# EEE5015: Machine Learning & Artificial Intelligence

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- Logistic Regression
  - Logistic regression
  - Regularization
  - Cross validation
- Multi-class classification
  - Linear regression
  - Logistic regression

# Logistic Regression

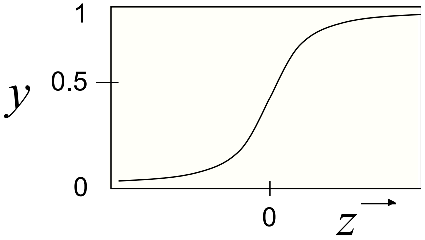
An alternative: replace the $sign(\cdot)$ with the sigmoid or logistic function

We assumed a particular functional form: sigmoid applied to a linear function of the data

$$y(\mathbf{x}) = \sigma\left(\mathbf{w}^T\mathbf{x} + w_0\right)$$

where the sigmoid is defined as

$$\sigma(z) = \frac{1}{1 + \exp(-z)}$$




The output is a smooth function of the inputs and the weights. It can be seen as a smoothed and differentiable alternative to $sign(\cdot)$

# Logistic Regression

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where the sigmoid is defined as

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- One parameter per data dimension (feature) and the bias
- Features can be discrete or continuous
- Output of the model: value $y \in [0, 1]$
- Allows for gradient-based learning of the parameters

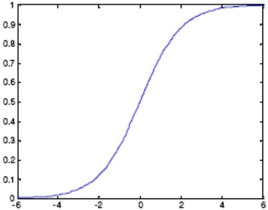
# Shape of the Logistic Function

Let's look at how modifying **w** changes the shape of the function

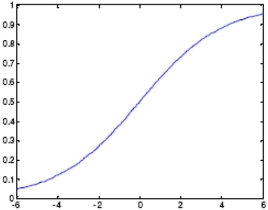
1D example:

$$y = \sigma\left(w_1 x + w_0\right)$$

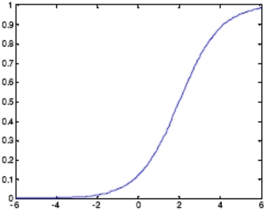
$w_0 = 0, w_1 = 1$




$w_0 = 0, w_1 = 0.5$




$w_0 = -2, w_1 = 1$



# Probabilistic Interpretation

If we have a value between 0 and 1, let's use it to model class probability

$$p(C = 0|\mathbf{x}) = \sigma(\mathbf{w}^T\mathbf{x} + w_0) \quad \text{with} \quad \sigma(z) = \frac{1}{1 + \exp(-z)}$$

Substituting we have

$$p(C = 0|\mathbf{x}) = \frac{1}{1 + \exp\left(-\mathbf{w}^T\mathbf{x} - w_0\right)}$$

Suppose we have two classes, how can I compute $p(C = 1|\mathbf{x})$?

Use the marginalization property of probability

$$p(C = 1|\mathbf{x}) + p(C = 0|\mathbf{x}) = 1$$

Thus

$$p(C = 1|\mathbf{x}) = 1 - \frac{1}{1 + \exp\left(-\mathbf{w}^T\mathbf{x} - w_0\right)} = \frac{\exp(-\mathbf{w}^T\mathbf{x} - w_0)}{1 + \exp\left(-\mathbf{w}^T\mathbf{x} - w_0\right)}$$

# Decision Boundary for Logistic Regression

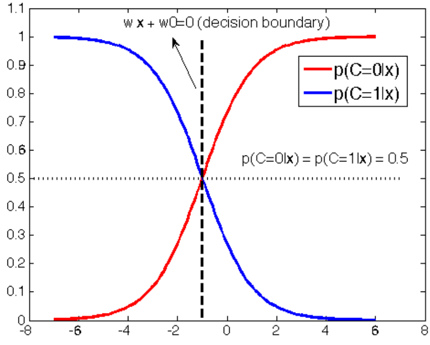
What is the decision boundary for logistic regression?

$p(C = 1|\mathbf{x}, \mathbf{w}) = p(C = 0|\mathbf{x}, \mathbf{w}) = 0.5$

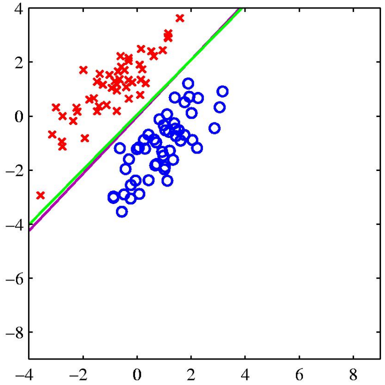
$p(C = 0|\mathbf{x}, \mathbf{w}) = \sigma\left(\mathbf{w}^T\mathbf{x} + w_0\right) = 0.5$, where $\sigma(z) = \frac{1}{1+\exp(-z)}$

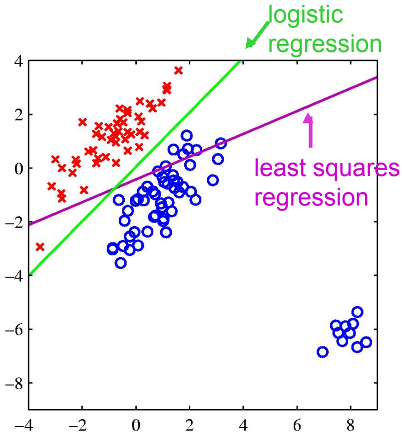
Decision boundary: $\mathbf{w}^T\mathbf{x} + w_0 = 0$

Logistic regression has a linear decision boundary



# Logistic Regression vs Least Squares Regression




If the right answer is 1 and the model says 1.5, it loses, so it changes the boundary to avoid being “too correct” (tilts away from outliers)

# Example

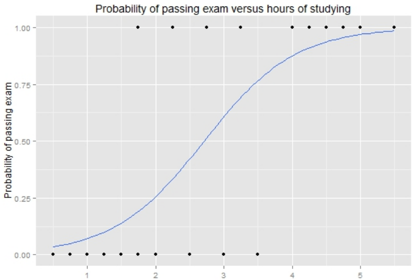
**Problem**: Given the number of hours a student spent learning, will (s)he pass the exam?

Training data (top row: $x^{(i)}$, bottom row: $t^{(i)}$)

| Hours | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 | 1.75 | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 | 3.25 | 3.50 | 4.00 | 4.25 | 4.50 | 4.75 | 5.00 | 5.50 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pass | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |

Learn **w** for our model, i.e., logistic regression (coming up)

Make predictions:




| Hours of study | Probability of passing exam |
|---|---|
| 1 | 0.07 |
| 2 | 0.26 |
| 3 | 0.61 |
| 4 | 0.87 |
| 5 | 0.97 |

# Learning?

When we have a d-dim input $\mathbf{x} \in \Re^d$

How should we learn the weights $\mathbf{w} = (w_0, w_1, \cdots, w_d)$?

We have a probabilistic model

Let's use maximum likelihood

# Probability vs Likelihood

- You can estimate a **probability** of an event using the function that describes the probability distribution and its parameters.
- For example, you can estimate the outcome of a fair coin flip by using the Bernoulli distribution and the probability of success 0.5. In this ideal case, you already know how the data is distributed.

- But the real world is messy. Often you don’t know the exact parameter values, and you may not even know the probability distribution that describes your specific use case.
- Instead, you have to estimate the function and its parameters from the data.
- The **likelihood** describes the relative evidence that the data has a particular distribution and its associated parameters.

# Probability vs Likelihood

- We can describe the **likelihood** as a function of an observed value of the data x, and the distributions' unknown parameter θ.

$$f(x, \theta)$$

- In short, when estimating the **probability**, you go from a distribution and its parameters to the event.

$$Probability\colon\ p(event \mid distribution)$$

- When estimating the **likelihood**, you go from the data to the distribution and its parameters.

$$Likelihood\colon\ L(distribution \mid data\ )$$

# Probability vs Likelihood

- Recall that a coin flip is a Bernoulli trial, which can be described in the following function.

$$P(X = x) \ = p^x(1 - p)^{1-x}$$

- The probability p is a parameter of the function.

- To be consistent with the likelihood notation, we write down the formula for the likelihood function with theta instead of p.

$$L(x, \theta) \ = \theta^x(1 - \theta)^{1-x}$$

# Probability vs Likelihood

- Let's say we throw the coin 3 times. It comes up heads the first 2 times. The last time it comes up tails. What is the likelihood that hypothesis A given the data?

- Now, we need a hypothesis about the parameter theta. We assume that the coin is fair. The probability of obtaining heads is 0.5.

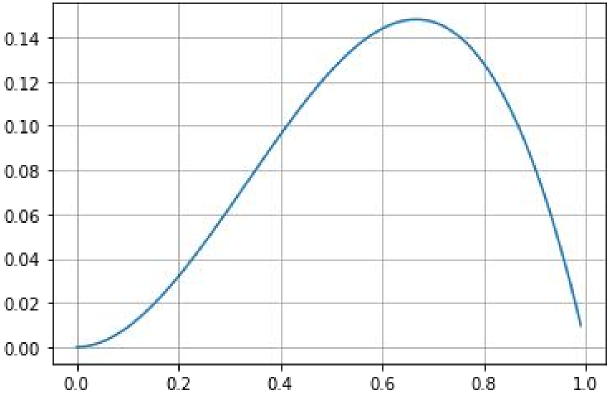
$$
\begin{aligned}
P(X = 1) &= 0.5^1(1-0.5)^{1-1} = 0.5 \\
P(X = 1) &= 0.5^1(1-0.5)^{1-1} = 0.5 \\
P(X = 1) &= 0.5^0(1-0.5)^{1-0} = 0.5
\end{aligned}
$$

- Multiplying all of these gives us the following value

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

# Probability vs Likelihood

- For three coin tosses with 2 heads, the plot would look like this by varying the parameter from 0 to 1, for which the likelihood is maximized at 2/3.




- In other words: Given the fact that 2 of our three coin tosses landed up heads, it seems more likely that the true probability of getting heads is 2/3.

# Probability vs Likelihood

- Mathematically, we can denote the maximum likelihood estimation as a function that results in the theta maximizing the likelihood.

$$\theta_{ML} = arg\, max_{\theta}\, L(\theta, x) = \prod_{i=1}^{n} p(x_i, \theta)$$

- The variable x represents the range of examples drawn from the unknown data distribution, which we would like to approximate and n the number of examples.

- For most practical applications, maximizing the log-likelihood is often a better choice because the logarithm reduced operations by one level. Multiplications become additions; powers become multiplications, etc.

$$\theta_{ML} = arg\, max_{\theta}\, l(\theta, x) = \sum_{i=1}^{n} log(p(x_i, \theta))$$

# Conditional Likelihood

Assume $t \in \{0, 1\}$, we can write the probability distribution of each of our training points $p(t^{(1)}, \cdots, t^{(N)} | \mathbf{x}^{(1)}, \cdots \mathbf{x}^{(N)}; \mathbf{w})$

Assuming that the training examples are sampled IID: independent and identically distributed, we can write the *likelihood function*:

$$L(\mathbf{w}) = p(t^{(1)}, \cdots, t^{(N)} | \mathbf{x}^{(1)}, \cdots \mathbf{x}^{(N)}; \mathbf{w}) = \prod_{i=1}^{N} p(t^{(i)} | \mathbf{x}^{(i)}; \mathbf{w})$$

We can write each probability as (will be useful later):

$$\begin{aligned} p(t^{(i)} | \mathbf{x}^{(i)}; \mathbf{w}) &= p(C = 1 | \mathbf{x}^{(i)}; \mathbf{w})^{t^{(i)}} p(C = 0 | \mathbf{x}^{(i)}; \mathbf{w})^{1 - t^{(i)}} \\ &= \left(1 - p(C = 0 | \mathbf{x}^{(i)}; \mathbf{w})\right)^{t^{(i)}} p(C = 0 | \mathbf{x}^{(i)}; \mathbf{w})^{1 - t^{(i)}} \end{aligned}$$

We can learn the model by maximizing the likelihood

$$\max_{\mathbf{w}} L(\mathbf{w}) = \max_{\mathbf{w}} \prod_{i=1}^{N} p(t^{(i)} | \mathbf{x}^{(i)}; \mathbf{w})$$

Easier to maximize the log likelihood $\log L(\mathbf{w})$

# Loss Function

$$L(\mathbf{w}) = \prod_{i=1}^{N} p(t^{(i)}|\mathbf{x}^{(i)}) \qquad \text{(likelihood)}$$

$$= \prod_{i=1}^{N} \left(1 - p(C=0|\mathbf{x}^{(i)})\right)^{t^{(i)}} p(C=0|\mathbf{x}^{(i)})^{1-t^{(i)}}$$

We can convert the maximization problem into minimization so that we can write the loss function:

$$\ell_{log}(\mathbf{w}) = -\log L(\mathbf{w})$$

$$= -\sum_{i=1}^{N} \log p(t^{(i)}|\mathbf{x}^{(i)}; \mathbf{w})$$

$$= -\sum_{i=1}^{N} t^{(i)} \log(1 - p(C=0|\mathbf{x}^{(i)}, \mathbf{w})) - \sum_{i=1}^{N} (1 - t^{(i)}) \log p(C=0|\mathbf{x}^{(i)}; \mathbf{w})$$

It's a convex function of $\mathbf{w}$. Can we get the global optimum?

# Gradient Descent

$$\min_{\mathbf{w}} \ell(\mathbf{w}) = \min_{\mathbf{w}} \left\{ -\sum_{i=1}^{N} t^{(i)} \log(1 - p(C = 0|\mathbf{x}^{(i)}, \mathbf{w})) - \sum_{i=1}^{N} (1 - t^{(i)}) \log p(C = 0|\mathbf{x}^{(i)}, \mathbf{w}) \right\}$$

Gradient descent: iterate and at each iteration compute steepest direction towards optimum, move in that direction, step-size $\lambda$

$$w_j^{(t+1)} \leftarrow w_j^{(t)} - \lambda \frac{\partial \ell(\mathbf{w})}{\partial w_j}$$

You can write this in vector form

$$\triangledown \ell(\mathbf{w}) = \left[ \frac{\partial \ell(\mathbf{w})}{\partial w_0}, \cdots, \frac{\partial \ell(\mathbf{w})}{\partial w_k} \right]^T, \qquad \text{and} \qquad \triangle(\mathbf{w}) = -\lambda \triangledown \ell(\mathbf{w})$$

But where is $\mathbf{w}$?

$$p(C = 0|\mathbf{x}) = \frac{1}{1 + \exp(-\mathbf{w}^T \mathbf{x} - w_0)}, \quad p(C = 1|\mathbf{x}) = \frac{\exp(-\mathbf{w}^T \mathbf{x} - w_0)}{1 + \exp(-\mathbf{w}^T \mathbf{x} - w_0)}$$

# Let's Compute the Updates

The loss is

$$\ell_{log-loss}(\mathbf{w}) = -\sum_{i=1}^{N} t^{(i)} \log p(C = 1|\mathbf{x}^{(i)}, \mathbf{w}) - \sum_{i=1}^{N} (1 - t^{(i)}) \log p(C = 0|\mathbf{x}^{(i)}, \mathbf{w})$$

where the probabilities are

$$p(C = 0|\mathbf{x}, \mathbf{w}) = \frac{1}{1 + \exp(-z)} \qquad p(C = 1|\mathbf{x}, \mathbf{w}) = \frac{\exp(-z)}{1 + \exp(-z)}$$

and $z = \mathbf{w}^T \mathbf{x} + w_0$

We can simplify

$$\begin{aligned}
\ell(\mathbf{w})_{log-loss} &= \sum_i t^{(i)} \log(1 + \exp(-z^{(i)})) + \sum_i t^{(i)} z^{(i)} + \sum_i (1 - t^{(i)}) \log(1 + \exp(-z^{(i)})) \\
&= \sum_i \log(1 + \exp(-z^{(i)})) + \sum_i t^{(i)} z^{(i)}
\end{aligned}$$

Now it's easy to take derivatives

# Updates

$$\ell(\mathbf{w}) = \sum_i t^{(i)} z^{(i)} + \sum_i \log(1 + \exp(-z^{(i)}))$$

Now it's easy to take derivatives

Remember $z = \mathbf{w}^T \mathbf{x} + w_0$

$$\frac{\partial \ell}{\partial w_j} = \sum_i \left( t^{(i)} x_j^{(i)} - x_j^{(i)} \cdot \frac{\exp(-z^{(i)})}{1 + \exp(-z^{(i)})} \right)$$

What's $x_j^{(i)}$? The $j-$th dimension of the $i-$th training example $\mathbf{x}^{(i)}$

And simplifying

$$\frac{\partial \ell}{\partial w_j} = \sum_i x_j^{(i)} \left( t^{(i)} - p(C = 1 | \mathbf{x}^{(i)}; \mathbf{w}) \right)$$

Don't get confused with indices: $j$ for the weight that we are updating and $i$ for the training example

# Gradient Descent

Putting it all together (plugging the update into gradient descent):
Gradient descent for logistic regression:

$$w_j^{(t+1)} \leftarrow w_j^{(t)} - \lambda \sum_i x_j^{(i)} \left( t^{(i)} - p(C = 1 | \mathbf{x}^{(i)}; \mathbf{w}) \right)$$

where:

$$p(C = 1 | \mathbf{x}^{(i)}; \mathbf{w}) = \frac{\exp(-\mathbf{w}^T \mathbf{x} - w_0)}{1 + \exp(-\mathbf{w}^T \mathbf{x} - w_0)} = \frac{1}{1 + \exp(\mathbf{w}^T \mathbf{x} + w_0)}$$

This is all there is to learning in logistic regression. Simple, huh?

# Regularization

We can define priors on parameters $\mathbf{w}$

This is a form of regularization

Helps avoid large weights and overfitting

$$\max_{\mathbf{w}} \log \left[ p(\mathbf{w}) \prod_i p(t^{(i)} | \mathbf{x}^{(i)}, \mathbf{w}) \right]$$

What's $p(\mathbf{w})$?

# Regularized Logistic Regression

For example, define prior: normal distribution, zero mean and identity covariance $p(\mathbf{w}) = \mathcal{N}(0, \alpha^{-1}\mathbf{I})$

This prior pushes parameters towards zero

Including this prior the new gradient is

$$w_j^{(t+1)} \leftarrow w_j^{(t)} - \lambda \frac{\partial \ell(\mathbf{w})}{\partial w_j} - \lambda \alpha w_j^{(t)}$$

where $t$ here refers to iteration of the gradient descent

The parameter $\alpha$ is the importance of the regularization, and it's a hyper-parameter

How do we decide the best value of $\alpha$ (or a hyper-parameter in general)?

# Use of Validation Set

Tuning hyper-parameters:

**Never use test data for tuning the hyper-parameters**

We can divide the set of training examples into two disjoint sets: training and validation

Use the first set (i.e., training) to estimate the weights $\mathbf{w}$ for different values of $\alpha$

Use the second set (i.e., validation) to estimate the best $\alpha$, by evaluating how well the classifier does on this second set

This tests how well it generalizes to unseen data

# Cross-Validation

Leave-p-out cross-validation:

- We use $p$ observations as the validation set and the remaining observations as the training set.
- This is repeated on all ways to cut the original training set.
- It requires $\mathcal{C}_n^p$ for a set of $n$ examples

Leave-1-out cross-validation: When $p = 1$, does not have this problem

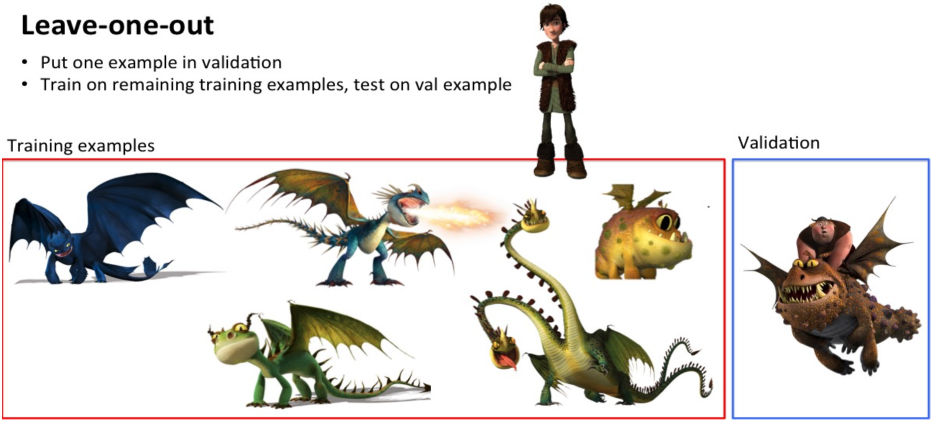
k-fold cross-validation:

- The training set is randomly partitioned into k equal size subsamples.
- Of the k subsamples, a single subsample is retained as the validation data for testing the model, and the remaining $k - 1$ subsamples are used as training data.
- The cross-validation process is then repeated $k$ times (the folds).
- The k results from the folds can then be averaged (or otherwise combined) to produce a single estimate

# Cross-Validation (with Pictures)

Train your model:

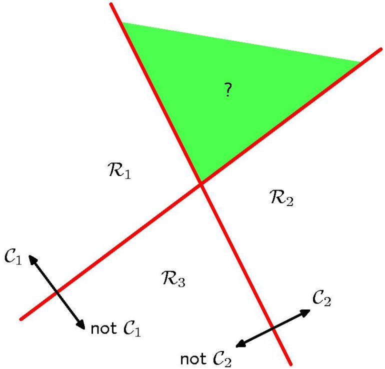
Leave-one-out cross-validation:
k-fold cross-validation:



- Multi-class classification with:
  - Least squares regression
  - Logistic regression

# Discriminant Functions for $K > 2$ classes

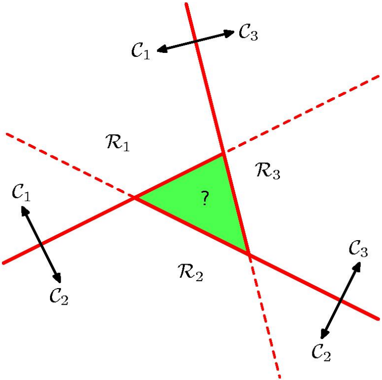
- First idea: Use $K$ classifiers, each solving a two class problem of separating point in a class $C_k$ from points not in the class.
- Known as **1 vs all** or **1 vs the rest** classifier




- PROBLEM: More than one good answer for green region!

# Discriminant Functions for $K > 2$ classes

- Another simple idea: Introduce $K(K-1)/2$ two-way classifiers, one for each possible pair of classes
- Each point is classified according to majority vote amongst the disc. func.
- Known as the **1 vs 1 classifier**




- PROBLEM: Two-way preferences need not be transitive

# K-Class Discriminant

- We can avoid these problems by considering a single K-class discriminant comprising $K$ functions of the form

$$y_k(\mathbf{x}) = \mathbf{w}_k^T \mathbf{x} + w_{k,0}$$

  and then assigning a point $\mathbf{x}$ to class $C_k$ if

$$\forall j \neq k \qquad y_k(\mathbf{x}) > y_j(\mathbf{x})$$

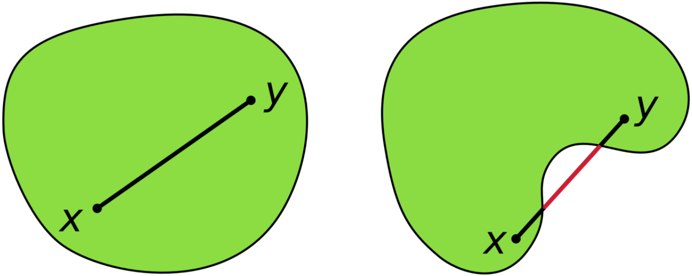
- Note that $\mathbf{w}_k^T$ is now a vector, not the $k$-th coordinate
- The decision boundary between class $C_j$ and class $C_k$ is given by $y_j(\mathbf{x}) = y_k(\mathbf{x})$, and thus it's a $(D-1)$ dimensional hyperplane defined as

$$(\mathbf{w}_k - \mathbf{w}_j)^T \mathbf{x} + (w_{k0} - w_{j0}) = 0$$

- What about the binary case? Is this different?
- What is the shape of the overall decision boundary?

# K-Class Discriminant

- The decision regions of such a discriminant are always **singly connected** and **convex**
- In Euclidean space, an object is **convex** if for every pair of points within the object, every point on the straight line segment that joins the pair of points is also within the object

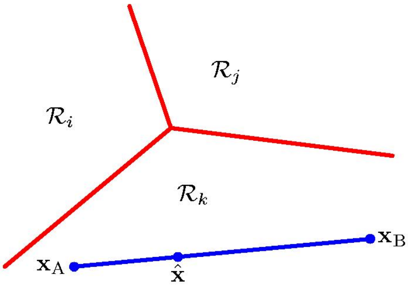



- Which object is convex?

# K-Class Discriminant

- The decision regions of such a discriminant are always **singly connected** and **convex**
- Consider 2 points $\mathbf{x}_A$ and $\mathbf{x}_B$ that lie inside decision region $R_k$
- Any convex combination $\hat{\mathbf{x}}$ of those points also will be in $R_k$

$$\hat{\mathbf{x}} = \lambda \mathbf{x}_A + (1 - \lambda)\mathbf{x}_B$$



# Proof

- A convex combination point, i.e., $\lambda \in [0, 1]$

$$\hat{\mathbf{x}} = \lambda \mathbf{x}_A + (1 - \lambda)\mathbf{x}_B$$

- From the linearity of the classifier $y(\mathbf{x})$

$$y_k(\hat{\mathbf{x}}) = \lambda y_k(\mathbf{x}_A) + (1 - \lambda) y_k(\mathbf{x}_B)$$

- Since $\mathbf{x}_A$ and $\mathbf{x}_B$ are in $R_k$, it follows that $y_k(\mathbf{x}_A) > y_j(\mathbf{x}_A)$, $y_k(\mathbf{x}_B) > y_j(\mathbf{x}_B)$, $\forall j \neq k$
- Since $\lambda$ and $1 - \lambda$ are positive, then $\hat{\mathbf{x}}$ is inside $R_k$
- Thus $R_k$ is singly connected and convex

# Multi-class Classification with Linear Regression

- From before we have:

$$y_k(\mathbf{x}) = \mathbf{w}_k^T \mathbf{x} + w_{k,0}$$

  which can be rewritten as:

$$\mathbf{y}(\mathbf{x}) = \tilde{\mathbf{W}}^T \tilde{\mathbf{x}}$$

  where the $k$-th column of $\tilde{\mathbf{W}}$ is $[w_{k,0}, \mathbf{w}_k^T]$, and $\tilde{\mathbf{x}}$ is $[1, \mathbf{x}^T]^T$

- **Training:** How can I find the weights $\tilde{\mathbf{W}}$ with the standard sum-of-squares regression loss?

**1-of-K encoding:**

For multi-class problems (with $K$ classes), instead of using $t = k$ (target has label $k$) we often use a **1-of-K encoding**, i.e., a vector of $K$ target values containing a single 1 for the correct class and zeros elsewhere

*Example:* For a 4-class problem, we would write a target with class label 2 as:

$$\mathbf{t} = [0, 1, 0, 0]^T$$

# Multi-class Classification with Linear Regression

- Sum-of-least-squares loss:

$$\begin{aligned} \ell(\tilde{\mathbf{W}}) &= \sum_{n=1}^{N} ||\tilde{\mathbf{W}}^T \tilde{\mathbf{x}}^{(n)} - \mathbf{t}^{(n)}||^2 \\ &= ||\tilde{\mathbf{X}}\tilde{\mathbf{W}} - \mathbf{T}||_F^2 \end{aligned}$$

where the $n$-th row of $\tilde{\mathbf{X}}$ is $[\tilde{\mathbf{x}}^{(n)}]^T$, and $n$-th row of $\mathbf{T}$ is $[\mathbf{t}^{(n)}]^T$

- Setting derivative wrt $\tilde{\mathbf{W}}$ to 0, we get:

$$\tilde{\mathbf{W}} = (\tilde{\mathbf{X}}^T \tilde{\mathbf{X}})^{-1} \tilde{\mathbf{X}}^T \mathbf{T}$$

# Multi-class Logistic Regression

- Associate a set of weights with each class, then use a normalized exponential output

$$p(C_k|\mathbf{x}) = y_k(\mathbf{x}) = \frac{\exp(z_k)}{\sum_j \exp(z_j)}$$

where the activations are given on

$$z_k = \mathbf{w}_k^T \mathbf{x}$$

- The function $\frac{\exp(z_k)}{\sum_j \exp(z_j)}$ is called a softmax function

# Multi-class Logistic Regression

- The likelihood

$$p(\mathbf{T}|\mathbf{w}_1, \cdots, \mathbf{w}_k) = \prod_{n=1}^{N}\prod_{k=1}^{K} p(C_k|\mathbf{x}^{(n)})^{t_k^{(n)}} = \prod_{n=1}^{N}\prod_{k=1}^{K} y_k^{(n)}(\mathbf{x}^{(n)})^{t_k^{(n)}}$$

  with

$$p(C_k|\mathbf{x}) = y_k(\mathbf{x}) = \frac{\exp(z_k)}{\sum_j \exp(z_j)}$$

  where k-th row of $\mathbf{T}$ is 1-of-K encoding of example k and

$$z_k = \mathbf{w}_k^T \mathbf{x} + w_{k0}$$

- 

- Derive the loss by computing the negative log-likelihood:

$$E(\mathbf{w}_1, \cdots, \mathbf{w}_K) = -\log p(\mathbf{T}|\mathbf{w}_1, \cdots, \mathbf{w}_K) = -\sum_{n=1}^{N}\sum_{k=1}^{K} t_k^{(n)} \log[y_k^{(n)}(\mathbf{x}^{(n)})]$$

  This is known as the **cross-entropy** error for multiclass classification

- How do we obtain the weights?

# Training Multi-class Logistic Regression

- How do we obtain the weights?

$$E(\mathbf{w}_1, \cdots, \mathbf{w}_K) = -\log p(\mathbf{T}|\mathbf{w}_1, \cdots, \mathbf{w}_K) = -\sum_{n=1}^{N}\sum_{k=1}^{K} t_k^{(n)} \log[y_k^{(n)}(\mathbf{x}^{(n)})]$$

- Do gradient descent, where the derivatives are

$$\frac{\partial y_j^{(n)}}{\partial z_k^{(n)}} = \underline{\delta(k,j)} y_j^{(n)} - y_j^{(n)} y_k^{(n)}$$

and

$$\frac{\partial E}{\partial z_k^{(n)}} = \sum_{j=1}^{K} \frac{\partial E}{\partial y_j^{(n)}} \cdot \frac{\partial y_j^{(n)}}{\partial z_k^{(n)}} = y_k^{(n)} - t_k^{(n)}$$

$$\frac{\partial E}{\partial w_{k,i}} = \sum_{n=1}^{N}\sum_{j=1}^{K} \frac{\partial E}{\partial y_j^{(n)}} \cdot \frac{\partial y_j^{(n)}}{\partial z_k^{(n)}} \cdot \frac{\partial z_k^{(n)}}{\partial w_{k,i}} = \boxed{\sum_{n=1}^{N} (y_k^{(n)} - t_k^{(n)}) \cdot x_i^{(n)}}$$

- The derivative is the error times the input

# Softmax for 2 Classes

- Let's write the probability of one of the classes

$$p(C_1|\mathbf{x}) = y_1(\mathbf{x}) = \frac{\exp(z_1)}{\sum_j \exp(z_j)} = \frac{\exp(z_1)}{\exp(z_1) + \exp(z_2)}$$

- I can equivalently write this as

$$p(C_1|\mathbf{x}) = y_1(\mathbf{x}) = \frac{\exp(z_1)}{\exp(z_1) + \exp(z_2)} = \frac{1}{1 + \exp\left(-(z_1 - z_2)\right)}$$

- So the logistic is just a special case that avoids using redundant parameters
- Rather than having two separate set of weights for the two classes, combine into one

$$z' = z_1 - z_2 = \mathbf{w}_1^T\mathbf{x} - \mathbf{w}_2^T\mathbf{x} = \mathbf{w}^T\mathbf{x}$$

- The over-parameterization of the softmax is because the probabilities must add to 1.

# Logistic Regression wrap-up

**Advantages:**

Easily extended to multiple classes

Natural probabilistic view of class predictions

Quick to train

Fast at classification

Good accuracy for many simple data sets

Resistant to overfitting

Can interpret model coefficients as indicators of feature importance

**Less good:**

Linear decision boundary (too simple for more complex problems?)