Artificial Intelligence (CS303)

Lecture 9: Inference in Bayesian Networks

Hints for this lecture

- **Calculate/estimate posterior probability distribution.**

Outline of this lecture

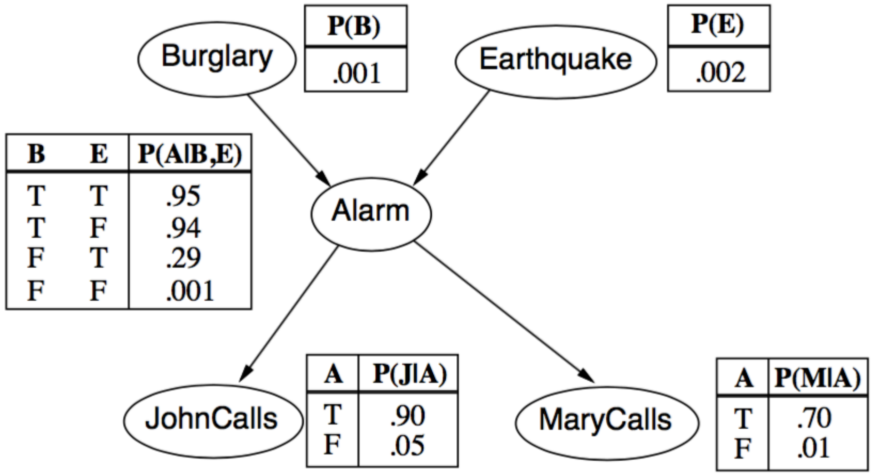
- **Problem Statement**
- **Exact Inference**
- **Approximate Inference**

I. Problem Statement

What are we looking for?

- Given a Bayesian Network, and an (or some) observed events, which specifies the value for evidence variables, we want to know the probability distribution of one (or several) query variables X, P (X | events).

- E. g. : $\mathbf{P}(Burglary \mid JohnCalls = true, MaryCalls = true)$



More Examples

Simple queries: compute posterior marginal $\mathbf{P}(X_i|\mathbf{E}=\mathbf{e})$
e.g., $P(NoGas|Gauge=empty, Lights=on, Starts=false)$

Conjunctive queries: $\mathbf{P}(X_i, X_j|\mathbf{E}=\mathbf{e}) = \mathbf{P}(X_i|\mathbf{E}=\mathbf{e})\mathbf{P}(X_j|X_i, \mathbf{E}=\mathbf{e})$

Optimal decisions: decision networks include utility information;
probabilistic inference required for $P(outcome|action, evidence)$

Value of information: which evidence to seek next?

Sensitivity analysis: which probability values are most critical?

Explanation: why do I need a new starter motor?

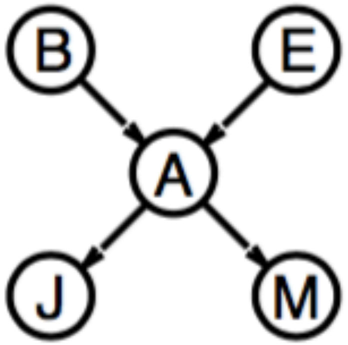
II. Exact Inference

Enumeration

Simple query on the burglary network:

$\mathbf{P}(B|j,m)$
$= \mathbf{P}(B,j,m)/P(j,m)$
$= \alpha\mathbf{P}(B,j,m)$
$= \alpha \sum_e \sum_a \mathbf{P}(B,e,a,j,m)$

Need to consider all values of "hidden variables", e.g., *alarm*=true, *alarm*=false



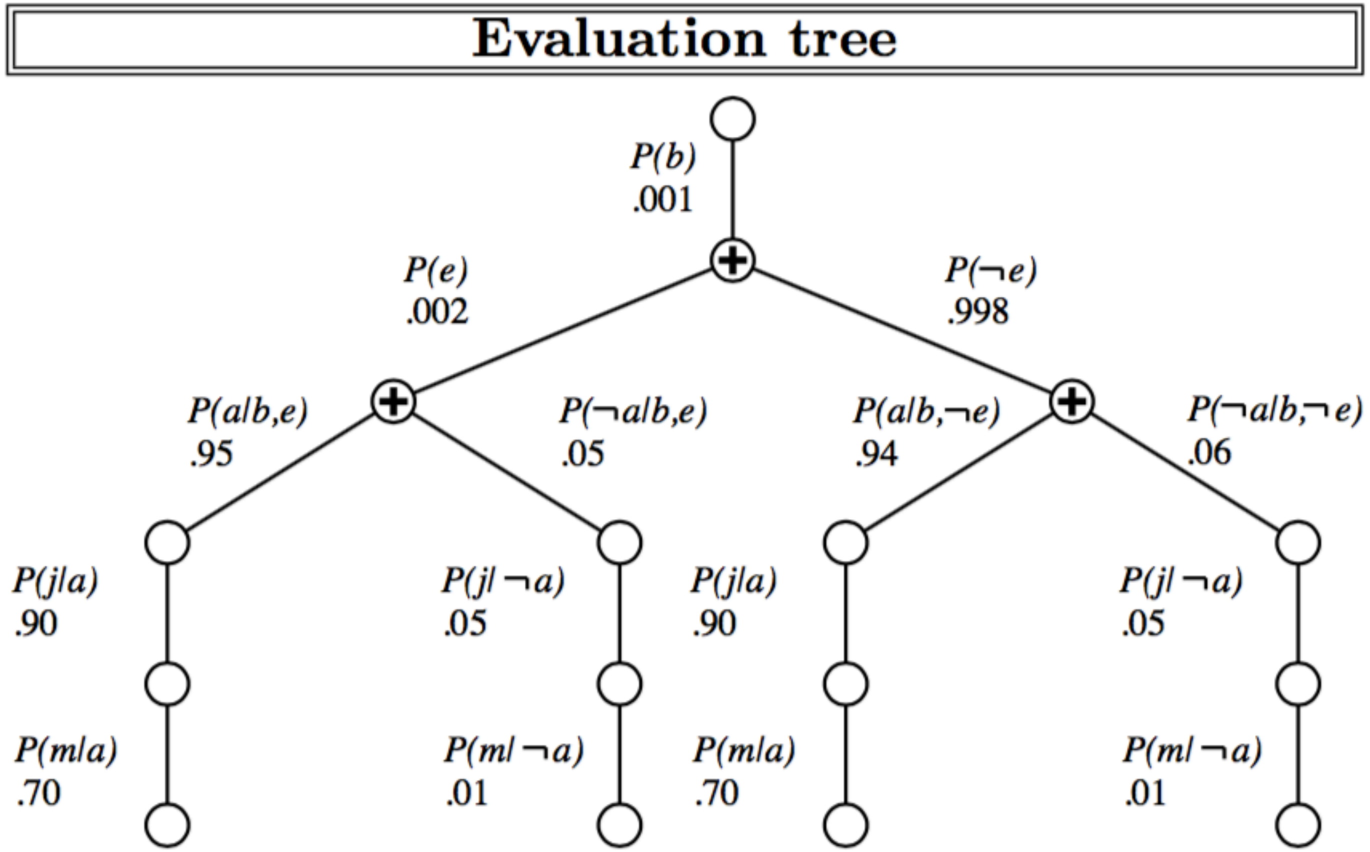
Rewrite full joint entries using product of CPT entries:

$\mathbf{P}(B|j,m)$
$= \alpha \sum_e \sum_a \mathbf{P}(B)P(e)\mathbf{P}(a|B,e)P(j|a)P(m|a)$
$= \alpha\mathbf{P}(B) \sum_e P(e) \sum_a \mathbf{P}(a|B,e)P(j|a)P(m|a)$

Recursive depth-first enumeration: $O(n)$ space, $O(d^n)$ time

d=2 for Boolean variables

Enumeration



Enumeration is inefficient: repeated computation
e.g., computes $P(j|a)P(m|a)$ for each value of e

Enumeration by Variable Elimination

Variable elimination: carry out summations right-to-left, storing intermediate results (factors) to avoid recomputation

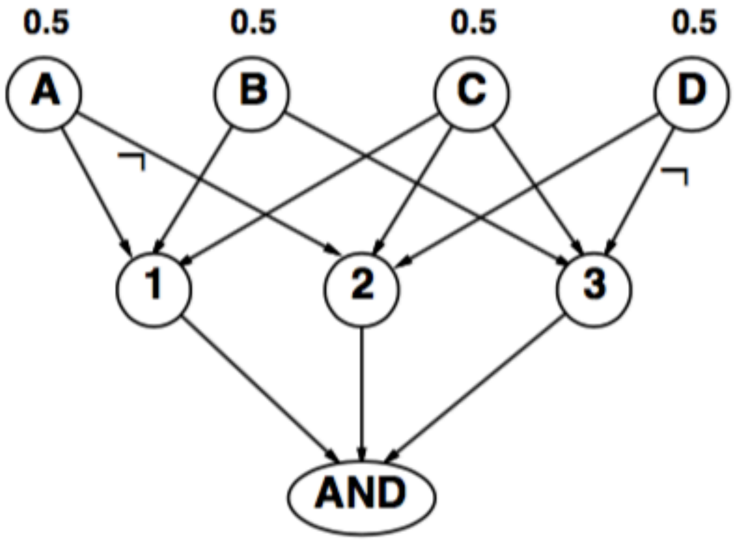
$$
\begin{aligned}
&\mathbf{P}(B|j,m) \\
&\quad = \alpha \underbrace{\mathbf{P}(B)}_{B} \Sigma_e \underbrace{P(e)}_{E} \Sigma_a \underbrace{\mathbf{P}(a|B,e)}_{A} \underbrace{P(j|a)}_{J} \underbrace{P(m|a)}_{M} \\
&\quad = \alpha \mathbf{P}(B) \Sigma_e P(e) \Sigma_a \mathbf{P}(a|B,e) P(j|a) f_M(a) \\
&\quad = \alpha \mathbf{P}(B) \Sigma_e P(e) \Sigma_a \mathbf{P}(a|B,e) f_J(a) f_M(a) \\
&\quad = \alpha \mathbf{P}(B) \Sigma_e P(e) \Sigma_a f_A(a,b,e) f_J(a) f_M(a) \\
&\quad = \alpha \mathbf{P}(B) \Sigma_e P(e) f_{\bar{A}JM}(b,e) \text{ (sum out } A\text{)} \\
&\quad = \alpha \mathbf{P}(B) f_{\bar{E}\bar{A}JM}(b) \text{ (sum out } E\text{)} \\
&\quad = \alpha f_B(b) \times f_{\bar{E}\bar{A}JM}(b)
\end{aligned}
$$

Complexity of Exact Inference

Singly connected networks (or polytrees):
- any two nodes are connected by at most one (undirected) path
- time and space cost of variable elimination are $O(d^k n)$

Multiply connected networks:
- can reduce 3SAT to exact inference $\Rightarrow$ NP-hard
- equivalent to **counting** 3SAT models $\Rightarrow$ #P-complete

1. A v B v C
2. C v D v ¬A
3. B v C v ¬D



III. Approximate Inference

Basic Idea

- **Sampling/Monte Carlo/Stochastic Simulation…**

Basic idea:

1) Draw N samples from a sampling distribution S
2) Compute an approximate posterior probability $\hat{P}$
3) Show this converges to the true probability P

Outline:

- Sampling from an empty network
- Rejection sampling: reject samples disagreeing with evidence
- Likelihood weighting: use evidence to weight samples
- Markov chain Monte Carlo (MCMC): sample from a stochastic process whose stationary distribution is the true posterior

Sampling from an empty network

function PRIOR-SAMPLE(bn) **returns** an event sampled from bn
  inputs: bn, a belief network specifying joint distribution $\mathbf{P}(X_1, \ldots, X_n)$

  $\mathbf{x} \leftarrow$ an event with n elements
  for $i = 1$ **to** n **do**
    $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
      given the values of $Parents(X_i)$ in $\mathbf{x}$
  return $\mathbf{x}$

Assume the joint distribution could be easily sampled

Sampling from an empty network

Probability that PRIORSAMPLE generates a particular event

$$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i|parents(X_i)) = P(x_1 \ldots x_n)$$

i.e., the true prior probability

E.g., $S_{PS}(t, f, t, t) = 0.5 \times 0.9 \times 0.8 \times 0.9 = 0.324 = P(t, f, t, t)$

Let $N_{PS}(x_1 \ldots x_n)$ be the number of samples generated for event $x_1, \ldots, x_n$

Then we have

$$\begin{aligned} \lim_{N\to\infty} \hat{P}(x_1, \ldots, x_n) &= \lim_{N\to\infty} N_{PS}(x_1, \ldots, x_n)/N \\ &= S_{PS}(x_1, \ldots, x_n) \\ &= P(x_1 \ldots x_n) \end{aligned}$$

That is, estimates derived from PRIORSAMPLE are consistent

Shorthand: $\hat{P}(x_1, \ldots, x_n) \approx P(x_1 \ldots x_n)$

Rejection Sampling

$\hat{\mathbf{P}}(X|\mathbf{e})$ estimated from samples agreeing with $\mathbf{e}$

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function REJECTION-SAMPLING(X, e, bn, N) returns an estimate of P(X|e)
   local variables: N, a vector of counts over X, initially zero

   for j = 1 to N do
      x ← PRIOR-SAMPLE(bn)
      if x is consistent with e then
         N[x] ← N[x]+1 where x is the value of X in x
   return NORMALIZE(N[X])
```

In case distribution of one or more variable is difficult to sample

E.g., estimate $\mathbf{P}(Rain|Sprinkler = true)$ using 100 samples
27 samples have $Sprinkler = true$
Of these, 8 have $Rain = true$ and 19 have $Rain = false$.

$\hat{\mathbf{P}}(Rain|Sprinkler = true) = \text{NORMALIZE}(\langle 8, 19\rangle) = \langle 0.296, 0.704\rangle$

Similar to a basic real-world empirical estimation procedure

$$\begin{aligned}
\hat{\mathbf{P}}(X|\mathbf{e}) &= \alpha \mathbf{N}_{PS}(X, \mathbf{e}) && \text{(algorithm defn.)} \\
&= \mathbf{N}_{PS}(X, \mathbf{e})/N_{PS}(\mathbf{e}) && \text{(normalized by } N_{PS}(\mathbf{e})\text{)} \\
&\approx \mathbf{P}(X, \mathbf{e})/P(\mathbf{e}) && \text{(property of PRIORSAMPLE)} \\
&= \mathbf{P}(X|\mathbf{e}) && \text{(defn. of conditional probability)}
\end{aligned}$$

Hence rejection sampling returns consistent posterior estimates

Problem: hopelessly expensive if $P(\mathbf{e})$ is small

$P(\mathbf{e})$ drops off exponentially with number of evidence variables!

Likelihood Weighting

Idea: fix evidence variables, sample only nonevidence variables, and weight each sample by the likelihood it accords the evidence

Fix the evidence variables to reduce the sampling space

function LIKELIHOOD-WEIGHTING(X, $\mathbf{e}$, bn, N) **returns** an estimate of $P(X|\mathbf{e})$
  local variables: $\mathbf{W}$, a vector of weighted counts over X, initially zero

  for $j = 1$ to N **do**
    $\mathbf{x}, w \leftarrow$ WEIGHTED-SAMPLE(bn)
    $\mathbf{W}[x] \leftarrow \mathbf{W}[x] + w$ where x is the value of X in $\mathbf{x}$
  return NORMALIZE($\mathbf{W}[X]$)

---

function WEIGHTED-SAMPLE(bn, $\mathbf{e}$) **returns** an event and a weight

  $\mathbf{x} \leftarrow$ an event with n elements; $w \leftarrow 1$
  for $i = 1$ **to** n **do**
    if X_i has a value x_i in $\mathbf{e}$
      then $w \leftarrow w \times P(X_i = x_i \mid parents(X_i))$
      else $x_i \leftarrow$ a random sample from $\mathbf{P}(X_i \mid parents(X_i))$
  return $\mathbf{x}$, w

Markov Chain Monte Carlo (MCMC)

"State" of network = current assignment to all variables.

Generate next state by sampling one variable given Markov blanket
Sample each variable in turn, keeping evidence fixed

function MCMC-ASK(X, $\mathbf{e}$, bn, N) **returns** an estimate of $P(X|\mathbf{e})$
  local variables: $\mathbf{N}[X]$, a vector of counts over X, initially zero
    $\mathbf{Z}$, the nonevidence variables in bn
    $\mathbf{x}$, the current state of the network, initially copied from $\mathbf{e}$

  initialize $\mathbf{x}$ with random values for the variables in $\mathbf{Y}$
  for $j = 1$ to N **do**
    for each Z_i in $\mathbf{Z}$ **do**
      sample the value of Z_i in $\mathbf{x}$ from $\mathbf{P}(Z_i|mb(Z_i))$
        given the values of $MB(Z_i)$ in $\mathbf{x}$
      $\mathbf{N}[x] \leftarrow \mathbf{N}[x] + 1$ where x is the value of X in $\mathbf{x}$
  return NORMALIZE($\mathbf{N}[X]$)

Markov Blanket

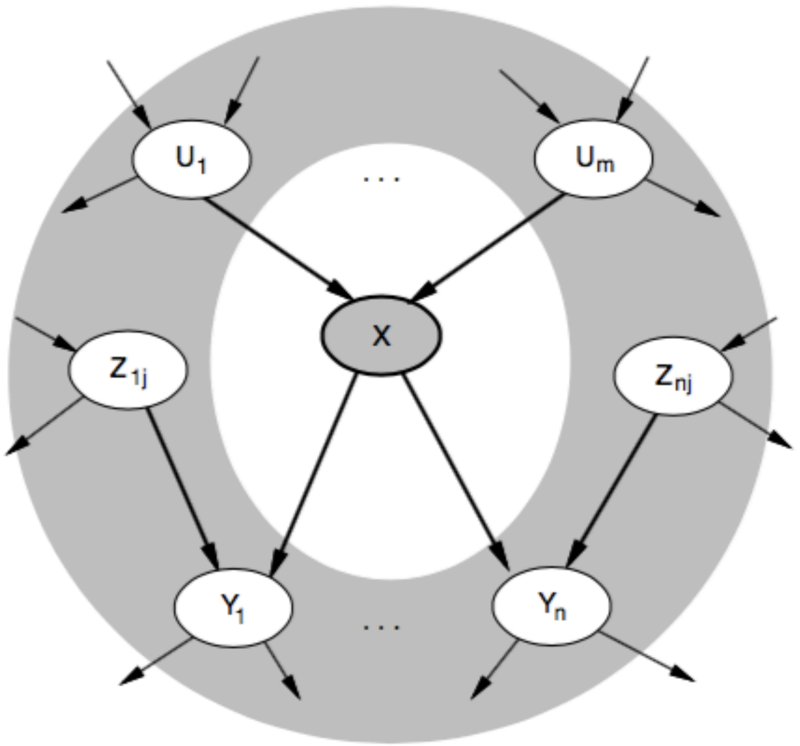
Further reduce the sampling space by only considering variables in Markov blanket

Can also choose a variable to sample at random each time

Markov Chain Monte Carlo (MCMC)

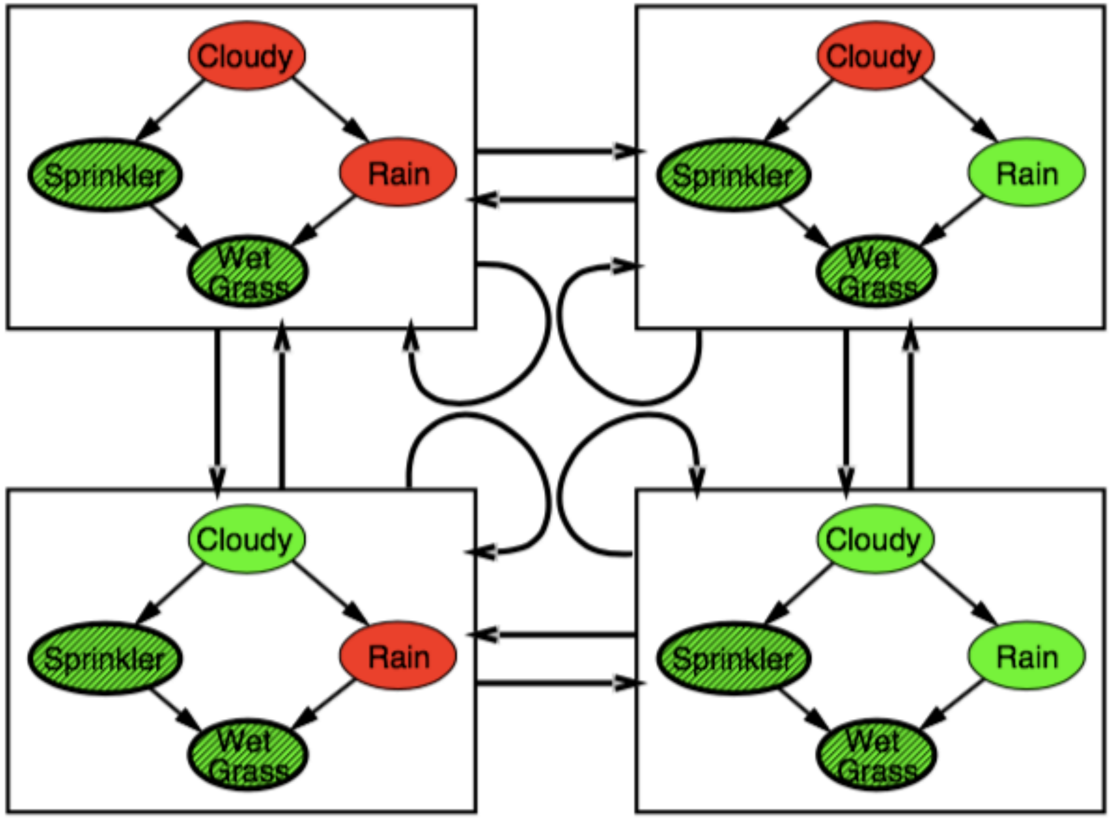
Markov blanket

Each node is conditionally independent of all others given its Markov blanket: parents + children + children's parents



Markov Chain Monte Carlo (MCMC)

With $Sprinkler = true, WetGrass = true$, there are four states:



Theorem: chain approaches stationary distribution:
long-run fraction of time spent in each state is exactly proportional to its posterior probability

Wander about for a while, average what you see

Summary

Exact inference by variable elimination:

- polytime on polytrees, NP-hard on general graphs
- space $=$ time, very sensitive to topology

Approximate inference by LW, MCMC:

- LW does poorly when there is lots of (downstream) evidence
- LW, MCMC generally insensitive to topology
- Convergence can be very slow with probabilities close to 1 or 0
- Can handle arbitrary combinations of discrete and continuous variables

To be continued