Student-*t*, and Clayton CBL portfolios. VaR and CVaR are estimated empirically at the 1% level. Economic measures are average turnover and final wealth for the portfolio at the end of the sample, assuming a € 100 initial investment with the proportional transaction cost set to 1 basis point (bp). Bold values show the five best portfolios for each measure.

ulate the stock returns. For each out-of-sample iteration, we obtain the optimal weights and, using the realizations of stock returns over the out-of-sample, we calculate the portfolio returns based on a daily rebalancing strategy. Having obtained the returns for the optimal portfolio strategies based on the suggested forecasting models, we compare the out-of-sample descriptive statistics, risk-adjusted ratios, and economic performance of these portfolios.

In the BL approach, the τ parameter is the confidence level for investors' views. In the original BL approach, this parameter is set to a value close to zero (He & Litterman, 1999). This dominates the equilibrium model and places a small weight on investors' views. However, because we are using the VECM/VAR model, we expect to gain from modelling cointegration relations between stocks and possible improvements of the mean forecasts. As we also implement reward/risk optimization, the BL conditional mean modeling will have an effect on the results of the portfolio back-testing. In this study, we suggest to set τ to 0.5, i.e., the equilibrium and the view models have equal weights. We investigate the appropriateness of this choice by performing robustness analyses on the scaling parameter (see Tables S2 and S3 in the supplementary material).

In addition to the equally-weighted (EQW) and historical portfolios, where we apply the portfolio optimization techniques to historical and observed asset returns, we define, as another benchmark, a modified BL approach which is different from the suggested CBL model. First, we use the historical method to obtain the prior covariance $\boldsymbol{\Sigma}$. Second, we estimate the returns' posterior distribution by drawing M simulations from the multivariate Gaussian distribution. Since we apply the investor's view model similar to the CBL model, we consider the BL approach a benchmark on how the equilibrium model is estimated. This provides us a comparison of different distributional assumptions (e.g., symmetric or asymmetric).[9]

Regarding the tests in general, we have compared the performance of our empirical models in three different settings. We have not recognized transaction costs, since all methods use the same one-step ahead forecasts with continuous reshuffling of the portfolios over time. We assume that the different strategies do not, on average, impose significant differences in the true transaction costs that would be imposed in real-world conditions. Hence, the results are comparable.

We divide our empirical investigation into three parts. First, we apply the CBL models with CAPM equilibrium returns and construct the unconstrained and constrained optimal portfolios. We also evaluate different copula families, including Clayton, Student-*t*, and a mixed version where pair-copula families are selected from Gaussian, Student-*t*, Clayton, Frank, Joe, and Gumbel. Second, in Section 4.2, we present and analyze the results of the CBL models with risk-adjusted equilibrium returns. Then, in Section 4.3, we present a multi-period analysis in which the portfolio strategies are compared based on several sub-periods.

4.1. CAPM equilibrium

Table 1 reports the out-of-sample performance for the CBL portfolios. In the mixed version, pair-wise copulas are selected from the Gaussian, Student-*t*, Clayton, Frank, Gumbel, and Joe families based on the mBICV criterion.

[9] For a comparison of alternative investor view modeling, we also construct the BL and CBL portfolios using the momentum strategy suggested by Fabozzi et al. (2006) (see supplementary material, Table S4).

Table 3
Multi-period analysis with CAPM equilibrium.

Portfolio Strategy	2000–2006			2007–2009			2010–2019			2020		
	Ave. Return	Std. Devation	CVaR	Ave. Return	Std. Devation	CVaR	Ave. Return	Std. Devation	CVaR	Ave. Return	Std. Devation	CVaR
Panel A: Benchmark Portfolios												
(i) EQW	0.024	1.48	5.38	-0.003	2.04	7.79	0.034	1.27	4.50	0.000	2.15	9.34
(ii) Historical-based												
Min CVaR	0.015	**1.16**	**4.34**	-0.058	**1.49**	**5.68**	0.040	**1.03**	**3.49**	-0.039	**1.48**	**6.96**
Max SR	-0.011	1.41	5.86	-0.030	1.89	7.58	0.032	1.27	4.16	**0.063**	1.81	8.15
Constrained Max SR	-0.007	1.24	4.72	-0.044	**1.63**	**5.76**	0.031	1.13	3.77	0.040	**1.63**	**7.31**
Max STARR	-0.016	1.40	5.51	-0.064	2.05	8.20	0.034	1.32	4.43	0.048	1.90	8.27
Constrained Max STARR	-0.015	1.26	4.71	-0.064	1.78	**7.06**	0.036	1.18	3.97	0.021	**1.70**	7.55
Panel B: BL Portfolios												
Min CVaR	0.032	**1.24**	**4.59**	**0.006**	**1.61**	**6.21**	0.034	**1.06**	**3.73**	0.029	**1.76**	**7.19**
Max SR	0.084	1.70	6.24	**0.041**	2.10	8.46	**0.042**	1.27	4.73	**0.078**	2.16	8.33
Constrained Max SR	0.062	1.50	5.54	**0.040**	1.94	7.91	0.041	1.17	4.27	**0.074**	2.01	7.97
Max STARR	0.068	1.66	6.25	**0.057**	2.04	7.79	0.042	1.30	4.78	**0.095**	2.03	**7.18**
Constrained Max STARR	0.053	1.50	5.72	**0.050**	1.93	7.23	0.040	1.22	4.31	**0.091**	1.93	**7.20**
Panel C: Mixed CBL Portfolios												
Min CVaR	0.060	**1.12**	**3.54**	-0.017	**1.76**	7.76	0.040	**1.06**	**3.77**	-0.003	1.95	10.12
Max SR	**0.092**	1.50	5.28	-0.074	1.96	8.72	**0.048**	1.23	4.34	0.034	2.03	8.25
Constrained Max SR	**0.090**	1.34	4.46	-0.055	1.78	7.78	**0.042**	1.14	4.09	0.034	1.91	8.35
Max STARR	0.083	1.48	4.83	-0.181	2.79	17.03	0.031	1.25	4.30	0.021	2.22	9.20
Constrained Max STARR	0.072	1.34	4.33	-0.192	2.72	17.38	0.029	1.18	4.09	0.009	2.12	9.31
Panel D: Student-t CBL Portfolios												
Min CVaR	0.047	**1.08**	**3.47**	0.001	**1.74**	**7.21**	0.036	**1.02**	**3.53**	-0.050	1.82	9.07
Max SR	**0.096**	1.48	5.08	-0.068	1.91	7.93	**0.047**	1.22	4.35	0.031	2.03	8.24
Constrained Max SR	**0.086**	1.31	4.45	-0.048	1.77	7.75	**0.043**	1.13	4.01	0.009	1.88	8.04
Max STARR	0.082	1.46	5.03	-0.140	2.68	15.79	0.034	1.23	4.28	-0.010	2.21	9.77
Constrained Max STARR	0.072	1.32	4.48	-0.141	2.62	15.77	0.035	1.15	3.94	-0.026	2.14	9.73
Panel E: Clayton CBL Portfolios												
Min CVaR	0.056	**1.13**	**3.55**	-0.010	1.81	7.67	0.028	**1.06**	**3.70**	-0.041	**1.79**	8.91
Max SR	**0.089**	1.46	4.94	-0.067	1.87	7.69	**0.046**	1.21	4.30	0.046	2.00	8.14
Constrained Max SR	0.084	1.32	4.38	-0.051	1.76	7.62	0.040	1.14	4.03	0.028	1.85	7.94
Max STARR	0.078	1.49	4.70	-0.187	2.93	17.42	0.028	1.26	4.41	-0.010	2.43	9.19
Constrained Max STARR	0.071	1.35	4.36	-0.185	2.82	17.34	0.028	1.18	4.13	0.009	2.18	9.11

Notes: The portfolios consist of 50 components of the Eurostoxx 50 index. The results are obtained by applying rolling window estimation with a training sample size of 500 days. Panel A reports the results for the benchmark model including the EQW and historical-based portfolios. Panel B reports the results for the BL approach. Panels C–E report the results for the CBL model with mixed, Student-*t*, and Clayton copula. CVaR is estimated empirically at the 1% level. Bold values show the five best portfolios for each measure.

Regarding the benchmarks, in Panel A, the historical-based Min CVaR portfolio outperforms EQW in terms of volatility and downside risk. In particular, it results in an out-of-sample CVaR of 4.45%. Though the historical-based reward/risk maximization cannot increase the out-of-sample average return and the risk-adjusted ratios, it results in a less volatile portfolio with lower tail risk compared with the EQW. For instance, for Max SR, the out-of-sample standard deviation and CVaR are 1.45% and 5.65%, respectively. The constrained SR and STARR portfolios both reduce volatility and downside risk compared with the unconstrained strategies. In this case, constrained Max SR has a volatility of 1.27% and a CVaR of 4.75%.

Panel B in Table 1 provides the results for the BL portfolios. All the BL-based strategies achieve a higher average return as well as a better risk-adjusted and economic performance. For instance, the BL-based Max SR portfolio has an average return of 0.057% and an accumulated wealth of € 492.46 at the end of out-of-sample, assuming a € 100 initial investment after considering the proportional transaction costs. In general, the BL approach with CAPM equilibrium improves the portfolio return, but at the cost of increasing the portfolio volatility and downside risk.

In Panel C of Table 1, most of the mixed CBL portfolios can outperform both EQW and historical-based counterparts in terms of higher average returns and risk-adjusted ratios. Comparing the EQW with the CBL Min CVaR portfolios in reducing downside risk, we see that the CBL model reduces out-of-sample CVaR to 3.41%. Although the Max SR and constrained Max SR portfolios result in higher volatility and downside risk than their historical counterpart, improvement in average return leads to considerably higher Sharpe ratios (0.030 and 0.032). Similar to those in Panel B, both the constrained Max SR and the constrained STARR maximization from the mixed CBL model reduce out-of-sample standard deviation and CVaR compared to non-constrained reward/risk optimization.

Panels D and E in Table 1 provide the results for the CBL portfolios with Student-*t* and Clayton Rvine copulas. Comparing the Min CVaR CBL portfolios, the Clayton copula results in lower CVaR (3.27%), while the mixed copula leads to higher average return and economic performance. In general, there is almost no gain from changing the copula family to either Clayton or Student-*t* copulas in terms of risk-adjusted ratios. However, most of the CBL portfolios achieve less volatility and downside risk compared to the BL-implied portfolios.

In summary, these results show that (1) there are advantages to incorporating copula modeling into the BL approach to maximize the investor's utility function, in particular, for reducing the portfolio tail risk, and (2) most of the constrained optimal portfolios result in lower risk (both volatility and downside risk) compared to their unconstrained counterparts.

4.2. Risk-adjusted equilibrium

As mentioned before, to estimate the risk-adjusted equilibrium for the CBL models, we consider a copula-based mean-CVaR port-

Table 4
Multi-period analysis with risk-adjusted equilibrium.

Portfolio Strategy	2000-2006			2007-2009			2010-2019			2020		
	Ave. Return	Std. Devation	CVaR	Ave. Return	Std. Devation	CVaR	Ave. Return	Std. Devation	CVaR	Ave. Return	Std. Devation	CVaR
Panel A: BL Portfolios												
Min CVaR	0.033	1.16	4.26	-0.007	**1.50**	**6.19**	**0.044**	1.03	3.51	-0.008	1.65	7.70
Max SR	0.064	1.38	5.48	**0.019**	1.89	7.56	0.039	1.08	3.74	-0.003	1.57	6.91
Constrained Max SR	0.055	1.29	5.03	**0.017**	1.73	6.94	0.039	1.05	3.65	-0.004	1.56	6.89
Max STARR	0.053	1.37	5.40	0.009	1.91	8.23	0.037	1.11	3.87	-0.066	1.68	6.59
Constrained Max STARR	0.044	1.30	4.97	**0.016**	1.82	7.93	0.040	1.08	3.82	-0.035	1.79	6.58
Panel B: Mixed CBL Portfolios												
Min CVaR	0.048	**1.00**	**2.98**	0.001	**1.50**	**5.92**	0.041	**1.00**	**3.40**	0.037	1.63	7.43
Max SR	**0.090**	1.13	3.44	-0.026	1.66	6.64	**0.043**	1.04	3.57	**0.057**	**1.58**	**7.16**
Constrained Max SR	**0.078**	1.07	**3.18**	-0.019	**1.58**	6.25	**0.044**	1.01	**3.42**	**0.055**	**1.57**	**7.17**
Max STARR	**0.075**	1.15	3.46	-0.004	1.74	6.51	0.039	1.08	3.68	**0.057**	1.74	8.06
Constrained Max STARR	0.064	1.10	3.26	-0.012	1.67	6.33	0.041	1.05	3.56	0.054	1.73	8.01
Panel C: Student-t CBL Portfolios												
Min CVaR	0.050	**1.00**	**3.00**	0.010	1.59	6.47	0.041	**0.985**	**3.38**	0.036	**1.55**	**7.19**
Max SR	0.089	1.11	3.44	-0.015	1.68	6.91	**0.045**	1.03	3.62	**0.050**	**1.52**	6.88
Constrained Max SR	**0.075**	**1.06**	**3.31**	-0.011	1.60	6.41	0.042	**1.00**	**3.48**	0.045	**1.51**	6.89
Max STARR	0.073	1.15	3.78	0.010	1.73	6.63	**0.043**	1.06	3.65	0.039	1.58	**7.22**
Constrained Max STARR	0.067	1.09	3.60	0.005	1.69	6.69	0.041	**1.03**	3.50	0.039	**1.56**	**7.12**
Panel D: Clayton CBL Portfolios												
Min CVaR	0.061	**1.06**	**3.29**	0.008	**1.52**	**5.83**	0.030	**1.01**	**3.58**	0.009	1.70	8.54
Max SR	0.087	1.15	3.57	-0.004	1.66	6.41	0.037	1.06	3.70	**0.050**	1.61	7.67
Constrained Max SR	**0.078**	1.10	3.45	-0.001	**1.58**	**5.97**	0.037	1.04	3.60	0.048	1.61	7.70
Max STARR	0.071	1.17	3.63	**0.032**	1.83	6.34	0.023	1.10	3.88	0.011	1.79	9.78
Constrained Max STARR	0.065	1.13	3.52	**0.017**	1.77	**6.21**	0.025	1.07	3.76	-0.044	1.89	10.50

Notes: The portfolios consist of 50 components of the Eurostoxx 50 index. The results are obtained by applying rolling window estimation with a training sample size of 500 days. Panel A reports the results for the BL approach. Panels B–D report the results for the CBL model with mixed, Student-*t*, and Clayton copula. CVaR is estimated empirically at the 1% level. Bold values show the five best portfolios for each measure.

folio that represents an investor's general utility function maximization. Then, instead of market capitalization weights, we use an asset's weights from this mean-CVaR optimization in Eq. (7) and estimate the return's one-step ahead conditional distribution from the CBL model.

Table 2 presents the out-of-sample performance for the portfolio strategies obtained from the CBL models with risk-adjusted equilibrium returns. In Panel A, the BL-based portfolios, computed using the risk-adjusted equilibrium, show lower out-of-sample volatility and downside risk compared to portfolios based on the CAPM equilibrium (see Table 1). In Panel B, when copula is set to mixed families, the portfolios from the CBL models do outperform both the benchmarks (Panel A, Table 1) and the BL-based strategies in terms of both minimizing portfolio volatility and downside risk as well as maximizing the risk-adjusted ratios. For instance, the Mixed CBL-based Min CVaR portfolio leads to an out-of-sample CVaR of 4.11%, which is lower than the benchmarks' and that of the CBL model with the CAPM equilibrium. Both the constrained and the unconstrained Max STARR portfolios from the mixed CBL model result in higher average returns and thus better economic performance than the same portfolios from the CBL model with CAPM Equilibrium. Applying the risk-adjusted equilibrium in the CBL approach results in both constrained and unconstrained Max SR portfolios with higher reward/risk ratios, and economic performance as well as lower volatility and downside risks than similar portfolios with CAPM equilibrium. In general, the results in Tables 1 and 2 show that the portfolios obtained from the CBL models with risk-adjusted equilibrium returns lead to better out-of-sample performance.

Panels C and D in Table 2 report the results of back-testing for the CBL-based portfolios with Student-*t* and Clayton copulas and risk-adjusted equilibrium returns. Similar to Table 1, there is a limited gain from changing the copula family in the CBL model. In most cases, the CBL model with Student-*t* copula produces both Max SR and Max STARR portfolios with higher average returns, risk-adjusted ratios, and final wealth. For instance, the Max SR portfolio from the CBL model with Student-*t* copulas in Panel C achieves the highest average returns (0.051%) and accumulated wealth (€ 537.79). In general, the mixed CBL portfolios achieve lower downside risk compared to their counterparts with Student-*t* and Clayton copulas. This indicates the advantage of modeling the dependence structure with a mixed copula model, that captures both symmetric and asymmetric tail dependence, to reduce portfolio downside risk. Comparing results in this table with those of Table 1, the risk-adjusted equilibrium model generally produces portfolios with lower volatility and higher returns.

In summary, there are gains from using risk-adjusted equilibrium returns in terms of both maximizing reward/risk ratios and minimizing portfolio downside risk. When using risk-adjusted equilibrium returns, the CVaR-constrained portfolios can achieve lower volatility and downside risk than the unconstrained portfolios, while providing similar reward/risk ratios. For a comparison of the economic performance of CBL portfolios (consisting of the Eurostoxx 50 components) with the benchmarks portfolios, see Figs. S4–S6 in the supplementary material.

4.3. Multi-period analysis

To perform a multi-period analysis, we divide the out-of-sample into several sub-periods. Using these holding periods, we can compare the short-term performance of the suggested portfolio strategies. To do so, we use the first 500 out-of-sample portfolio returns as a holding period and compute the performance measures. Then, rolling this holding period, we obtain the performance measures for 4833 sub-periods. We limit our multi-period analysis to the

Table 5
Robustness check - weekly frequency.

Portfolio Strategy	Ave. Return	Std. Devation	Sortino Ratio	Sharpe Ratio	CVaR	VaR	STARR Ratio	Mean /VaR	Ave. Turnover	Portfolio Wealth
Panel A: Benchmark Portfolios										
(i) EQW	0.180	3.12	0.074	0.058	13.25	8.75	0.014	0.021	**0.022**	338.56
(ii) Historical-based										
Min CVaR	0.150	2.71	0.068	0.055	11.92	8.01	0.013	0.019	**0.068**	285.09
Max SR	0.200	3.53	0.080	0.057	12.86	7.74	0.016	0.026	**0.152**	373.12
Constrained Max SR	0.196	3.28	0.085	0.060	11.93	**7.37**	0.016	0.027	**0.15**	385.41
Max STARR	0.201	3.58	0.076	0.056	14.46	8.50	0.014	0.024	0.234	362.04
Constrained Max STARR	0.194	3.25	0.080	0.060	12.44	7.88	0.016	0.025	**0.231**	372.21
Panel B: BL Portfolios (CAPM Equilibrium)										
Min CVaR	0.196	2.83	0.088	0.069	12.08	8.44	0.016	0.023	1.32	381.29
Max SR	0.214	3.65	0.081	0.059	13.76	9.93	0.016	0.022	1.66	341.06
Constrained Max SR	0.230	3.30	0.094	0.070	12.96	9.60	0.018	0.024	1.56	447.73
Max STARR	**0.292**	3.72	0.113	0.078	13.49	9.61	0.022	0.030	1.70	**695.37**
Constrained Max STARR	**0.275**	3.43	0.110	0.080	13.34	8.88	0.021	0.031	1.62	**659.61**
Panel C: BL Portfolios (Risk-adjusted Equilibrium)										
Min CVaR	0.171	2.84	0.077	0.060	12.21	8.63	0.014	0.020	1.22	303.61
Max SR	0.187	3.57	0.071	0.052	13.78	10.33	0.014	0.018	1.51	275.99
Constrained Max SR	0.200	3.28	0.080	0.061	13.29	9.87	0.015	0.020	1.39	345.81
Max STARR	0.210	3.68	0.080	0.057	14.20	9.84	0.015	0.021	1.58	331.64
Constrained Max STARR	0.210	3.40	0.082	0.062	13.81	10.08	0.015	0.021	1.48	365.25
Panel D: Mixed CBL Portfolios (CAPM Equilibrium)										
Min CVaR	0.189	**2.46**	0.102	0.077	9.75	**6.69**	0.019	0.028	1.22	395.88
Max SR	**0.266**	3.23	**0.125**	0.082	10.48	7.72	**0.025**	**0.034**	1.62	**642.01**
Constrained Max SR	**0.261**	2.93	**0.129**	**0.089**	10.33	7.37	**0.025**	**0.035**	1.53	**671.36**
Max STARR	0.227	3.11	0.106	0.073	10.90	8.00	0.021	0.028	1.62	461.28
Constrained Max STARR	0.229	2.86	0.112	0.080	10.49	7.56	0.022	0.030	1.54	504.58
Panel E: Mixed CBL Portfolios (Risk-adjusted Equilibrium)										
Min CVaR	0.218	**2.38**	**0.119**	**0.091**	**9.40**	**6.34**	**0.023**	**0.034**	1.09	532.70
Max SR	**0.248**	**2.62**	**0.128**	**0.094**	**9.56**	**6.89**	**0.026**	**0.036**	1.43	**645.35**
Constrained Max SR	0.242	**2.49**	**0.128**	**0.097**	**9.52**	**6.83**	**0.025**	**0.035**	1.30	639.22
Max STARR	0.218	2.65	0.111	0.083	**9.64**	7.10	**0.023**	0.031	1.44	487.03
Constrained Max STARR	0.214	**2.52**	0.113	**0.085**	**9.59**	**6.86**	0.022	0.031	1.35	487.96

Notes: The portfolios consist of 50 components of the Eurostoxx 50 index. The out-of-sample period runs from December 2002 to December 2020, consisting of 940 weeks. The results are obtained by applying rolling window estimation with a training sample size of 250 weeks. Panel A reports the results for the benchmark models including the EQW and historical approach. Panel B–C report the results for the BL approach. Panels D–E report the results for the CBL model with mixed copulas. VaR and CVaR are estimated empirically at the 1% level. Economic measures are average turnover and final wealth for the portfolio at the end of the sample, assuming a € 100 initial investment with the proportional transaction cost set to 1 basis point (bp). Bold values show the five best portfolios for each measure.

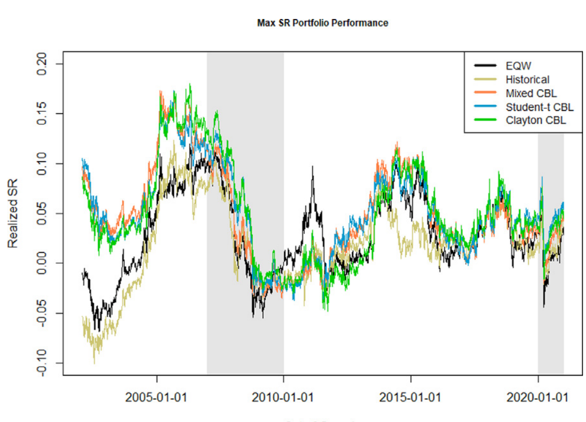

Fig. 2. This figure illustrates the realized SR for the Max SR portfolio strategies computed for each holding period consisting of 500 days. The benchmarks include the EQW and historical-based portfolios. The CBL portfolios are obtained using the Student-t, Clayton, and mixed copula families.

constrained and unconstrained Max SR portfolios obtained from the CBL model with risk-adjusted equilibrium returns.[10]

Fig. 2 plots the realized SRs for the Max SR portfolio strategies. Comparing the benchmarks including the EQW and historical-based portfolios, with the CBL-based strategies shows the gain from applying the CBL model to maximize the SR. For most of the holding periods ending before 2005, the benchmarks result in negative out-of-sample SRs, while the CBL portfolios produce higher average returns and, consequently, positive SRs. In particular, during this period, the mixed CBL Max SR portfolio produces a higher SR than the Student-t and Clayton copula models. As expected, during the global financial crisis (2007–2009), all the portfolios produce lower out-of-sample SRs. During this period, the worst SR is shown for the EQW portfolio (-0.059), while the Clayton CBL-based portfolio results in a higher minimum SR (-0.027). After the global financial crisis and until August 2011, the EQW performs better than all the other portfolios, with a maximum SR equal to 0.097. For most sub-periods after 2011 until 2020, the CBL portfolios outperform the benchmarks. During the COVID-19 pandemic, the minimum SR is reported for EQW (-0.044), while the Student-t CBL Max SR achieves the highest SR (0.087). Comparing with the benchmarks, the copula-based CBL portfolios result in higher SRs for most sub-periods (or holding periods). This indicates advantage of incorporating asymmetric tail dependence that can be captured in both the Clayton and the mixed CBL models.

Fig. 3 shows the result for the constrained Max SR portfolios. Even with tail constraints, the CBL portfolios can outperform the benchmarks. Similar to Fig. 2, the realized SR obtained from the constrained portfolios peaks during 2006. The highest SR (0.176) is reported for the Student-t CBL portfolio, while the historical-based optimization produces the lowest SR (-0.096). Similar to the un-

[10] Further results are available on request.

Table 6
Robustness check - monthly frequency.

Portfolio Strategy	Ave. Return	Std. Devation	Sortino Ratio	Sharpe Ratio	CVaR	VaR	STARR Ratio	Mean /VaR	Ave. Turnover	Portfolio Wealth
Panel A: Benchmark Portfolios										
(i) EQW	0.607	5.26	0.176	0.115	11.19	7.59	0.054	0.080	**0.050**	177.66
(ii) Historical-based										
Min CVaR	0.540	**4.13**	0.217	0.131	**8.17**	**6.21**	0.066	0.087	**0.090**	174.82
Max SR	**0.715**	**4.50**	**0.258**	**0.159**	**8.82**	**7.01**	**0.081**	**0.102**	**0.138**	**211.97**
Constrained Max SR	**0.731**	**4.34**	**0.280**	**0.168**	**8.24**	**6.78**	**0.089**	**0.108**	**0.130**	**218.13**
Max STARR	**0.679**	**4.37**	**0.270**	**0.156**	**7.96**	7.47	**0.085**	**0.091**	**0.172**	**204.57**
Constrained Max STARR	**0.644**	**4.28**	**0.261**	**0.150**	**7.83**	**7.12**	**0.082**	**0.090**	0.202	**196.76**
Panel B: BL Portfolios (CAPM Equilibrium)										
Min CVaR	0.571	5.34	0.135	0.107	12.96	8.14	0.044	0.070	1.52	165.27
Max SR	0.407	7.24	0.075	0.056	16.50	12.18	0.025	0.033	1.77	115.45
Constrained Max SR	0.429	6.63	0.086	0.065	15.25	10.21	0.028	0.042	1.71	125.58
Max STARR	0.597	7.06	0.117	0.085	15.78	11.75	0.038	0.051	1.77	148.82
Constrained Max STARR	0.624	6.40	0.133	0.097	14.54	9.47	0.043	0.066	1.63	163.03
Panel C: BL Portfolios (Risk-adjusted Equilibrium)										
Min CVaR	0.583	5.30	0.146	0.110	12.59	8.44	0.046	0.069	1.50	168.50
Max SR	0.370	7.36	0.066	0.050	17.20	12.76	0.022	0.029	1.77	109.10
Constrained Max SR	0.412	6.68	0.082	0.062	15.66	12.15	0.026	0.034	1.71	122.41
Max STARR	0.453	7.28	0.083	0.062	16.84	11.89	0.027	0.038	1.77	122.03
Constrained Max STARR	0.586	6.59	0.123	0.089	14.87	11.09	0.039	0.053	1.66	153.08
Panel D: Mixed CBL Portfolios (CAPM Equilibrium)										
Min CVaR	0.451	5.02	0.129	0.090	10.76	8.11	0.042	0.056	1.40	146.32
Max SR	0.277	6.46	0.054	0.043	16.38	10.92	0.017	0.025	1.70	105.44
Constrained Max SR	0.296	5.97	0.063	0.050	14.68	9.19	0.020	0.032	1.62	112.53
Max STARR	0.302	6.59	0.059	0.046	15.05	9.89	0.020	0.030	1.71	107.33
Constrained Max STARR	0.330	6.10	0.073	0.054	13.73	9.14	0.024	0.036	1.60	116.41
Panel E: Mixed CBL Portfolios (Risk-adjusted Equilibrium)										
Min CVaR	**0.754**	**4.44**	**0.254**	**0.170**	9.58	**6.20**	**0.079**	**0.122**	0.998	**220.86**
Max SR	0.383	6.17	0.098	0.062	12.77	8.43	0.030	0.045	1.53	125.43
Constrained Max SR	0.552	5.16	0.166	0.107	10.96	7.63	0.050	0.072	1.38	165.04
Max STARR	0.543	5.69	0.157	0.096	11.56	8.94	0.047	0.061	1.54	157.56
Constrained Max STARR	0.583	4.98	0.192	0.117	9.96	8.08	0.059	0.072	1.42	173.37

Notes: The portfolios consist of 50 components of the Eurostoxx 50 index. The out-of-sample period runs from September 2010 to December 2020, consisting of 123 months. The results are obtained by applying rolling window estimation with a training sample size of 150 months. Panel A reports the results for the benchmark models including the EQW and historical approach. Panel B–C report the results for the BL approach. Panels D–E report the results for the CBL model with mixed copulas. VaR and CVaR are estimated empirically at the 5% level. Economic measures are average turnover and final wealth for the portfolio at the end of the sample, assuming a € 100 initial investment with the proportional transaction cost set to 1 basis point (bp). Bold values show the five best portfolios for each measure.

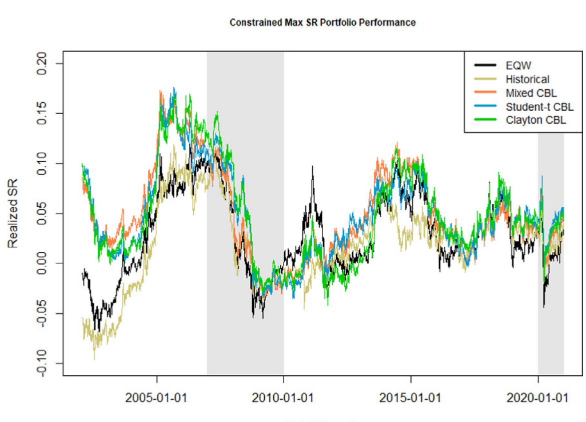

Fig. 3. This figure illustrates the realized SR for the constrained Max SR portfolio strategies computed for each holding period consisting of 500 days. The benchmarks include the EQW and historical-based portfolios. The CBL portfolios are obtained using the Student-*t*, Clayton, and mixed copula families.

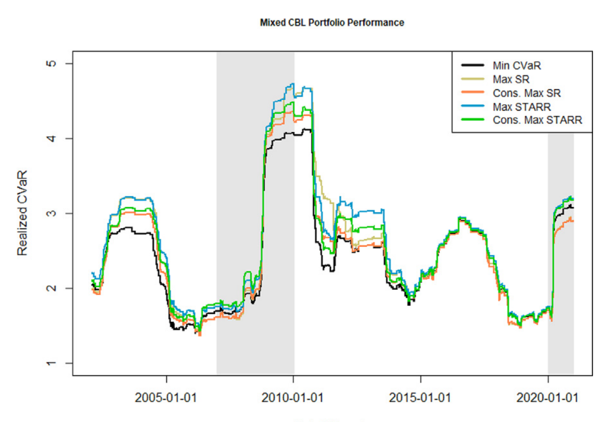

Fig. 4. This figure illustrates the realized CVaR for the mixed CBL portfolio strategies computed for each holding period consisting of 500 days.

constrained optimization, the Clayton CBL model leads to the most sub-periods (1674) with the highest SR, followed by the mixed and Student-*t* CBL models (1499 and 931, respectively)).

To show the improvements obtained from adding the CVaR constraint into the Max SR optimization, we plot the realized CVaR estimated empirically for the mixed CBL portfolios over the holding periods. As shown in Fig. 4, the constrained Max SR and STARR portfolios result in lower CVaR than the unconstrained portfolios. For a comparison of the (constrained) Max STARR CBL portfolios, see Figs. S2 and S3 in the supplementary material.

Table 3 reports performance measures obtained by holding the portfolios over four periods. During the 2000-2006 period, the BL portfolios (with CAPM equilibrium) achieve higher average returns than the benchmarks. In most cases, the reward/risk maximization, augmented with the BL-implied returns' posterior distribution, results in riskier optimal portfolios than those from the benchmarks. The CBL-based optimal portfolios perform better than those from

Table 7
Robustness check - stock market.

Portfolio Strategy	Ave. Return	Std. Devation	Sortino Ratio	Sharpe Ratio	CVaR	VaR	STARR Ratio	Mean /VaR	Ave. Turnover	Portfolio Wealth
Panel A: Benchmark Portfolios										
(i) EQW	0.033	1.23	0.034	0.027	5.04	3.51	0.007	0.009	**0.010**	364.84
(ii) Historical-based										
Min CVaR	0.038	**0.965**	0.051	0.039	**3.91**	2.70	0.010	0.014	**0.056**	**535.32**
Max SR	0.026	1.16	0.029	0.022	4.46	3.32	0.006	0.008	**0.118**	248.56
Constrained Max SR	0.025	1.04	0.031	0.024	4.14	2.92	0.006	0.008	**0.129**	251.27
Max STARR	0.027	1.26	0.028	0.021	4.72	3.69	0.006	0.007	**0.204**	234.07
Constrained Max STARR	0.023	1.10	0.028	0.021	4.24	3.05	0.005	0.008	0.233	213.44
Panel B: BL Portfolios (CAPM Equilibrium)										
Min CVaR	0.035	1.14	0.040	0.031	4.56	3.28	0.008	0.011	1.48	199.32
Max SR	**0.055**	1.33	0.056	0.041	4.95	3.64	0.011	0.015	1.37	524.82
Constrained Max SR	**0.051**	1.24	0.055	0.041	4.73	3.51	0.011	0.014	1.35	452.86
Max STARR	0.043	1.40	0.042	0.031	5.43	3.95	0.008	0.011	1.55	253.55
Constrained Max STARR	0.040	1.34	0.040	0.030	5.28	3.91	0.008	0.010	1.51	225.24
Panel C: BL Portfolios (Risk-adjusted Equilibrium)										
Min CVaR	0.044	1.06	0.056	0.041	4.09	3.01	0.011	0.015	1.26	374.70
Max SR	0.050	1.08	**0.062**	**0.046**	4.23	3.01	**0.012**	**0.017**	0.801	**633.76**
Constrained Max SR	0.046	1.04	0.058	0.044	4.11	2.89	0.011	0.016	0.805	523.42
Max STARR	0.040	1.21	0.045	0.033	4.51	3.44	0.009	0.012	1.25	278.36
Constrained Max STARR	0.041	1.18	0.047	0.034	4.41	3.38	0.009	0.012	1.21	302.68
Panel D: Mixed CBL Portfolios (CAPM Equilibrium)										
Min CVaR	**0.053**	1.13	**0.062**	**0.047**	4.48	3.02	**0.012**	**0.017**	1.29	**551.90**
Max SR	**0.055**	1.23	0.059	0.045	4.79	3.54	**0.012**	0.016	1.29	**598.15**
Constrained Max SR	**0.054**	1.16	**0.060**	**0.046**	4.62	3.27	**0.012**	0.016	1.23	**589.18**
Max STARR	0.048	1.33	0.046	0.036	5.27	3.66	0.009	0.013	1.47	350.42
Constrained Max STARR	0.048	1.28	0.047	0.037	5.18	3.54	0.009	0.014	1.42	370.99
Panel E: Mixed CBL Portfolios (Risk-adjusted Equilibrium)										
Min CVaR	0.044	**0.975**	0.059	0.045	**3.83**	**2.50**	0.011	**0.018**	1.17	412.21
Max SR	0.048	**0.990**	**0.063**	**0.048**	**3.95**	**2.69**	**0.012**	**0.018**	1.19	483.52
Constrained Max SR	0.046	**0.973**	**0.062**	**0.047**	**3.91**	**2.61**	**0.012**	**0.018**	1.15	463.97
Max STARR	0.045	1.03	0.056	0.043	4.15	**2.66**	0.011	**0.017**	1.35	375.41
Constrained Max STARR	0.045	**1.02**	0.057	0.044	**4.09**	**2.60**	0.011	**0.017**	1.30	388.08

Notes: The portfolios consist of 50 most capitalized components of the S&P 100 index. The out-of-sample period runs from July 2002 to December 2020, consisting of 5131 days. The results are obtained by applying rolling window estimation with a training sample size of 500 days. Panel A reports the results for the benchmark models including the EQW and historical approach. Panel B–C report the results for the BL approach. Panels D–E report the results for the CBL model with mixed copulas. VaR and CVaR are estimated empirically at the 1% level. Economic measures are average turnover and final wealth for the portfolio at the end of the sample, assuming a $100 initial investment with the proportional transaction cost set to 1 basis point (bp). Bold values show the five best portfolios for each measure.

the BL approach in terms of volatility and downside risk. During the 2007-2009 (global financial crisis) and 2020 (COVID-19 pandemic), both the BL and CBL approaches fail to produce less volatile portfolios with lower downside risk than those based on the historical approach. However, the BL-based (constrained) Max SR and STARR portfolios lead to higher average returns during the crisis periods.

Table 4 presents the results for the BL and CBL portfolios with risk-adjusted equilibrium over different periods. In regard to portfolio risk, the CBL-based portfolios generally outperform the BL-based strategies. Except for the 2007-2009 period, most of the CBL-based portfolios achieve higher average returns than the BL-based portfolios. Comparing the copula families, the Student-*t* Rvine copula results in portfolios with lower downside risk during the COVID-19 pandemic. However, during the 2007-2009 period, the Clayton copula performs better in reducing the out-of-sample CVaR.

5. Robustness checks

5.1. Data frequency

As a robustness check, we compare the performance of the portfolios obtained from the CBL model with the different copula families by changing the data frequency to weekly and monthly. We limit our analysis in this section to the CBL model with mixed copula families. For both frequencies, the data set runs from February 1998 to December 2020, resulting in 1190 weekly and 270 monthly observations. For the weekly (monthly) frequency, we use a rolling window of 250 (150) observations to construct the portfolio strategies.

Table 5 reports the out-of-sample performance of the portfolio strategies obtained from the weekly frequency. In Panels B and C, most of the BL portfolios result in higher average return and volatility than the benchmarks in Panel A. Those based on the CAPM equilibrium generally achieve higher accumulation of wealth. The mixed CBL-based Min CVaR optimization reduces out-of-sample CVaR to 9.40% compared to 11.92% of the historical approach. The Max SR and STARR portfolios also result in low CVaR values (9.56% and 9.64%, respectively). In Panel B, both the BL-based Max STARR portfolios increase the out-of-sample average return compared to their counterparts from the mixed copula CBL (Panels D and E) models. When applying the risk-adjusted equilibrium, the mixed CBL approach also reduces the average turnover, compared to the BL model. For a multi-period analysis with weekly frequency, see supplementary material Table S5.

Table 6 presents the results for the monthly frequency. Considering the benchmarks in Panel A, most of the historical-based portfolios result in lower portfolio volatility and higher average return. Both the BL and CBL approaches fail to produce portfolios with better performance measures than the historical-based. Most likely this is because the parameters in these models cannot be estimated with a sufficient precision which is needed for an improvement of portfolio performance. However, comparing the BL-based portfolios with those from the CBL model, we can conclude that the suggested CBL approach results in lower portfolio downside risk and volatility for both risk minimization and reward/risk maximization.

5.2. U.S. stock market

To investigate the performance of suggested models in another financial market, we apply the CBL approach to a data set consisting of the 50 most capitalized U.S. stocks that are constituents of the S&P 100 index. Similar to the Eurostoxx 50 index, we obtain the historical constituents of S&P 100, and from this list, we include those stocks with the highest market capitalization and construct several portfolio strategies.[11] Similar to Section 5.1, we only include CBL-based portfolios with mixed copula families.

Table 7 reports the results for the U.S. market. All the BL and CBL portfolios achieve higher average returns than both the EQW and historical-based strategies. Although the risk-adjusted equilibrium leads to portfolios with lower volatility and downside risk, the CAPM equilibrium generally increases the portfolio average return. The CBL-based Min CVaR portfolio (Panel E) shows the lowest out-of-sample CVaR (3.83%). Comparing to their unconstrained counterparts, all the tail-constrained portfolios result in lower volatility and downside risk, while achieving similar risk-adjusted ratios. Overall, these results indicate the advantages of (i) incorporating copula modeling with the BL approach, (ii) applying risk-adjusted equilibrium, and (iii) including tail constraints in the reward/risk maximization, in achieving less volatile portfolios with less downside risk. For a comparison of the economic performance of the CBL portfolios (consisting of the S&P 100 components) with the benchmarks, see Figures S7– S9 in the supplementary material.

6. Conclusions

In this study, we suggest and evaluate a novel CBL approach for modeling and forecasting portfolio returns. The CBL approach includes estimating the tail dependency structure and covariance matrix from truncated Rvine copulas. Using investors' views as well as CAPM-based and risk-adjusted equilibrium models, we obtain the BL posterior mean and covariance matrix. Then, using standardized residuals from copula modeling, we simulate one-step ahead asset returns. We consider three vine copula models: Student-t, Clayton, and mixed versions. We then use the mBICV criterion to select both the truncation level and the copula families (in the mixed version). Additionally, we impose tail constraints in the reward/risk maximization to reduce downside risk. We apply the models to the 50 components of the Eurostoxx 50 index (also S&P 100 for robustness check) and perform out-of-sample portfolio back-testing.

The results of the portfolio back-testing support using the suggested models. For the daily data frequency, we find gains from incorporating copula modeling into the BL approach to maximize the investor's utility function, particularly for the Min CVaR and Max SR portfolios. Most of the constrained optimal portfolios result in lower risk (both volatility and downside risk). In general, the CBL models outperform the benchmark portfolios for most out-of-sample measures. Higher portfolio returns and risk-adjusted performance are obtained from the CBL models with the risk-adjusted equilibrium compared with models with the CAPM-based equilibrium. Based on a multi-period analysis, we show the advantages (e.g., higher out-of-sample SR) of incorporating the asymmetric tail dependence captured in both the Clayton and the mixed CBL models.

The CBL approach can be extended in several aspects. First, other vine copula models (e.g., canonical and drawable vines) can be used with this approach. Second, because of its ability to model both lower and upper tail dependency, the CBL model can also be applied to other reward/risk maximization problems with different risk and reward measures. The CBL model can also be combined with multi-criteria portfolios where several risk and reward measures are included in a scenario-based optimization.

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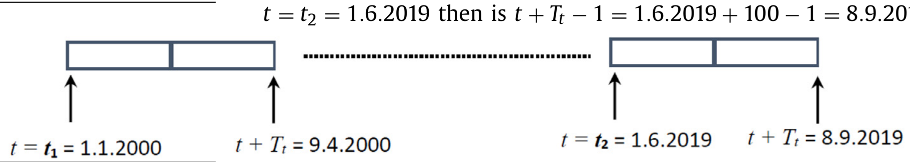
Appendix A. Advancement of the data window over time

The estimation sample O_t length is fixed (e.g., 500 days), and when changing t, it is updated with a new information set (since this is a rolling window). This is the same for the out-of-sample holdout interval H_t (e.g., for daily re-balancing $H_t = 1$.). Below we provide more clarifications:

T_t points to the starting point of the current test interval consisting of the observation set O_t for the estimation and the holdout set H_t for performance evaluation. Example: O_t, $H_t = 50$ time points each, hence $T_t = O_t + H_t = 100$. Let $t_1 = 1.1.2000$ and initialize $t = t_1$ before step 1 (Section 2.8). Then the final date of the first observation interval is $t + O_t - 1 = 1.1.2000 + 50 - 1 = 19.2.2000$ and that of the first holdout interval $t + O_t + H_t - 1 = 1.1.2000 + 100 - 1 = 9.4.2000$ (not separating workdays from holidays in this example). Iterating steps 1 - 4 through the data sample with step length Δ_t gives the first date of the last observation interval, for example $t_2 = 1.6.2019$. The final date of the last holdout interval at $t = t_2 = 1.6.2019$ then is $t + T_t - 1 = 1.6.2019 + 100 - 1 = 8.9.2019$.



Supplementary material

Supplementary material associated with this article can be found, in the online version, at 10.1016/j.ejor.2021.06.015.

[11] Due to the computational intensity, we refrained to use all the 100 stocks of the S&P 100 as input for the various models.

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