

Vision and Perception

Image in the frequency domain



SAPIENZA
UNIVERSITÀ DI ROMA

References

Basic reading:

- Szeliski textbook, Sections 3.4, 3.5

Additional reading:

The original Laplacian pyramid paper

- Burt and Adelson, “The Laplacian Pyramid as a Compact Image Code,” IEEE ToC 1983.

Overview of today's lecture

- Fourier series
- Frequency domain
- Fourier transform
- Frequency-domain filtering

The frequency domain

Is this claim true?



Jean Baptiste Joseph Fourier
(1768-1830)

The Fourier series claim (1807):

'Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.'

Well, almost.

- The theorem requires additional conditions.
- Close enough to be named after him.
- Very surprising result at the time.

Is this claim true?



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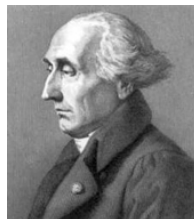
'Any univariate function can be rewritten as a weighted sum of sines and cosines of different frequencies.'

Well, almost.

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- Close enough to be named after him.
- Very surprising result at the time.



Malus



Lagrange



Legendre



Laplace

The committee examining his paper had expressed skepticism, in part due to not so rigorous proofs

Fourier series

Basic building block

$$A \sin(\omega x + \phi)$$

Fourier's claim: Add enough of these to get any *periodic* signal you want

Basic building block

$$A \sin(\omega x + \phi)$$

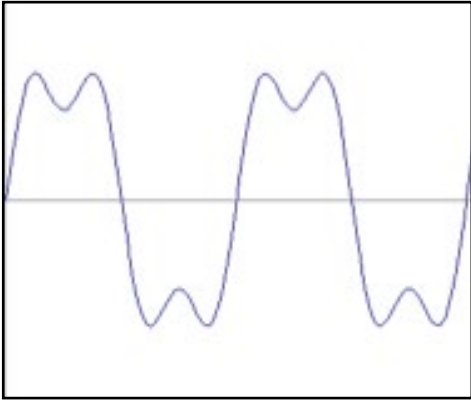
A diagram showing the equation $A \sin(\omega x + \phi)$ with five arrows pointing to its components: 'amplitude' points to A , 'sinusoid' points to \sin , 'angular frequency' points to ω , 'variable' points to x , and 'phase' points to ϕ .

$$T = \frac{2\pi}{|\omega|}$$

Fourier's claim: Add enough of these to get any *periodic* signal you want

Examples

How would you generate this function?



=

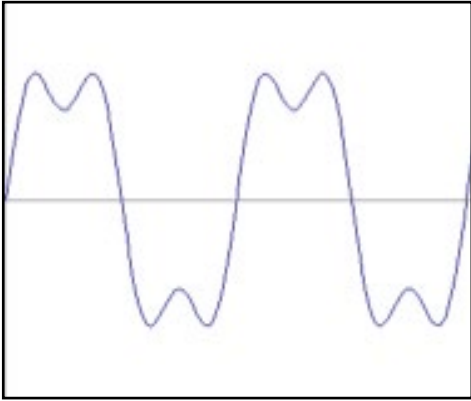
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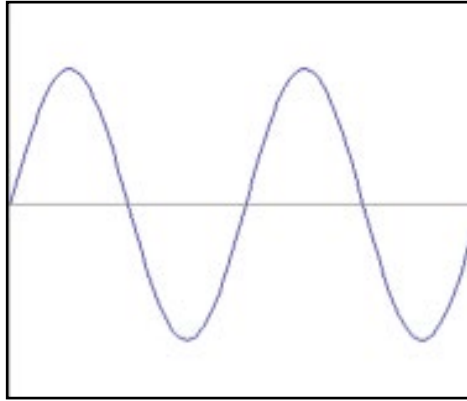
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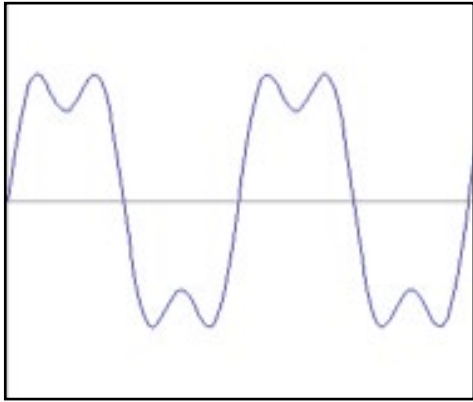
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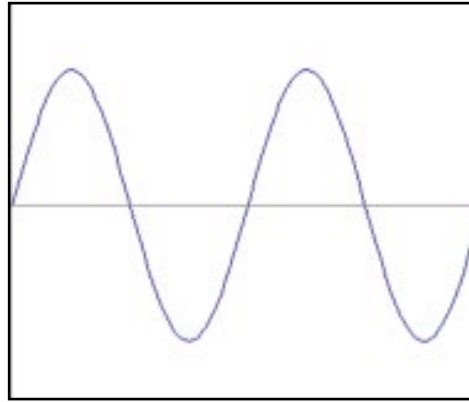
$\sin(2\pi x)$

Examples

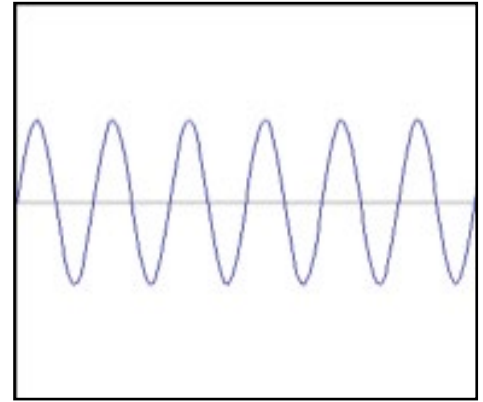
How would you generate this function?



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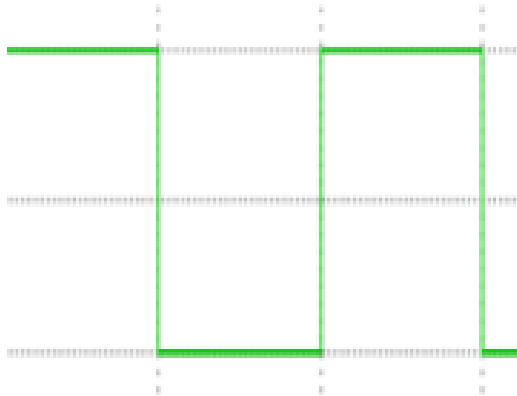
$$f(x) = \sin(2\pi x) + \frac{1}{3} \sin(2\pi 3x)$$

$$\sin(2\pi x)$$

$$\frac{1}{3} \sin(2\pi 3x)$$

Examples

How would you generate this function?



square wave

=

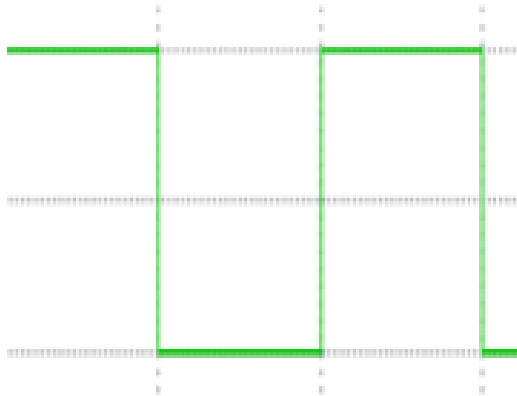
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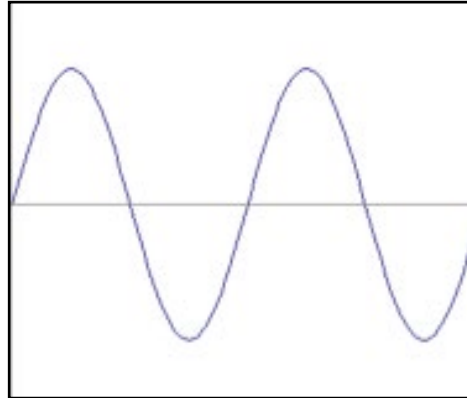
Examples

How would you generate this function?

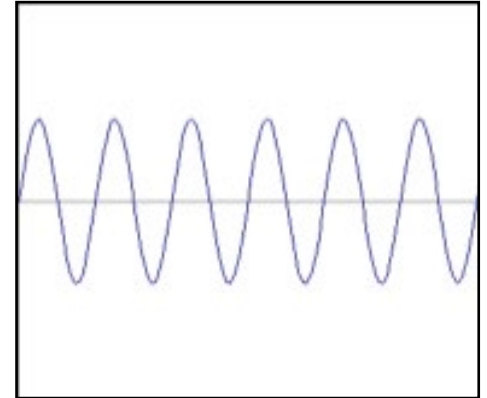


square wave

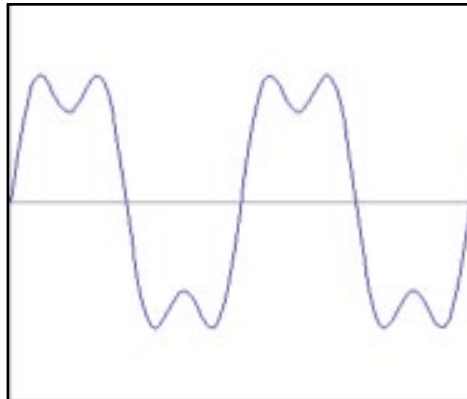
\approx



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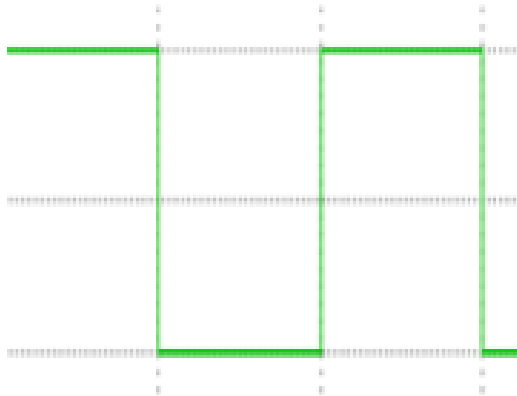


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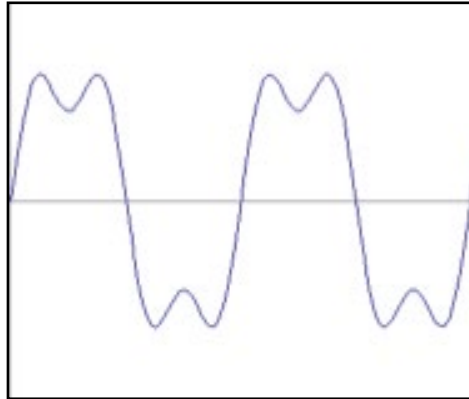
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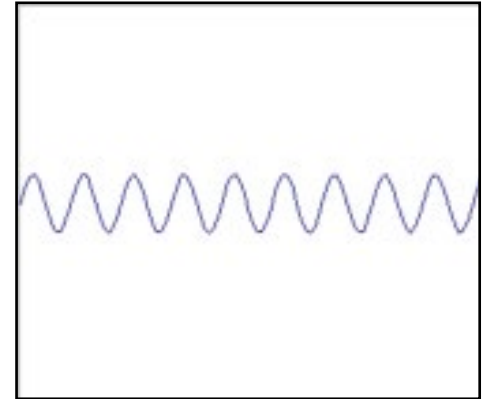


square wave

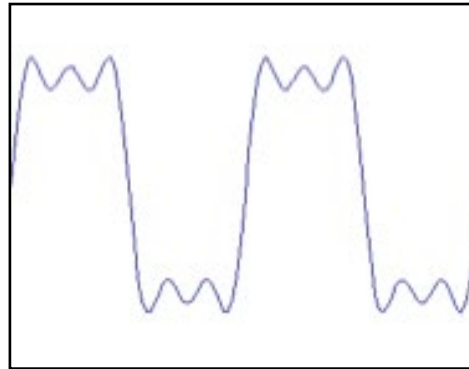
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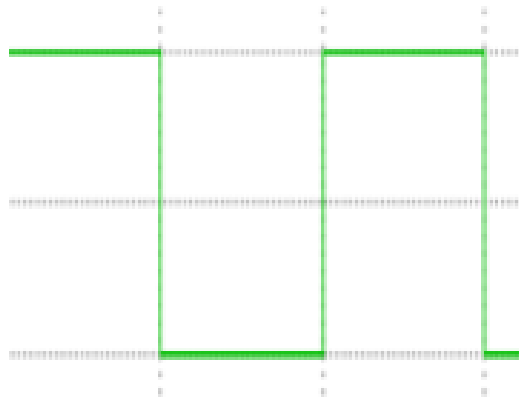


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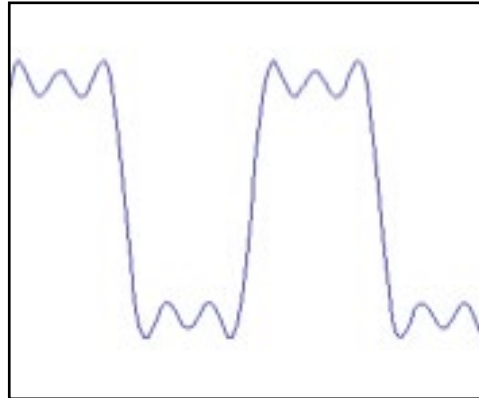


Examples

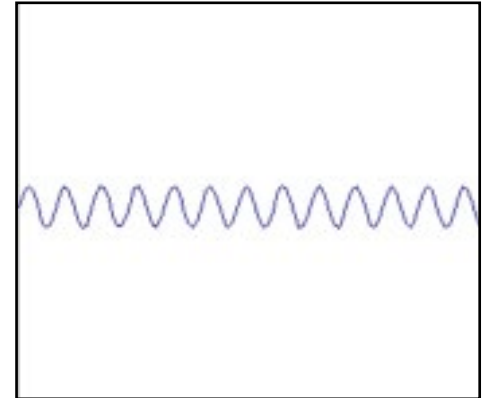
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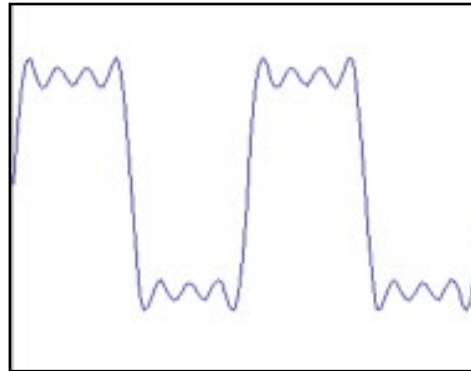
square wave

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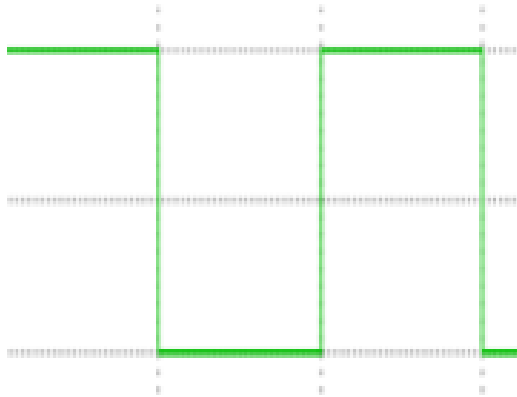


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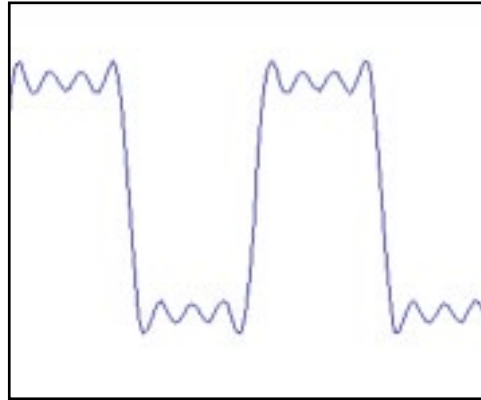
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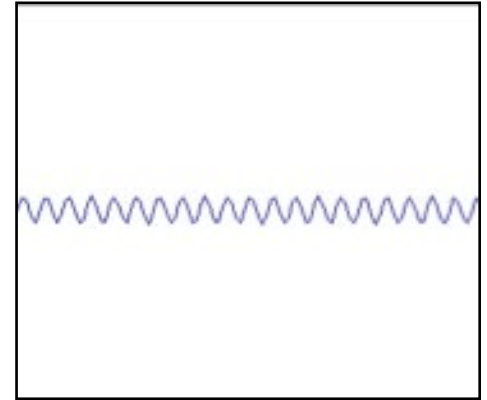


square wave

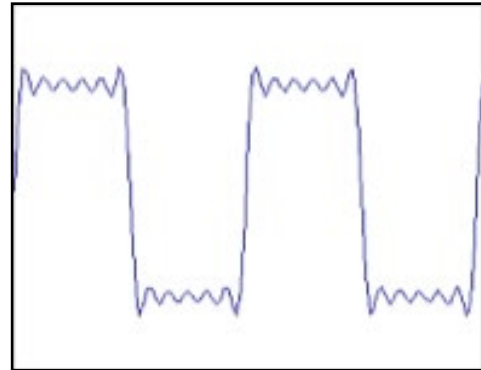
\approx



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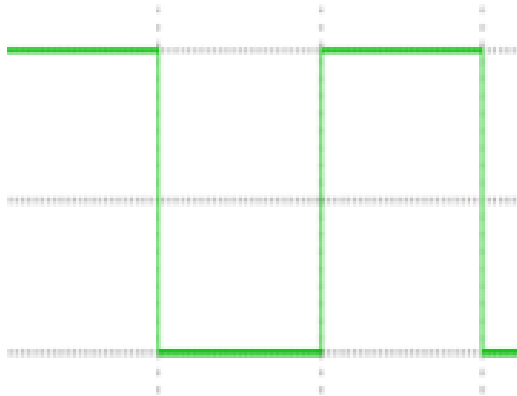


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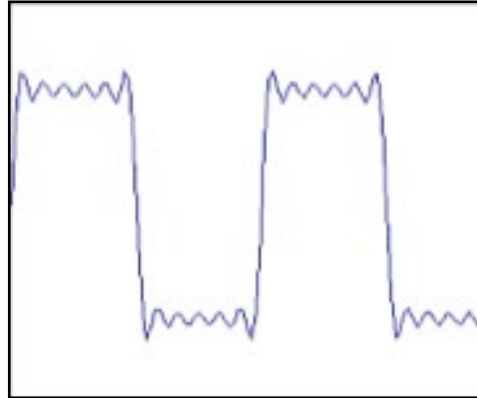
Examples

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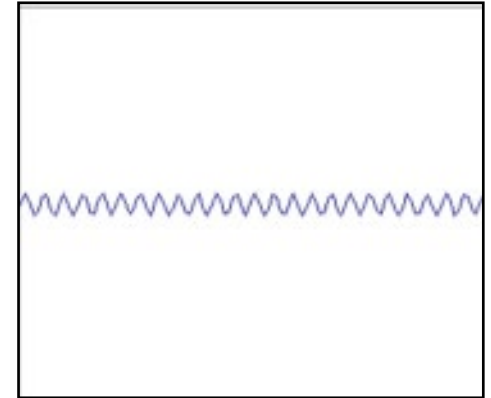


square wave

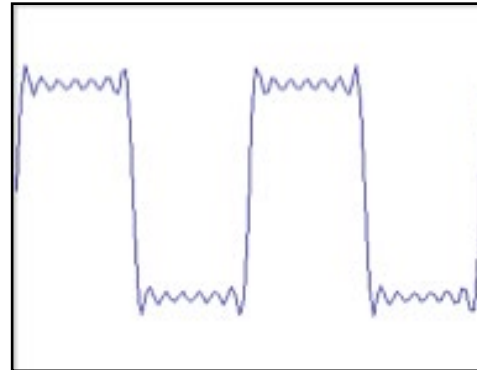
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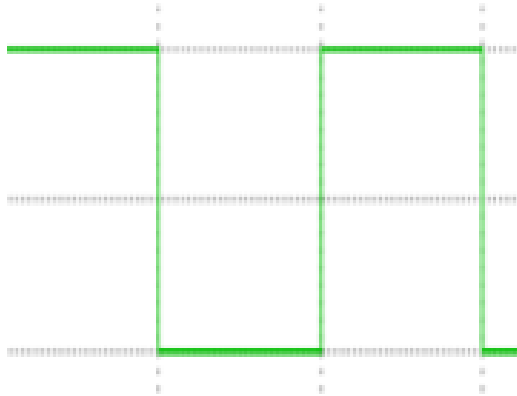


$=$



How would you express
this mathematically?

Examples



square wave

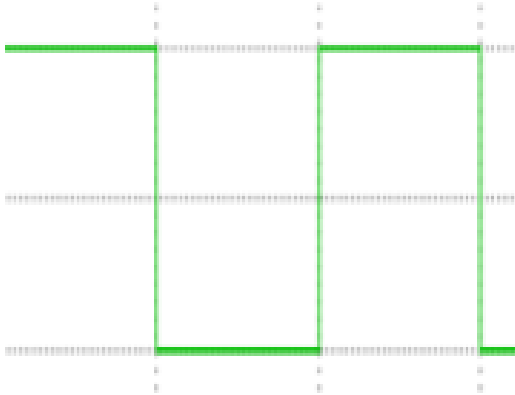
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$$A \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi kx)$$

infinite sum of sine waves

How would you visualize this in the frequency domain?

Examples



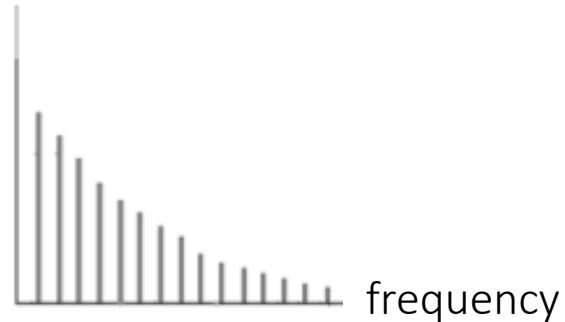
square wave

=

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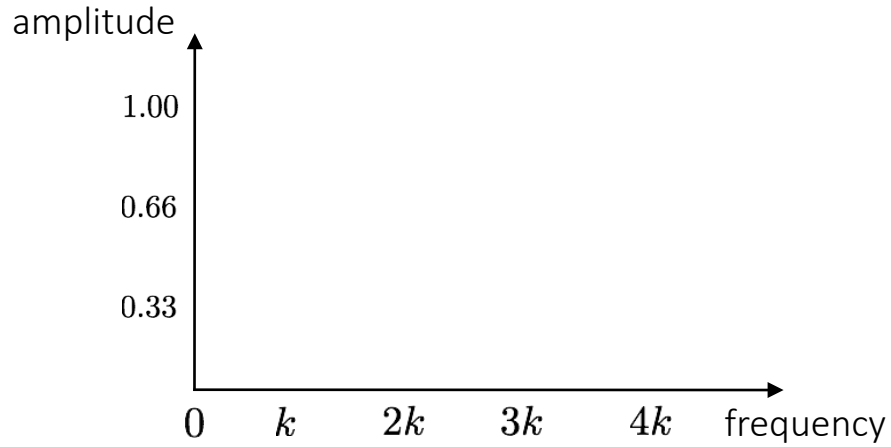
infinite sum of sine waves

Magnitude
(amplitude)



Frequency domain

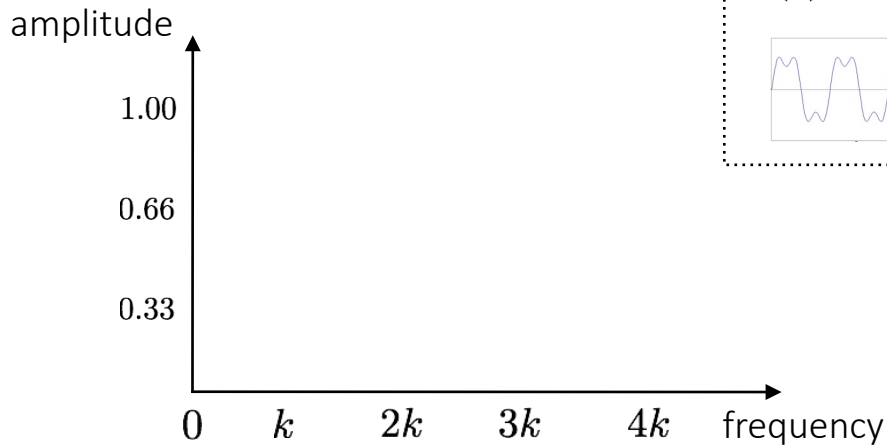
Visualizing the frequency spectrum



Visualizing the frequency spectrum

Recall the temporal domain visualization

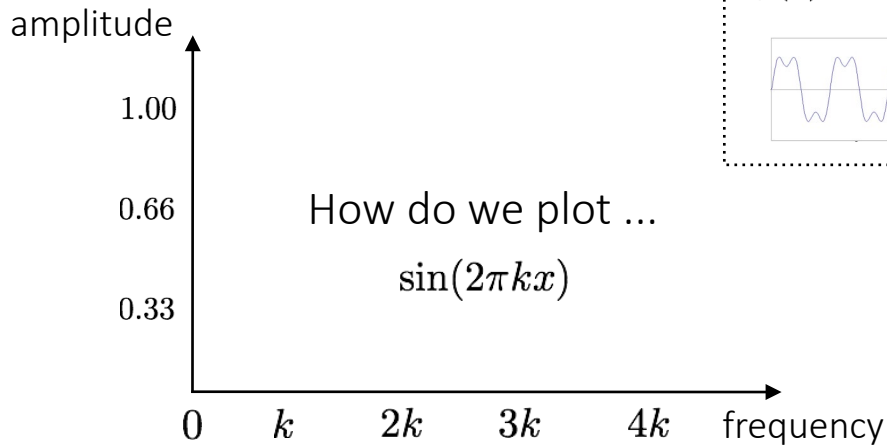
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Visualizing the frequency spectrum

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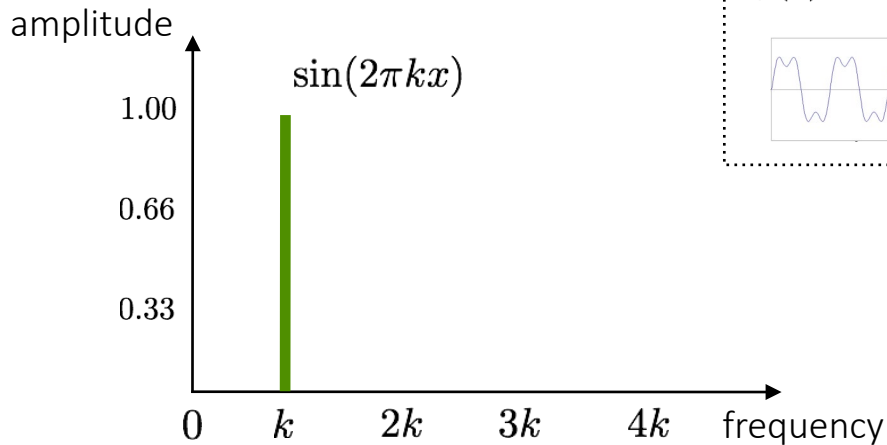
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Visualizing the frequency spectrum

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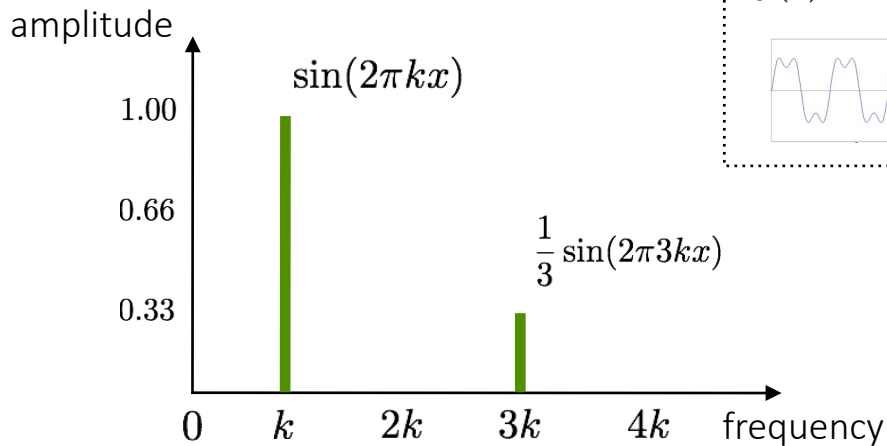
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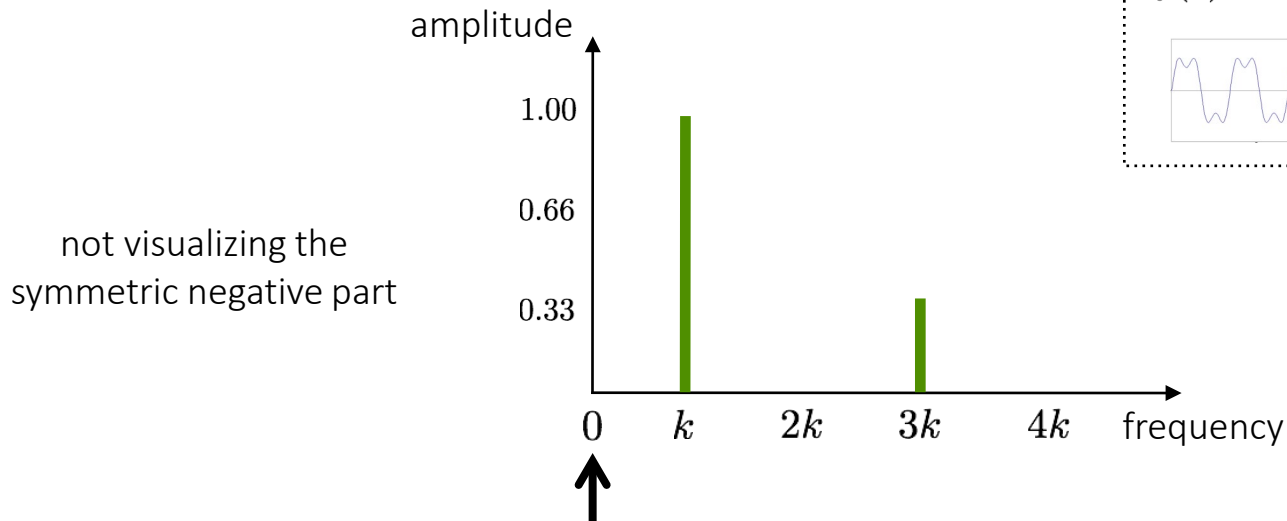
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Visualizing the frequency spectrum

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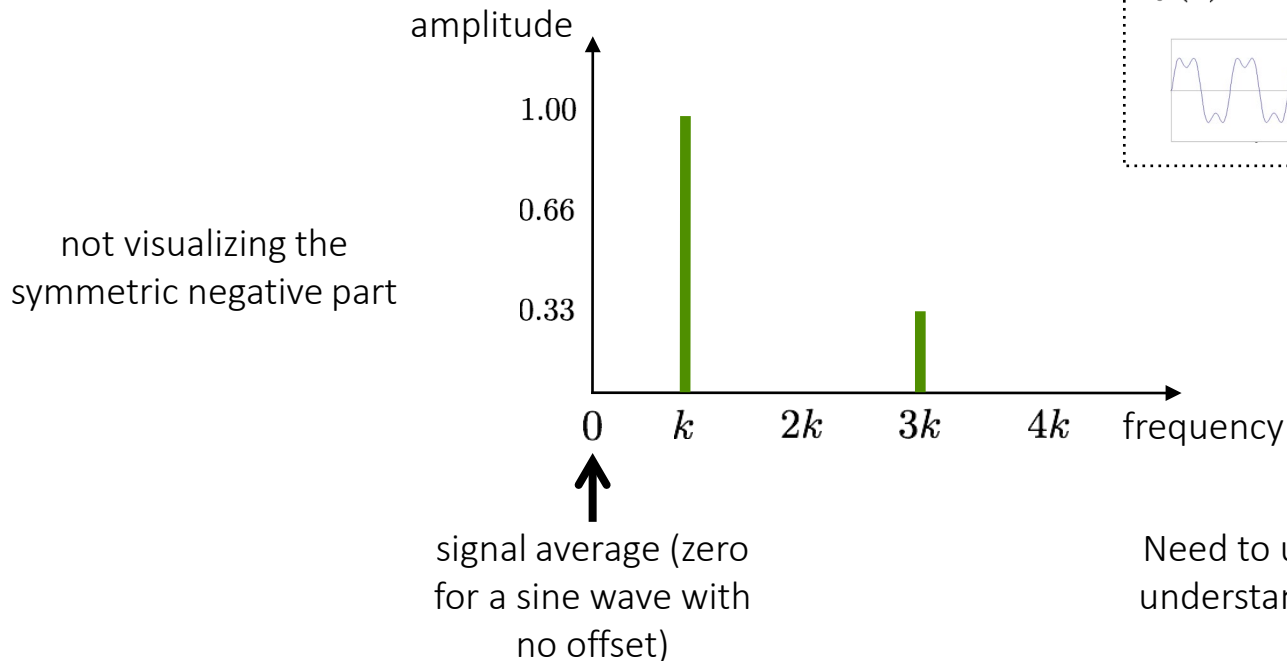
What is at zero frequency?

Need to understand this to understand the 2D version!

Visualizing the frequency spectrum

Recall the temporal domain visualization

$$f(x) = \sin(2\pi kx) + \frac{1}{3} \sin(2\pi 3kx)$$

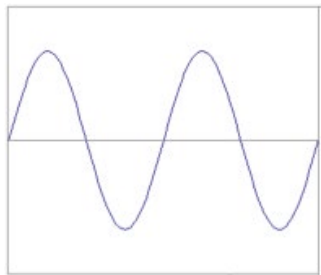


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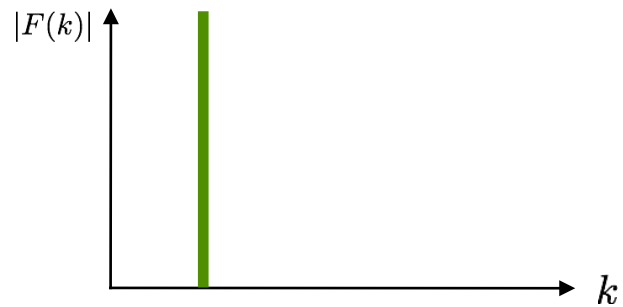
Examples

Spatial domain visualization

1D



Frequency domain visualization



2D

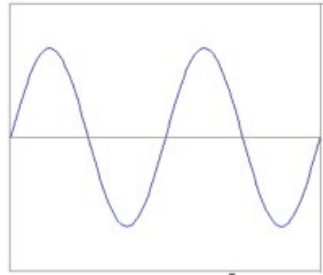


?

Examples

Spatial domain visualization

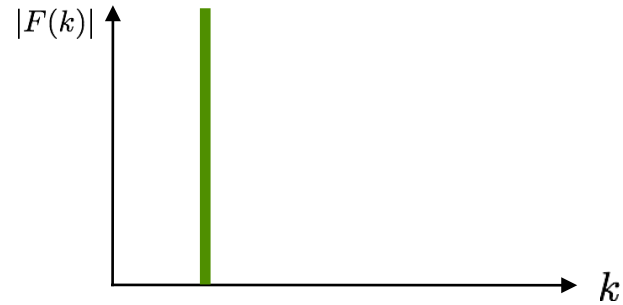
1D



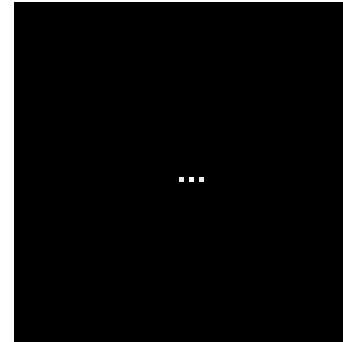
2D



Frequency domain visualization



k_y



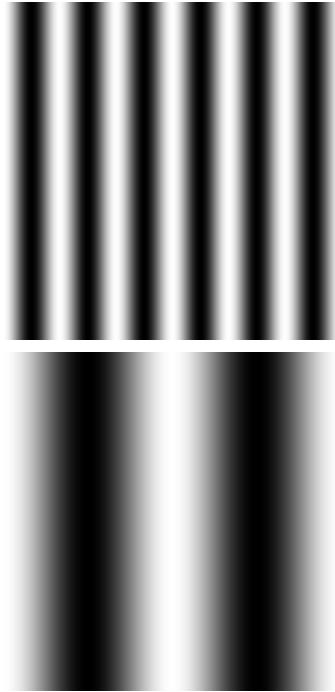
k_x

What do the three dots correspond to?

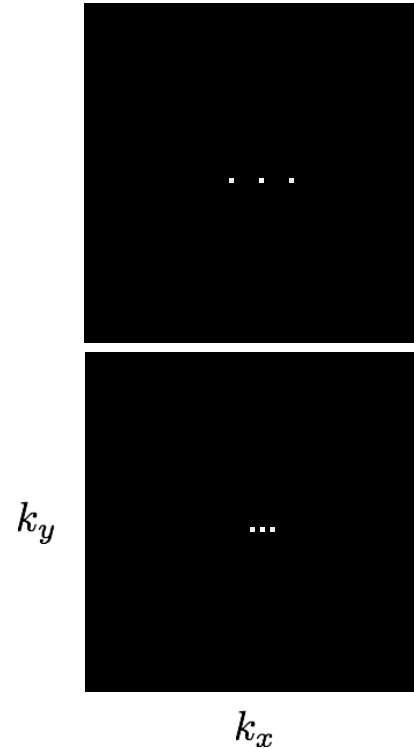
images that are pure cosines have particularly simple FT

Examples

Spatial domain visualization



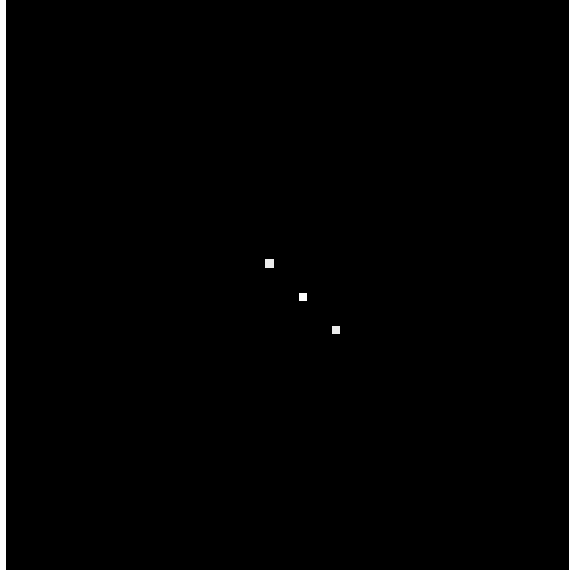
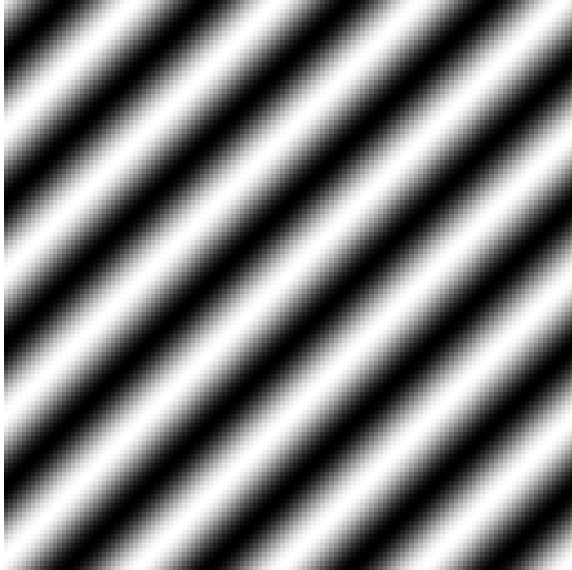
Frequency domain visualization



Scaling property of Fourier transform:
if we stretch a function by a factor in the time domain then squeeze the Fourier transform by the same factor in the frequency domain

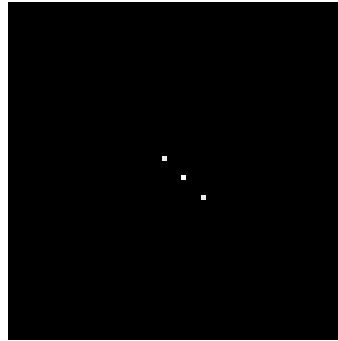
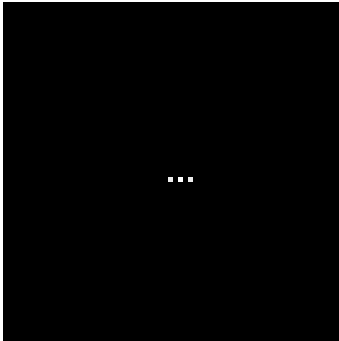
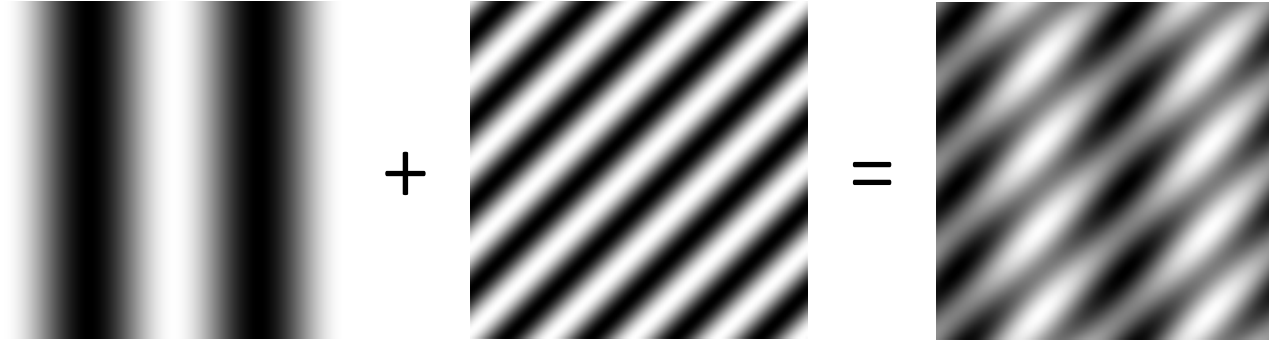
Examples

Rotation of the image results in equivalent rotation of its FT.



Has both an x and
y components

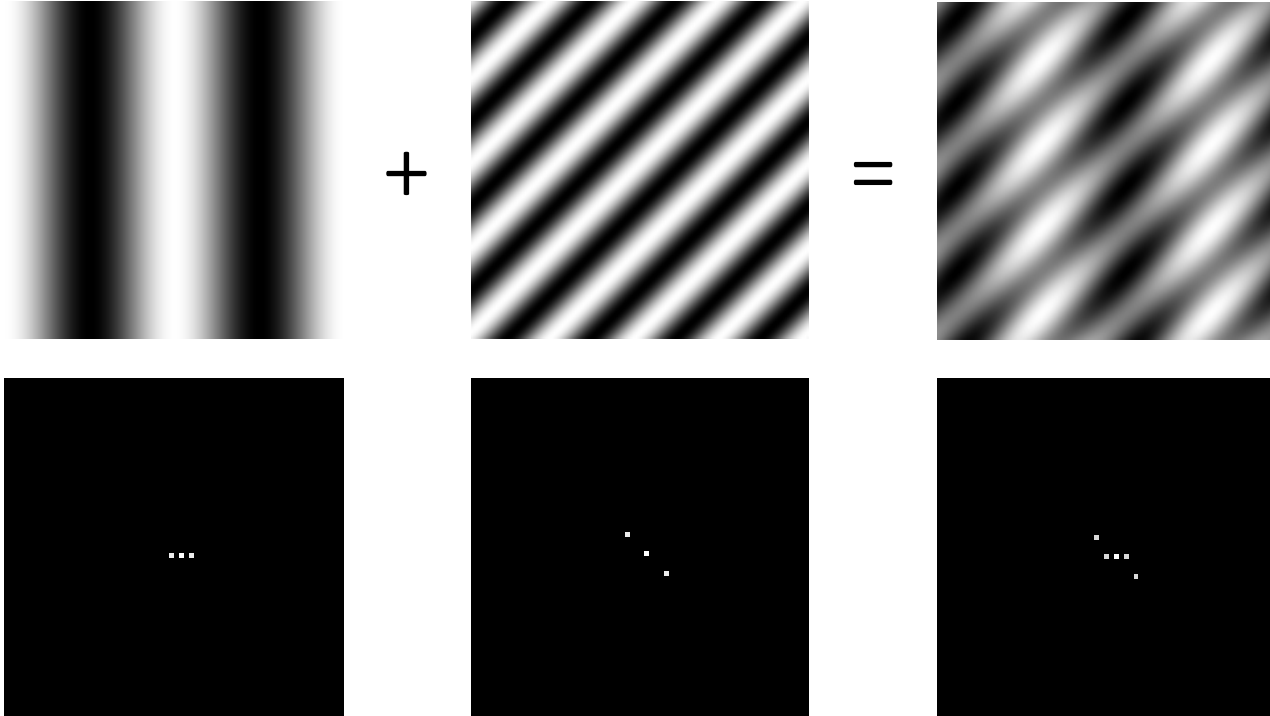
Examples



?

Linearity Property

Examples



Basic building block

$$A \sin(\omega x + \phi)$$

amplitude

sinusoid

angular frequency

variable

phase

What about non-periodic signals?

Fourier's claim: Add enough of these to get any periodic signal you want!

Background

- Any function that **periodically** repeats itself can be expressed as the **sum** of sines and/or cosines of different frequencies, each multiplied by a different coefficient (**Fourier series**).
- Even functions that are **not periodic** (but whose area under the curve is finite) can be expressed as the **integral** of sines and/or cosines multiplied by a weighting function (**Fourier transform**).

Fourier transform

Fourier transform

Fourier transform

inverse Fourier transform

continuous

$$F(k) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi kx} dx$$

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{j2\pi kx} dk$$

discrete

$$F(k) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi kx/N}$$

$k = 0, 1, 2, \dots, N-1$

$$f(x) = \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N}$$

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Where is the connection to the ‘summation of sine waves’ idea?

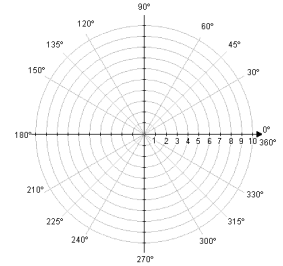
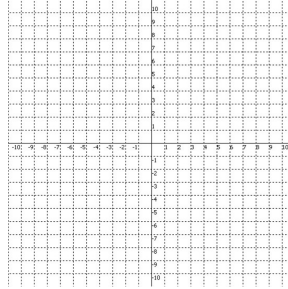
Recalling some basics

Complex numbers have two parts:

rectangular
coordinates

$$R + jI$$

real imaginary



Recalling some basics

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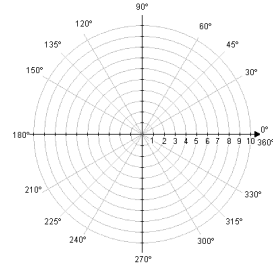
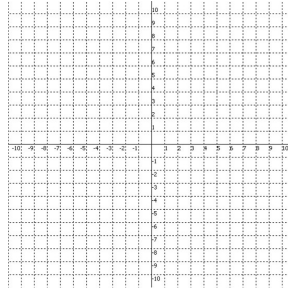
real imaginary

Alternative reparameterization:

polar
coordinates

$$r(\cos \theta + j \sin \theta)$$

how do we compute these?



polar transform

Recalling some basics

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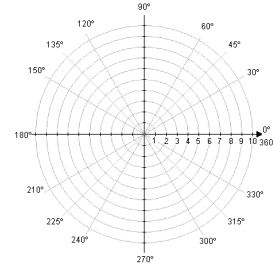
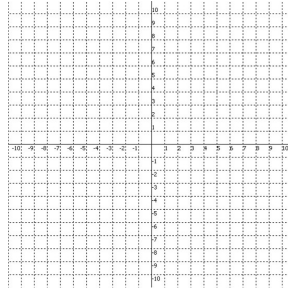
Alternative reparameterization:

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polar transform

$$\theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2}$$



polar transform

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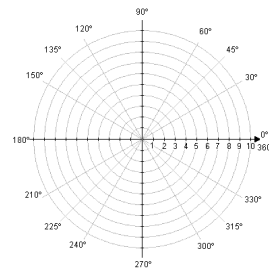
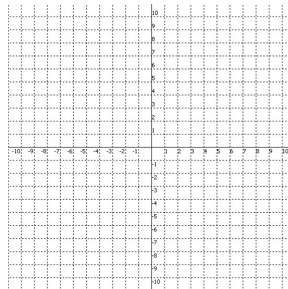
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polar transform

How do you write
these in exponential
form?

Recalling some basics

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real imaginary

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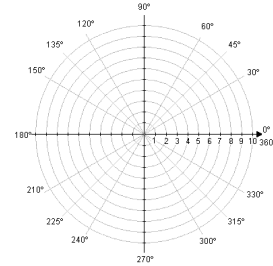
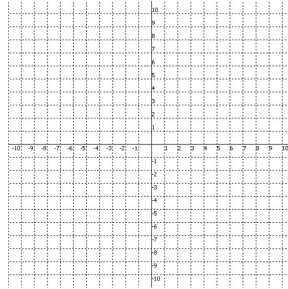
$$\theta = \tan^{-1}\left(\frac{I}{R}\right) \quad r = \sqrt{R^2 + I^2}$$

or
equivalently

$$re^{j\theta}$$

how did we get this?

exponential
form



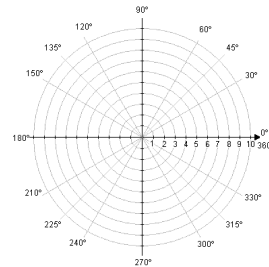
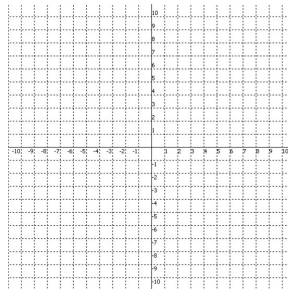
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polar transform

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or
equivalently

$$re^{j\theta}$$

Euler's formula

$$e^{j\theta} = \cos \theta + j \sin \theta$$

exponential
form

This will help us understand the Fourier transform equations

Fourier transform

Fourier transform

inverse Fourier transform

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$$f(x) = \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N}$$

$x = 0, 1, 2, \dots, N-1$

Where is the connection to the ‘summation of sine waves’ idea?

Fourier transform

Where is the connection to the 'summation of sine waves' idea?

$$f(x) = \sum_{k=0}^{N-1} F(k) e^{j2\pi kx/N}$$

Euler's formula
 $e^{j\theta} = \cos \theta + j \sin \theta$

sum over frequencies

$$f(x) = \sum_{k=0}^{N-1} F(k) \left\{ \cos(2\pi kx) + j \sin(2\pi kx) \right\}$$

scaling parameter

wave components

2D Fourier Transform

DFT can be computed for any-dimensional input function. In particular, the 2D DFT is useful when working with images

2D Discrete Fourier transform:

$$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$$

Discrete Inverse Fourier transform:

$$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$$

Spectrum and phase angle

Because the 2-D DFT is complex in general, it can be expressed in polar form:

$$F(u, v) = |F(u, v)|e^{j\theta(u, v)}$$

Where the magnitude

$$|F(u, v)| = \sqrt{R^2(u, v) + I^2(u, v)}$$

Is called the **Fourier spectrum** and

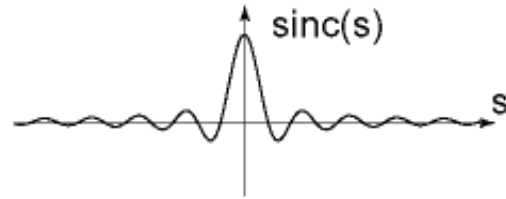
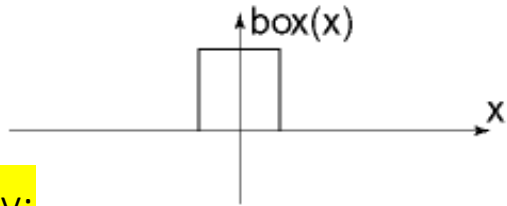
$$\theta(u, v) = \arctan\left(\frac{I(u, v)}{R(u, v)}\right)$$

Is called the **phase angle**

Fourier transform pairs

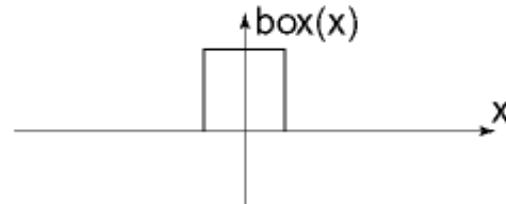
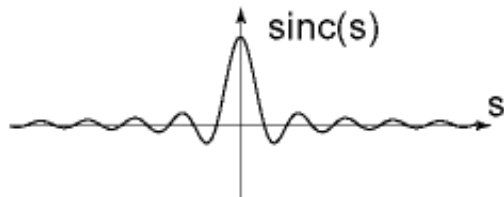
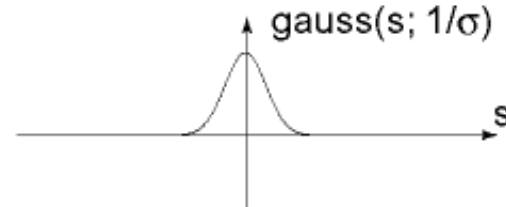
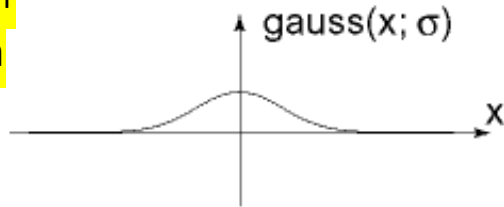
spatial domain

frequency domain



$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi}$$

Note the symmetry:
duality property of
Fourier transform



Computing the discrete Fourier transform (DFT)

Computing the discrete Fourier transform (DFT)

$$F(k) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi kx/N} \text{ is just a matrix multiplication:}$$

$$\mathbf{F} = \mathbf{W} \mathbf{f}$$

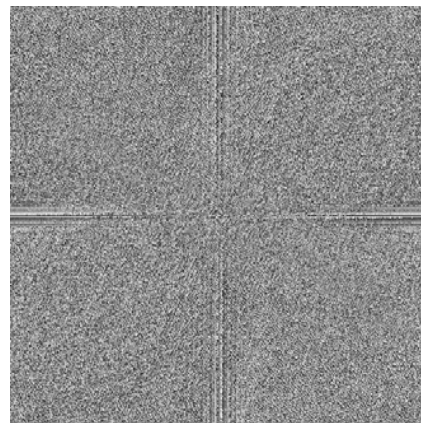
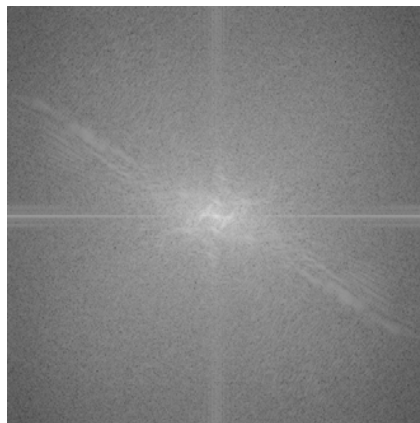
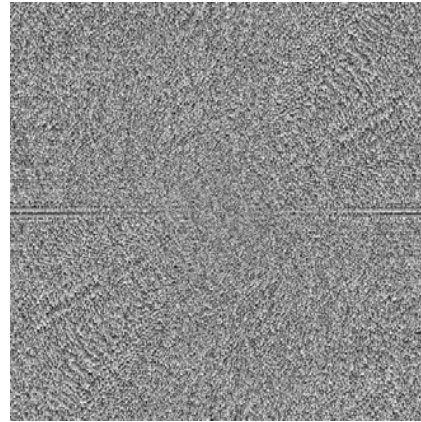
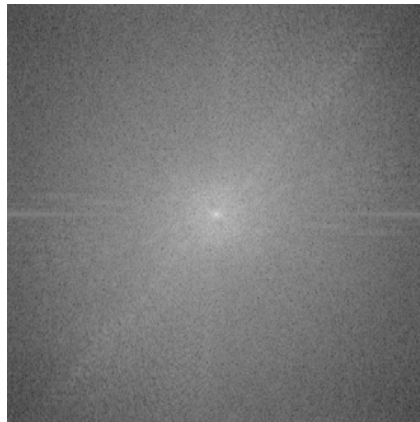
- Basically a matrix-vector product:

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{N-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & W_N^3 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & W_N^6 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \times \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

$$(W_N = e^{-j2\pi/N})$$

In practice this is implemented using the *fast Fourier transform* (FFT) algorithm.

Fourier transforms of natural images



original

amplitude

phase

Fourier transforms of natural images

Image phase matters!

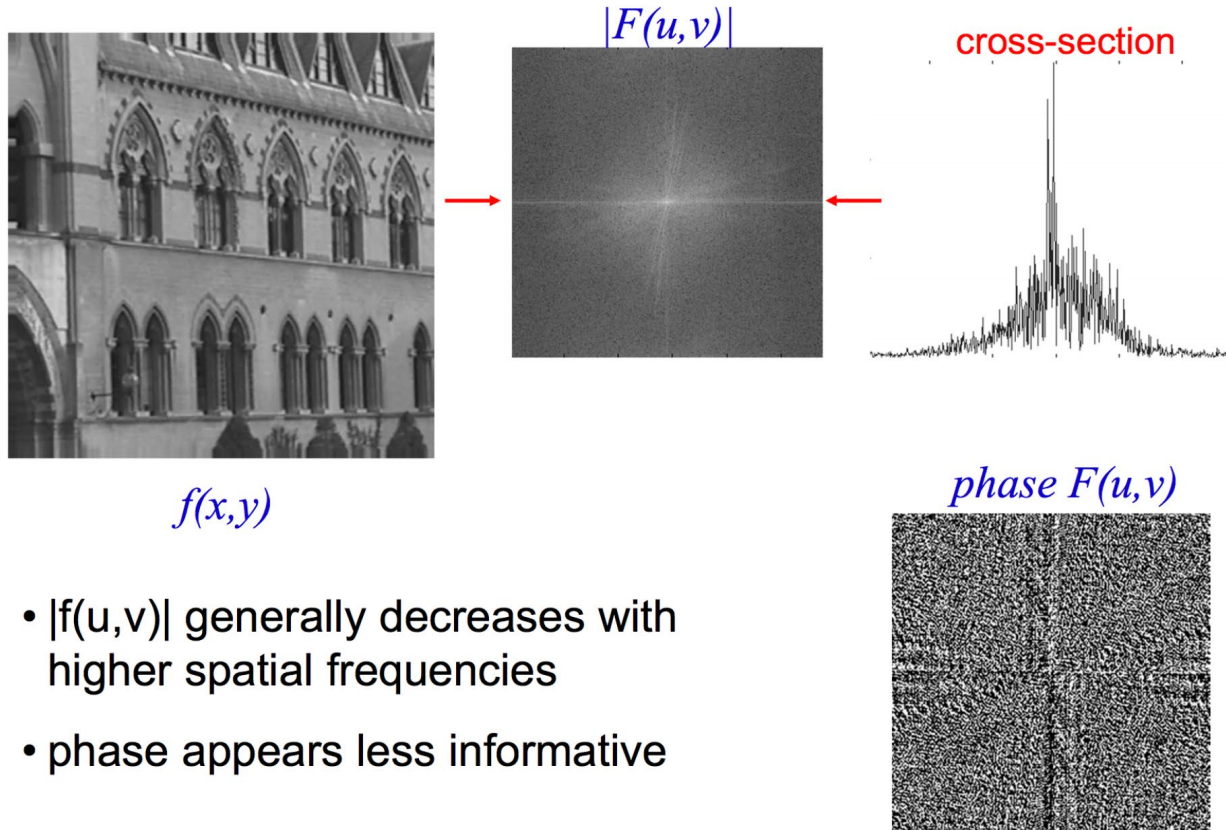


cheetah phase with zebra amplitude



zebra phase with cheetah amplitude

Discrete Fourier Transform - Visualization



Frequency-domain filtering

Why do we care about all this?

The convolution theorem

The Fourier transform of the convolution of two functions is the product of their Fourier transforms:

$$\mathcal{F}\{g * h\} = \mathcal{F}\{g\}\mathcal{F}\{h\}$$

The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms:

$$\mathcal{F}^{-1}\{gh\} = \mathcal{F}^{-1}\{g\} * \mathcal{F}^{-1}\{h\}$$

Convolution in spatial domain is equivalent to multiplication in frequency domain!

What do we use convolution for?

Convolution for 1D continuous signals

Definition of linear shift-invariant filtering as convolution:

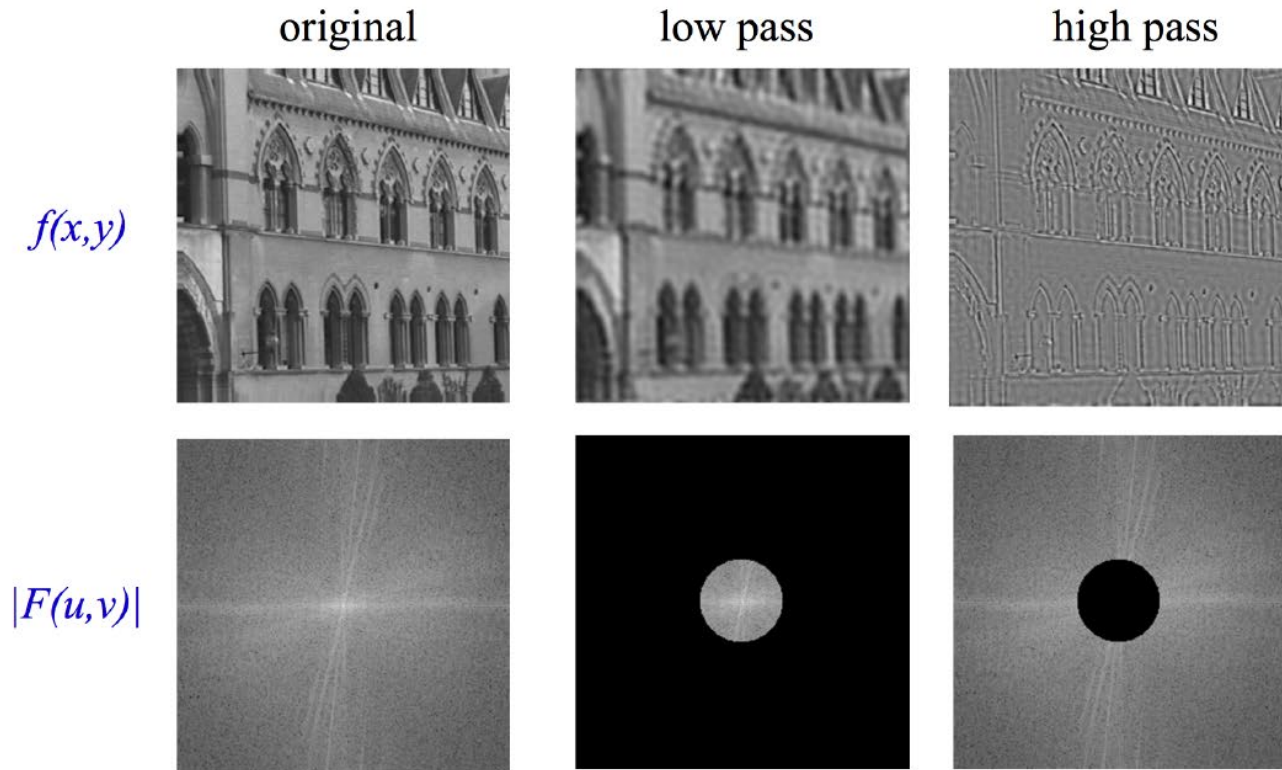
$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy$$

Diagram illustrating the convolution equation $(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)dy$. The equation is centered. Below the equation, three labels with arrows point to parts of the equation: "filtered signal" points to the left side $(f * g)(x)$, "filter" points to $f(y)$, and "input signal" points to $g(x - y)$.

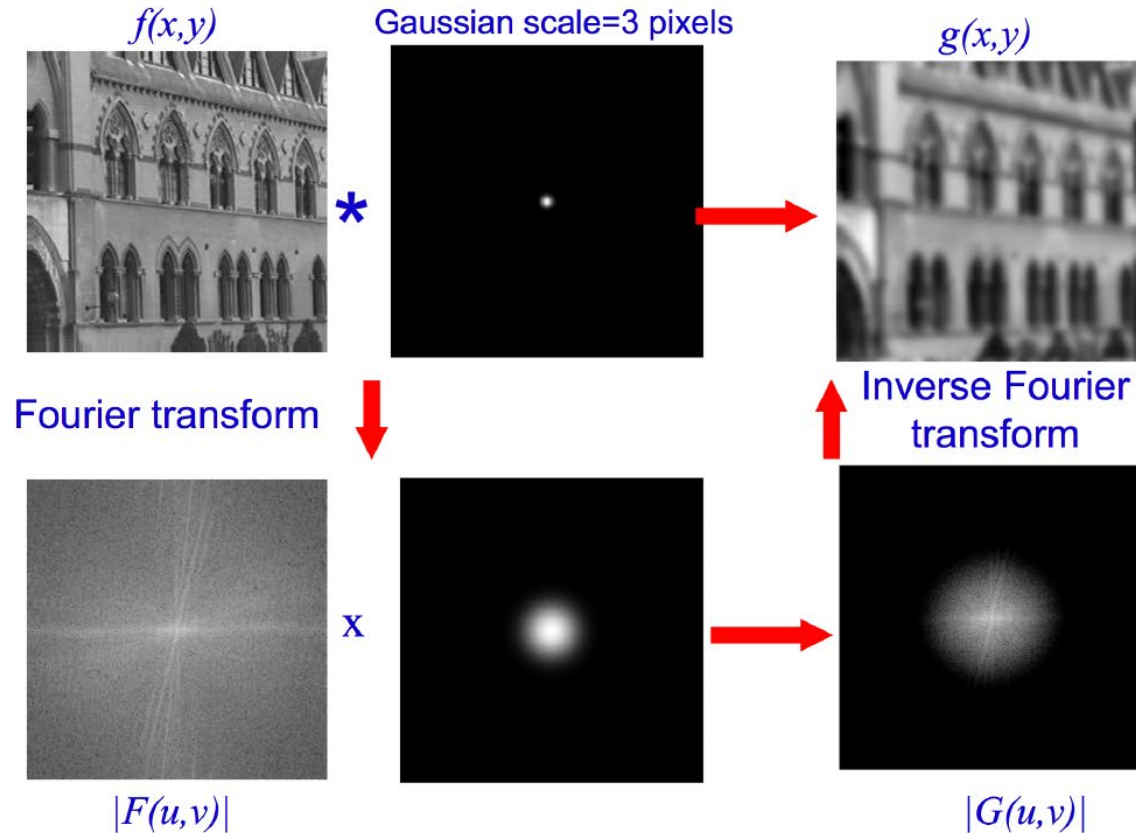
Why implement convolution in frequency domain?

Using the convolution theorem, we can interpret and implement all types of linear shift-invariant filtering as multiplication in frequency domain.

Image Filtering in the Frequency Domain

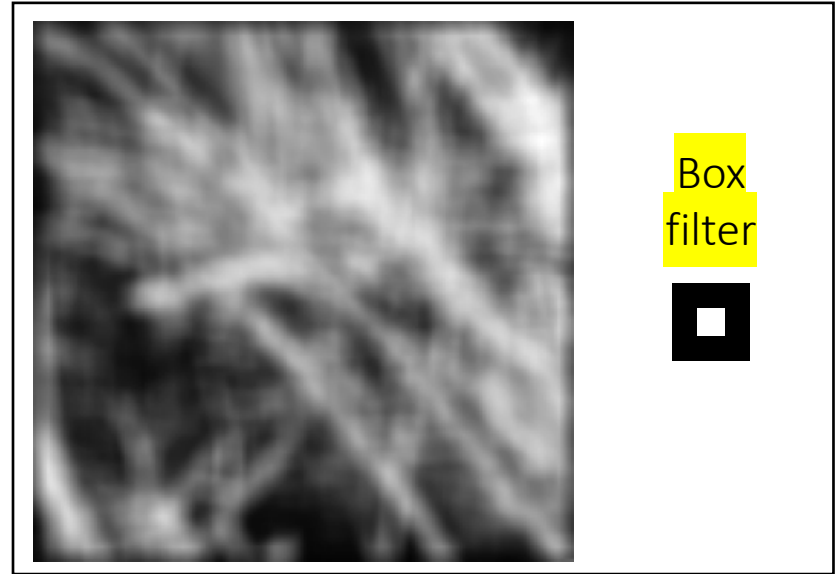


Blurring in the Time vs Frequency Domain

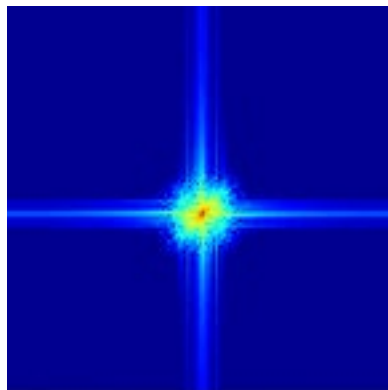
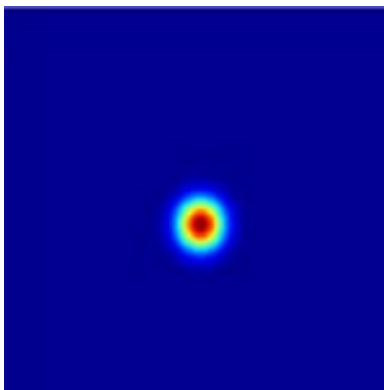
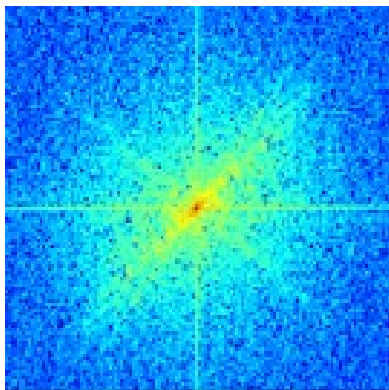
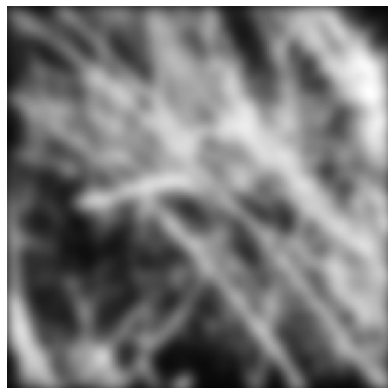
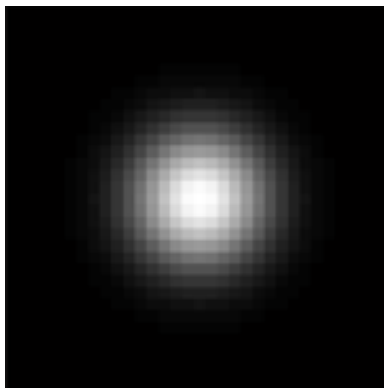


Revisiting blurring

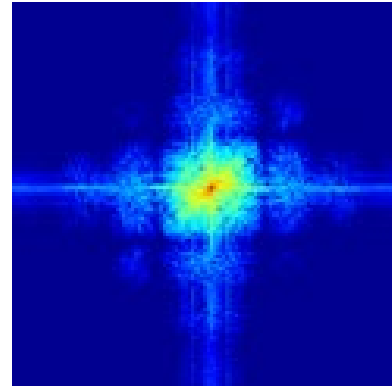
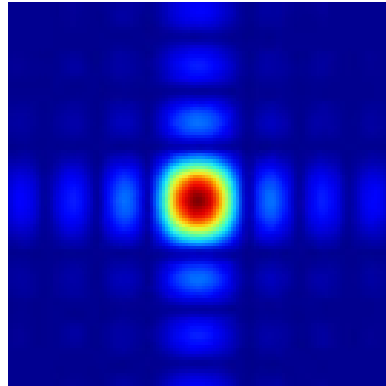
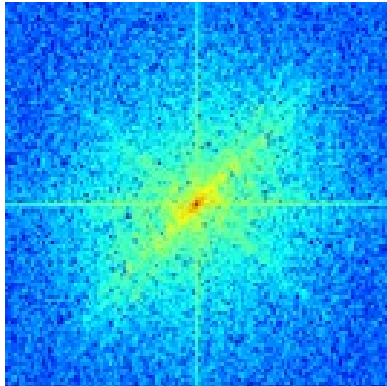
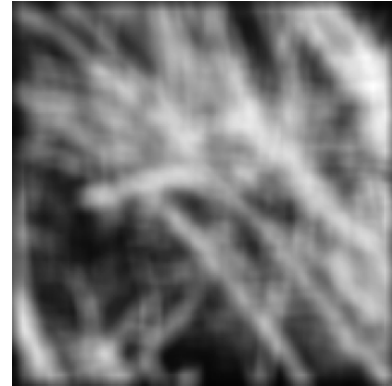
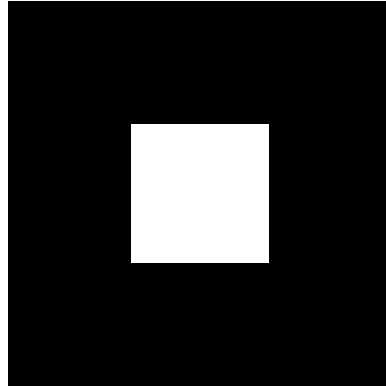
Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?



Gaussian blur



Box blur

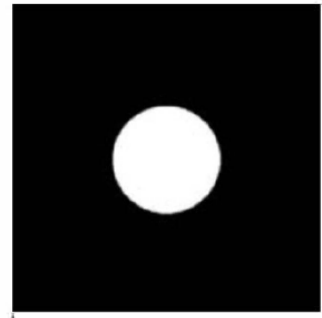


Ideal Low-pass filter

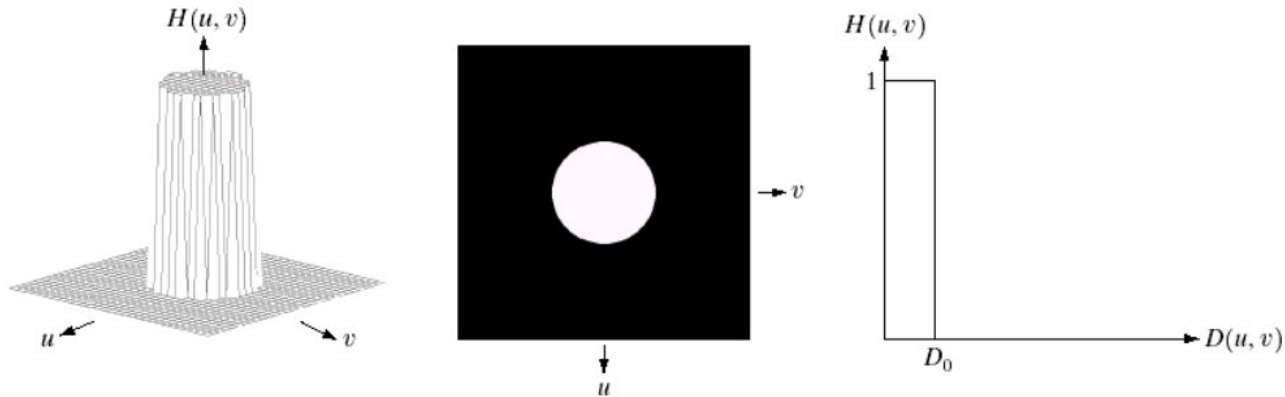
An ideal low-pass filter ILPF is defined by:

$$H(u, v) = \begin{cases} 1 & \text{if } D(u, v) \leq D_0 \\ 0 & \text{if } D(u, v) > D_0 \end{cases}$$

The point of transition between $H(u, v) = 1$ and $H(u, v) = 0$ is called the **cutoff frequency**



Ideal low pass filter



a b c

FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Ideal low pass filter

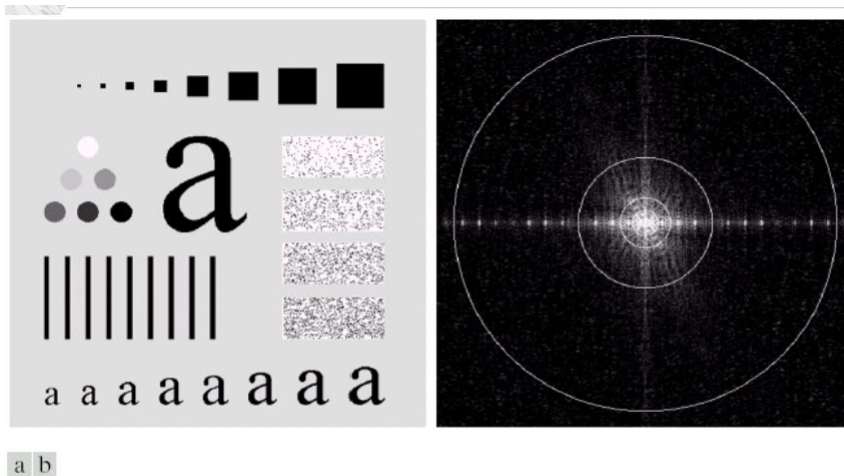
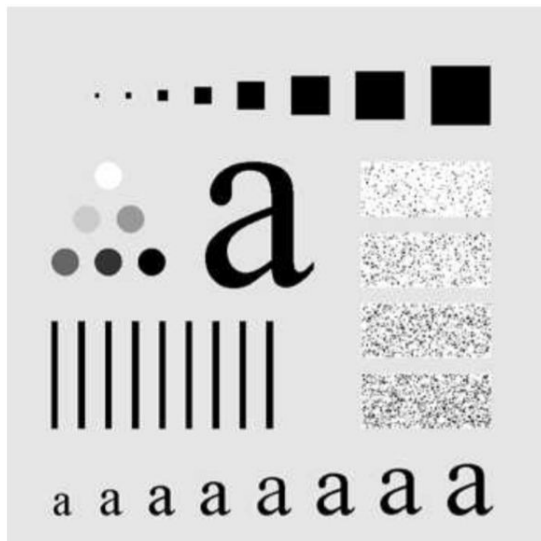
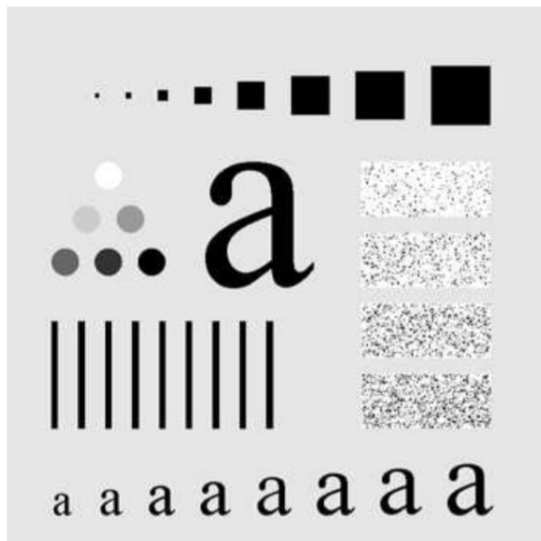


FIGURE 4.11 (a) An image of size 500×500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.



ILPF with cutoff
frequency =60



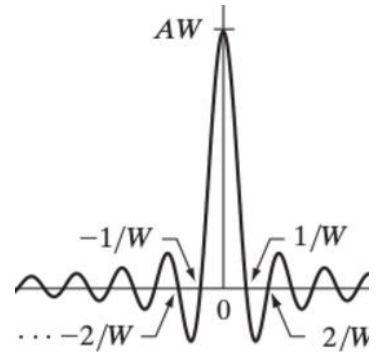
ILPF with cutoff
frequency =30

ILPF

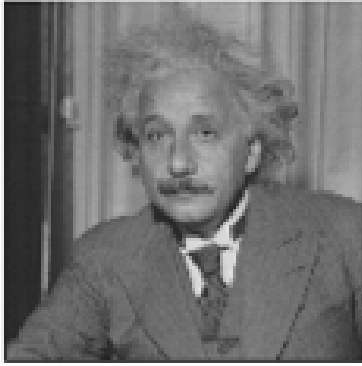
The blurring and ringing properties of ILPFs can be explained using the convolution theorem:

Because a cross section of the ILPF in the frequency domain looks like a box filter, a cross section of the corresponding spatial filter has the shape of a sinc.

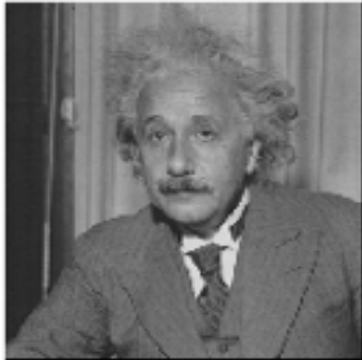
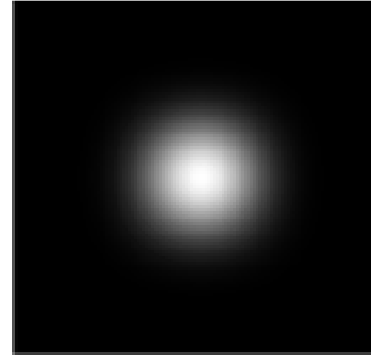
Convolving a sinc with an impulse copies the sinc at the location of the impulse. The sinc center lobe causes the blurring, while the outer lobes are responsible for ringing.



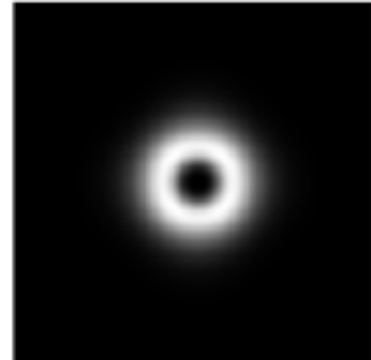
More filtering examples



?

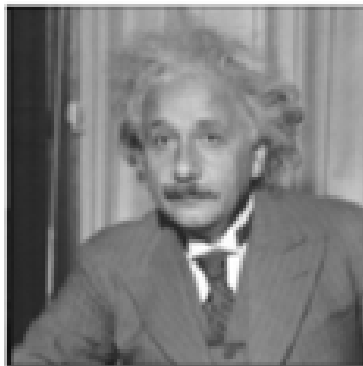
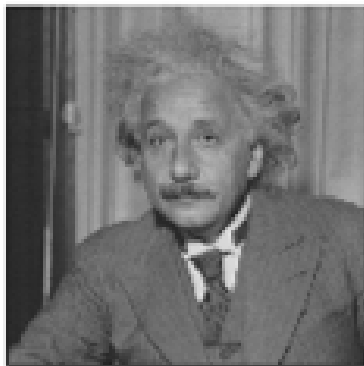


?

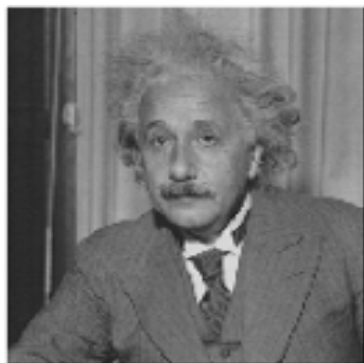
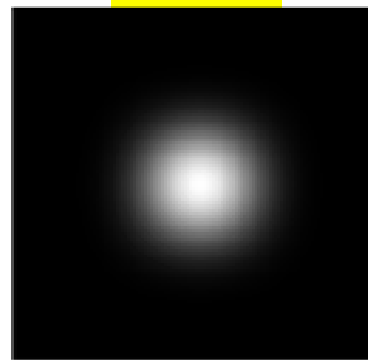


filters shown
in frequency-
domain

More filtering examples



low-pass



band-pass



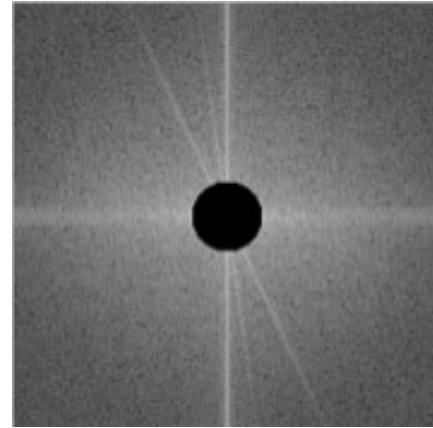
filters shown
in frequency-
domain

More filtering examples

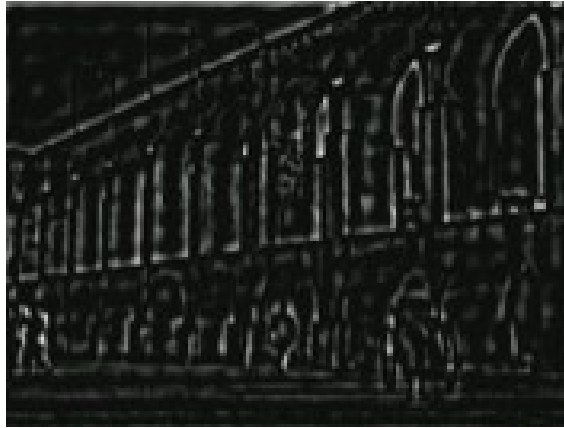


?

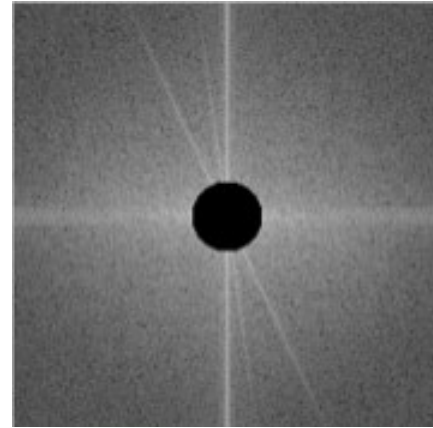
high-pass



More filtering examples

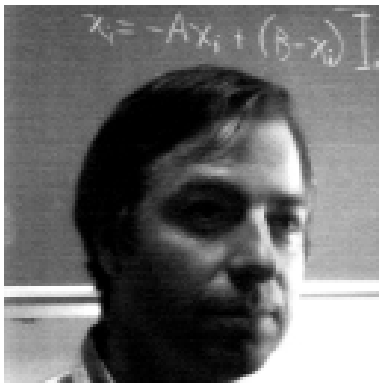


high-pass

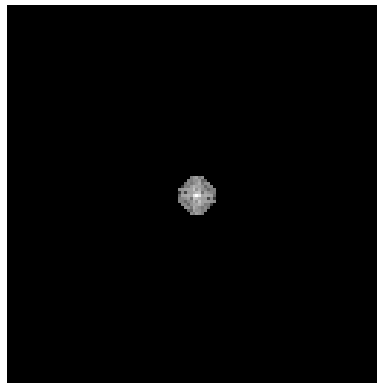


More filtering examples

original image

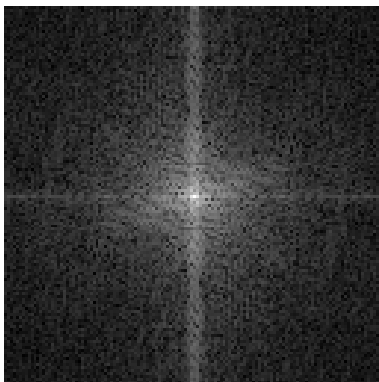


low-pass filter



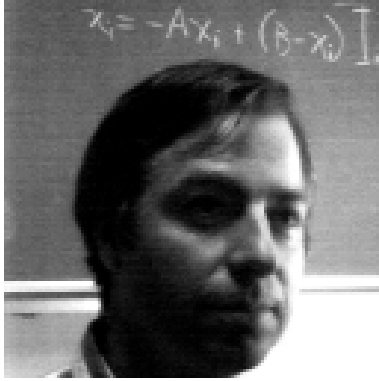
?

frequency magnitude

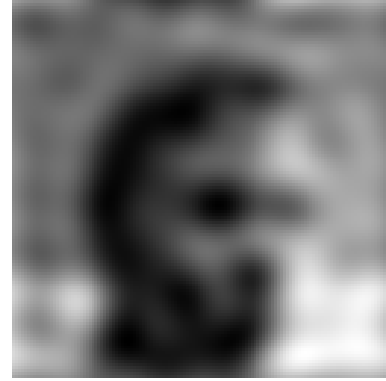
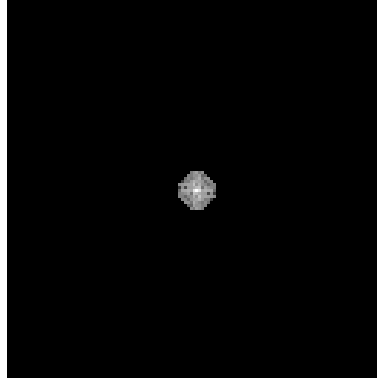


More filtering examples

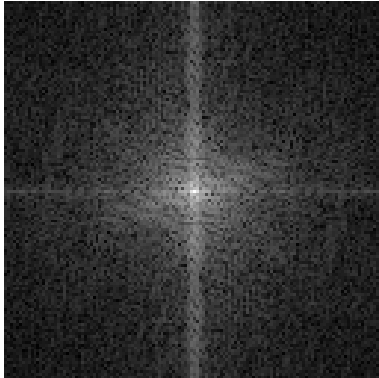
original image



low-pass filter

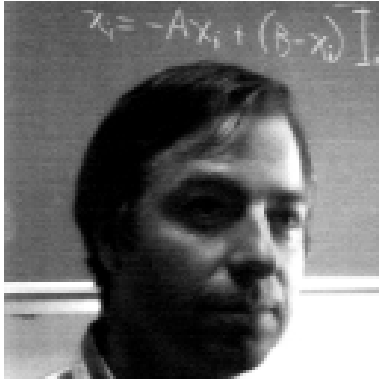


frequency magnitude

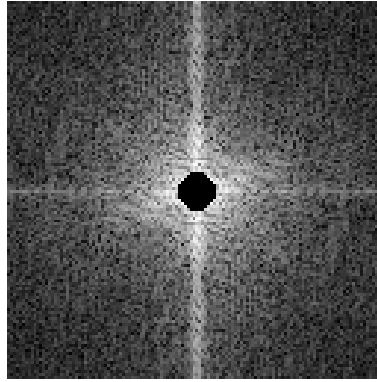


More filtering examples

original image

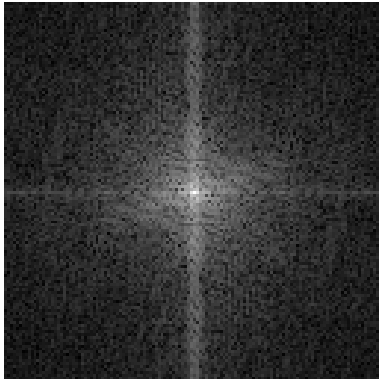


high-pass filter



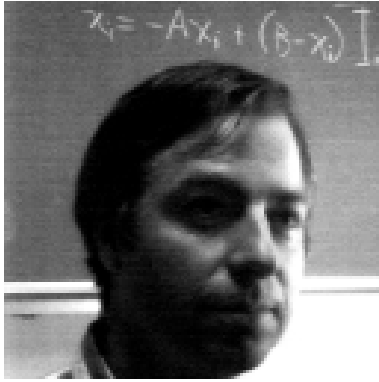
?

frequency magnitude

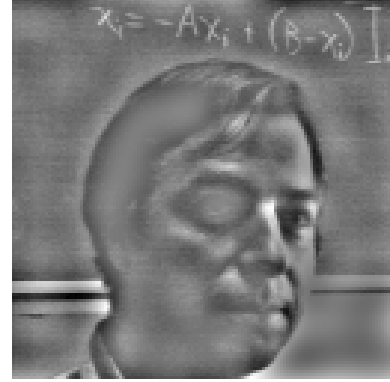
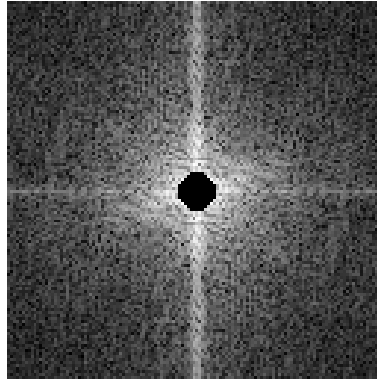


More filtering examples

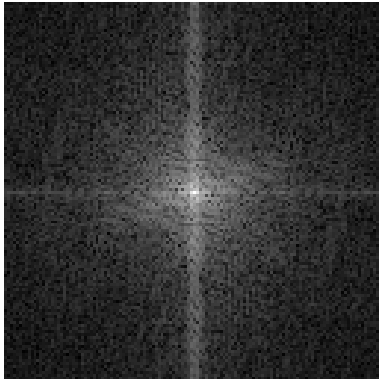
original image



high-pass filter

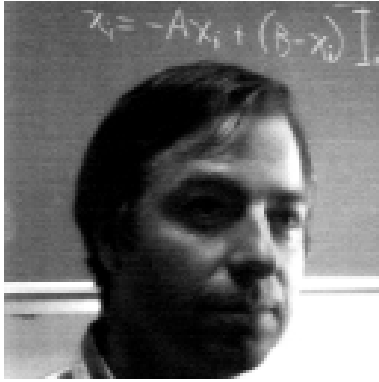


frequency magnitude

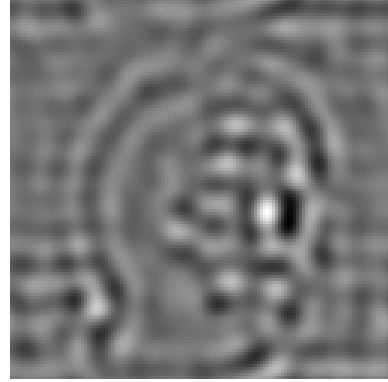
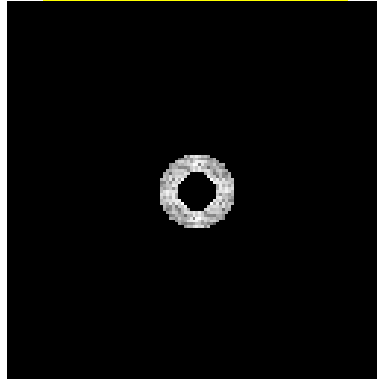


More filtering examples

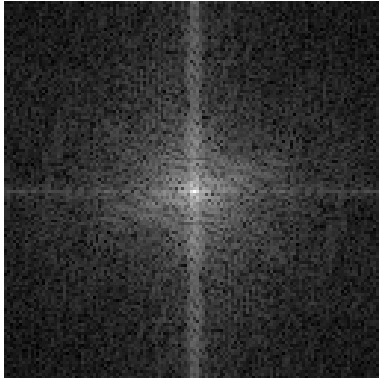
original image



band-pass filter

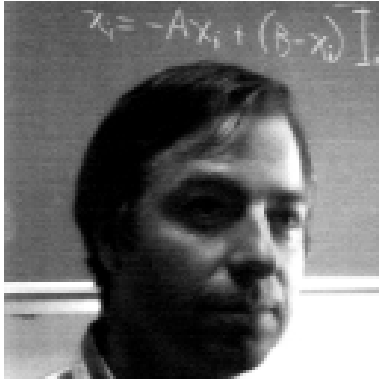


frequency magnitude

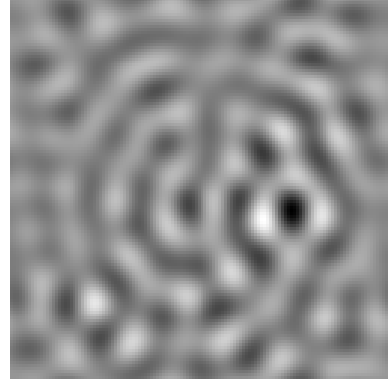
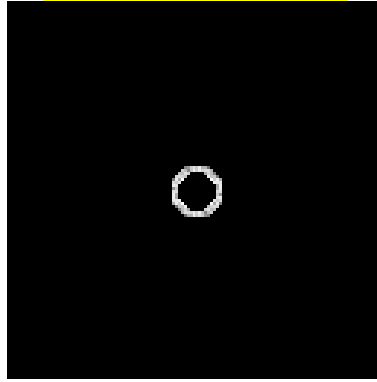


More filtering examples

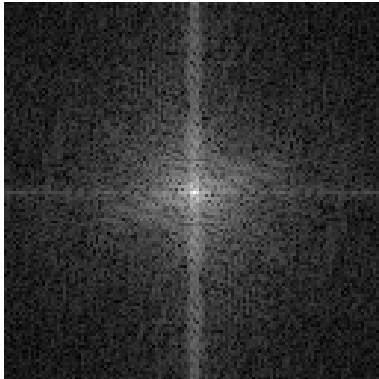
original image



band-pass filter

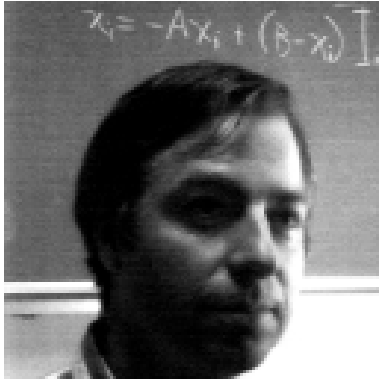


frequency magnitude

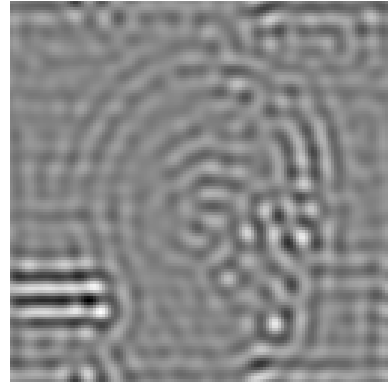
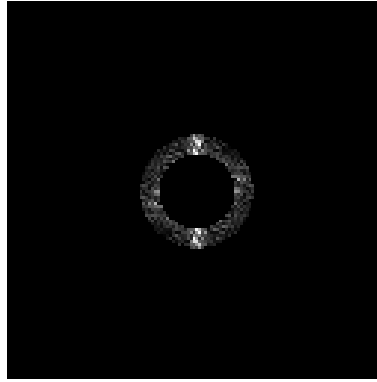


More filtering examples

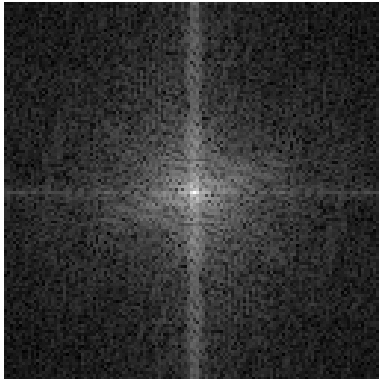
original image



band-pass filter

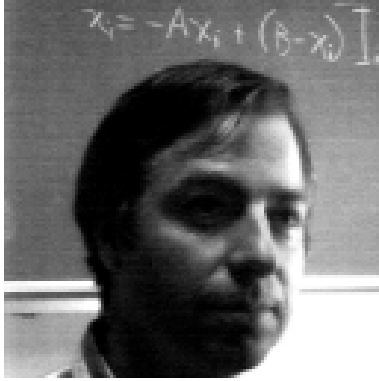


frequency magnitude

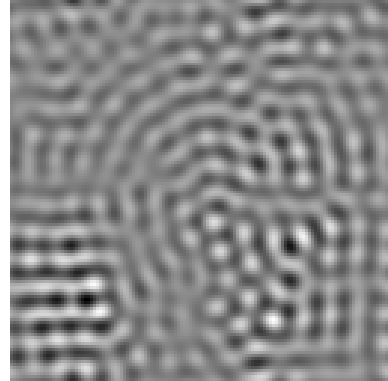
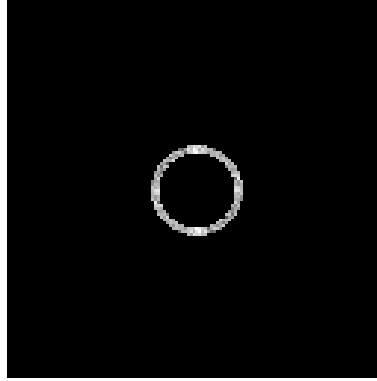


More filtering examples

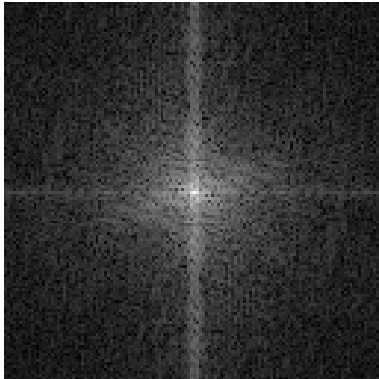
original image



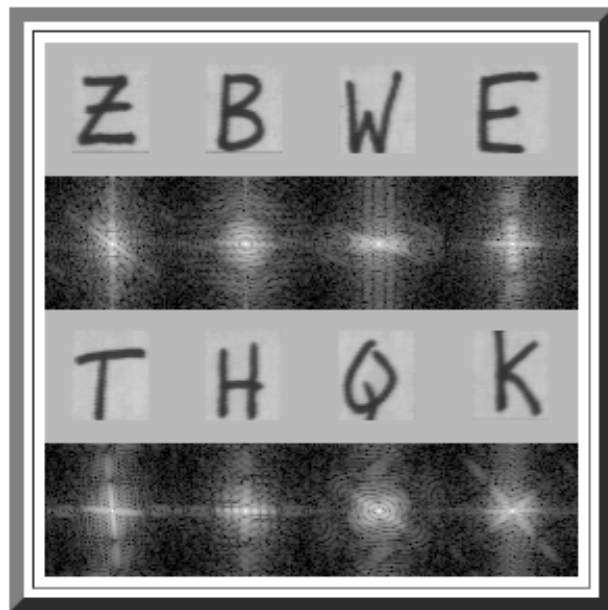
band-pass filter



frequency magnitude



Recognizing character



Revisiting sampling

The Nyquist-Shannon sampling theorem

A continuous signal can be perfectly reconstructed from its discrete version using linear interpolation, if sampling occurred with frequency:

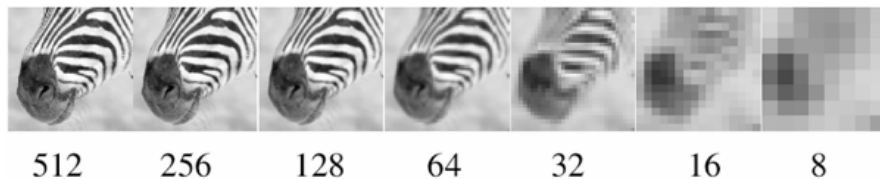
$$f_s \geq 2f_{\max}$$



This is called the
Nyquist frequency

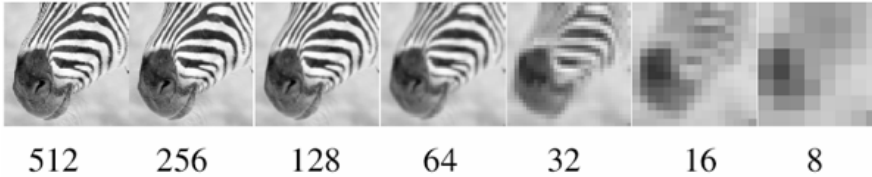
Equivalent reformulation: When downsampling, aliasing does not occur if samples are taken at the Nyquist frequency or higher.

Gaussian pyramid



How does the Nyquist-Shannon theorem relate to the Gaussian pyramid?

Gaussian pyramid



How does the Nyquist-Shannon theorem relate to the Gaussian pyramid?

- Gaussian blurring is low-pass filtering.
- By blurring we try to sufficiently decrease the Nyquist frequency to avoid aliasing.

How large should the Gaussian blur we use be?

- The cut-off frequency of the Gaussian filter is proportional to the standard deviation of the filter in the frequency domain
- The range is equal to double the standard deviation. In general we use a mask three times the standard deviation

Frequency-domain filtering in human vision



“Hybrid image”

Aude Oliva and Philippe Schyns

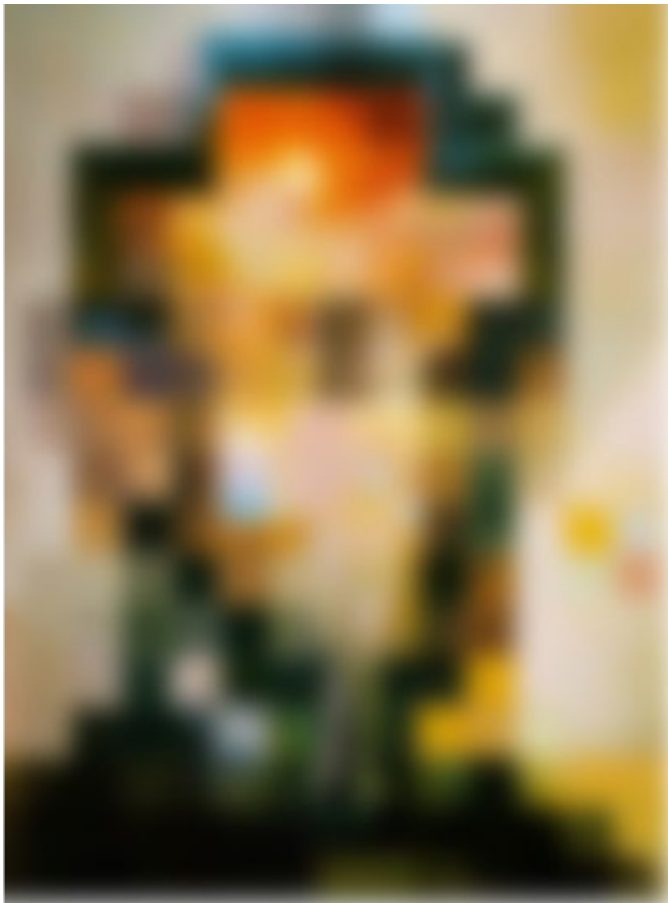
Frequency-domain filtering in human vision



*Gala Contemplating the
Mediterranean Sea Which at Twenty
Meters Becomes the Portrait of
Abraham Lincoln
(Homage to Rothko)*

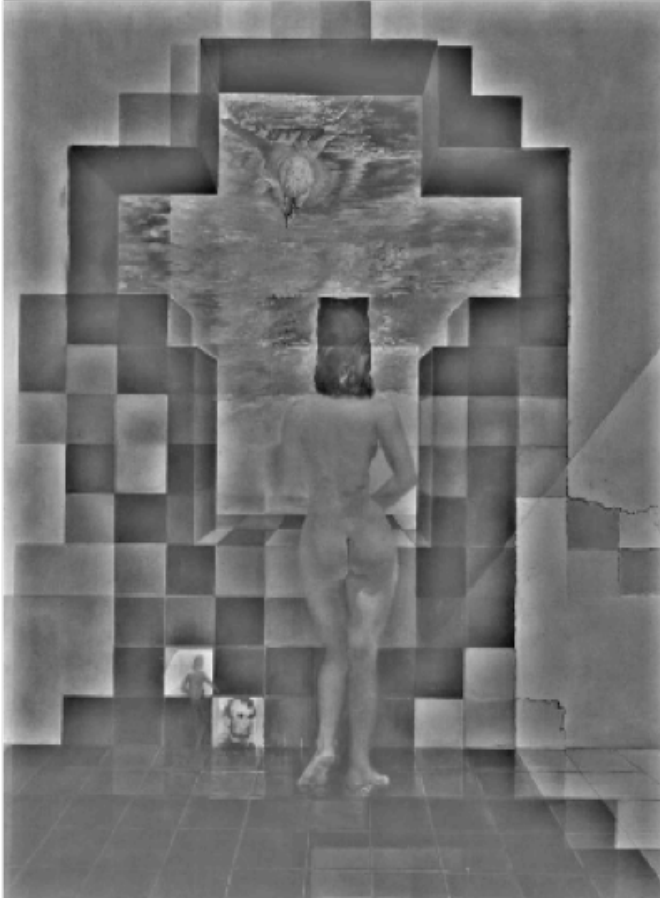
Salvador Dali, 1976

Frequency-domain filtering in human vision



Low-pass filtered version

Frequency-domain filtering in human vision



High-pass filtered version

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