Appendix A

Proof of $\int_{-\infty}^{\infty} 2M \frac{\sin 2\pi a M}{2\pi a M} da = 1$

We first consider the contour integration of a function $F(z) = e^{jz} f(z) = e^{jz}/z$ around a closed contour in the z-plane as shown in Figure A.1, where $z = x + jy$.

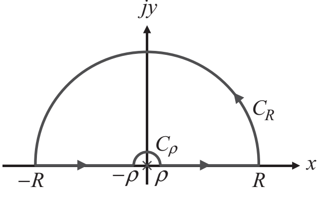

Figure A.1 A contour with a single pole at $z = 0$

Using Cauchy's residue theorem, the contour integral becomes

$$\oint \frac{e^{jz}}{z} dz = \int_{C_R} \frac{e^{jz}}{z} dz + \int_{-R}^{-\rho} \frac{e^{jx}}{x} dx + \int_{C_\rho} \frac{e^{jz}}{z} dz + \int_{\rho}^{R} \frac{e^{jx}}{x} dx = 0 \tag{A.1}$$

From Jordan's lemma, the first integral on the right of the first equality is zero if $R \to \infty$, i.e. $\lim_{R\to\infty} \int_{C_R} e^{jz} f(z) dz = 0$. Letting $z = \rho e^{j\theta}$ and $dz = j\rho e^{j\theta} d\theta$, where θ varies from π to 0, the third integral can be written as

$$\int_{C_\rho} \frac{e^{jz}}{z} dz = j \int_{\pi}^{0} e^{j(\rho e^{j\theta})} d\theta \tag{A.2}$$

Taking the limit as $\rho \to 0$, this becomes

$$\lim_{\rho \to 0} \left\{ j \int_{\pi}^{0} e^{j(\rho e^{j\theta})} d\theta \right\} = j\theta|_{\pi}^{0} = -j\pi \tag{A.3}$$

Now, consider the second and fourth integral together:

$$\int_{-R}^{-\rho} \frac{e^{jx}}{x} dx + \int_{\rho}^{R} \frac{e^{jx}}{x} dx = \int_{-R}^{-\rho} \frac{\cos x + j \sin x}{x} dx + \int_{\rho}^{R} \frac{\cos x + j \sin x}{x} dx \tag{A.4}$$

Since $\cos(x)/x$ is odd, the cosine terms cancel in the resulting integration. Thus, Equation (A.4) becomes

$$\int_{-R}^{-\rho} \frac{e^{jx}}{x} dx + \int_{\rho}^{R} \frac{e^{jx}}{x} dx = 2j \int_{\rho}^{R} \frac{\sin x}{x} dx \tag{A.5}$$

Combining the above results, for $R \to \infty$ and $\rho \to 0$, Equation (A.1) reduces to

$$\lim_{\substack{\rho \to 0 \\ R \to \infty}} \left\{ 2j \int_{\rho}^{R} \frac{\sin x}{x} dx \right\} = 2j \int_{0}^{\infty} \frac{\sin x}{x} dx = j\pi \tag{A.6}$$

Thus, we have the following result:

$$\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \tag{A.7}$$

We now go back to our problem. We have written

$$\lim_{M \to \infty} 2M \frac{\sin 2\pi aM}{2\pi aM} = \delta(a)$$

in Chapter 3. In order to justify this, the integral of the function

$$f(a) = 2M \frac{\sin 2\pi aM}{2\pi aM}$$

must be unity. We verify this using the above result. Letting $x = 2\pi aM$ and $dx = 2\pi M da$, we have

$$\int_{-\infty}^{\infty} f(a) da = \int_{-\infty}^{\infty} 2M \frac{\sin 2\pi aM}{2\pi aM} da = \int_{-\infty}^{\infty} 2M \frac{\sin x}{x} \frac{dx}{2\pi M} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx \tag{A.8}$$

From Equation (A.7),

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 2 \int_{0}^{\infty} \frac{\sin x}{x} dx = \pi$$

thus Equation (A.8) becomes

$$\int_{-\infty}^{\infty} 2M \frac{\sin 2\pi aM}{2\pi aM} da = 1 \tag{A.9}$$

This proves that the integral of the function in Figure A.2 (i.e. Figure 3.11) is unity.

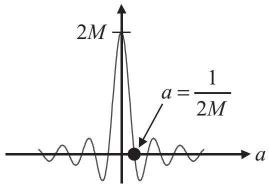

Figure A.2 Representation of the delta function using a sinc function