Signal Processing I & II

Chul Min Yeum
Assistant Professor
Civil and Environmental Engineering
University of Waterloo, Canada

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FACULTY OF ENGINEERING

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Reference

We will cover some key topics in Chapters 3 ~ 6 of the following reference:

Shin, K., & Hammond, J. K. (2008). Fundamentals of Signal Processing: for Sound and Vibration Engineers, John Wiley & Sons.

Chapter 3: Fourier Series

Chapter 4: Fourier Integrals (Fourier Transform) and Continuous-Time Linear Systems

Chapter 5: Time Sampling and Aliasing

Chapter 6: The Discrete Fourier Transform

Fast Fourier Transform

A fast Fourier transform (FFT) is an algorithm that computes the discrete Fourier transform (DFT) of a sequence, or its inverse (IDFT). Fourier analysis converts a signal from its original domain (often time or space) to a representation in the frequency domain and vice versa. It manages to reduce the complexity of computing the DFT from $O(n^2)$, which arises if one simply applies the definition of DFT, to $O(n \, log \, n)$, where n is the data size.

FFT in Matlab

fft

Fast Fourier transform

Syntax

```
Y = fft(X)
Y = fft(X,n)
Y = fft(X,n,dim)
```

Y = fft(X) computes the discrete Fourier transform (DFT) of X using a fast Fourier transform (FFT) algorithm

Y = fft(X,n) returns the n-point DFT. If no value is specified, Y is the same size as X.

Periodic Signals and Fourier Series

Periodic signals are analyzed using Fourier series. The basis of Fourier analysis of a periodic signal is the representation of such a signal by adding together sine and cosine functions of appropriate frequencies, amplitudes and relative phases. For a single sine wave

$$x(t) = Xsin(wt + \emptyset) = Xsin(2\pi ft + \emptyset)$$

where X is amplitude,

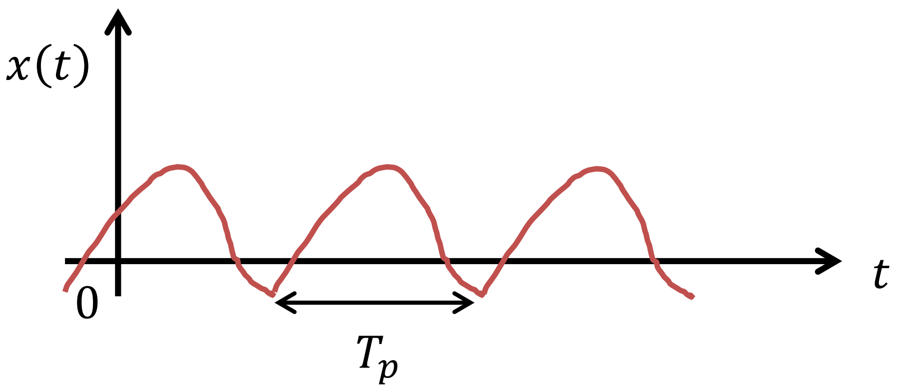
w is circular (angular) frequency in radians per unit time (rad/s),

f is (cyclical) frequency in cycles per unit time (Hz),

$\emptyset$ is phase angle with respect to the time origin in radians.

Fourier Series

A Fourier series is an expansion of a periodic function f(x) in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonality relationships of the sine and cosine functions. It decomposes any periodic function or periodic signal into the weighted sum of a (possibly infinite) set of simple oscillating functions, namely sines and cosines.



$$x(t) = x(t + nT_p)$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T_p}\right) + b_n \sin\left(\frac{2\pi n t}{T_p}\right)$$

Basic Trigonometric Equations

$$\int_{-\pi}^{\pi} \cos nt \ dt = 0$$

$$\int_{-\pi}^{\pi} \sin nt \ dt = 0$$

$$\cos mt \cos nt = \frac{1}{2}[\cos(m+n)\,t + \cos(m-n)\,t]$$

$$\sin mt \sin nt = \frac{1}{2}[\cos(m-n)\,t - \cos(m+n)\,t]$$

$$\sin mt \cos nt = \frac{1}{2}[\sin(m+n)\,t + \sin(m-n)\,t]$$

Orthogonality of Trigonometric Functions

$$\int_{-\pi}^{\pi} \cos mt \cos nt \ dt = \begin{cases} 0 \ if \ n \neq m \\ \pi \ if \ n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \sin nt \ dt = \begin{cases} 0 \ if \ n \neq m \\ \pi \ if \ n = m \end{cases}$$

$$\int_{-\pi}^{\pi} \sin mt \cos nt \ dt = \begin{cases} 0 \ if \ n \neq m \\ 0 \ if \ n = m \end{cases}$$

Fourier Coefficients

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt$$

$$a_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi mt}{T_p}\right)dt$$

$$b_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi mt}{T_p}\right)dt$$

Fourier Coefficients (Continue)

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

$$\frac{a_0}{2} = \frac{1}{T_p}\int_0^{T_p} x(t)dt = \boxed{\frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt} = \frac{a_0}{2} + \underbrace{\cancel{\frac{1}{T_p}\int_{-T_p/2}^{T_p/2} \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right) dt}}_{0}$$

$$a_m = \boxed{\frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi mt}{T_p}\right)dt} = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)\right)\cos\left(\frac{2\pi mt}{T_p}\right)dt$$

$$= \underbrace{\cancel{\frac{2}{T_p}\int_{-T_p/2}^{T_p/2}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T_p}\right)\right)\cos\left(\frac{2\pi mt}{T_p}\right)}}_{0} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right)\cos\left(\frac{2\pi mt}{T_p}\right)dt = \frac{2a_m}{T_p}\int_{-T_p/2}^{T_p/2}\cos\left(\frac{2\pi mt}{T_p}\right)\cos\left(\frac{2\pi mt}{T_p}\right)dt = a_m$$

$$b_m = \boxed{\frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi mt}{T_p}\right)dt}$$

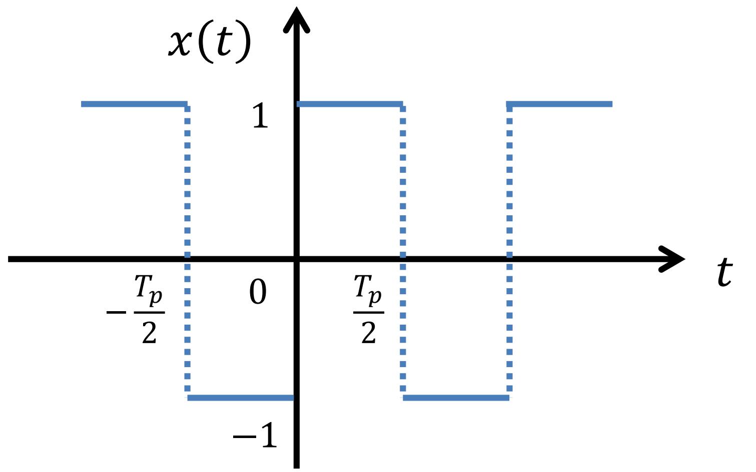
$$A = \frac{2\pi t}{T_p}$$

Example: Square Wave

$$x(t) = -1 \; if \; -\frac{T_p}{2} < t < 0$$

$$x(t) = \quad 1 \; if \; 0 < t < \frac{T_p}{2}$$

$$x(t + nT_p) = x(t)$$
$$where \; n = \pm 1, \pm 2, \dots$$



$$\frac{a_0}{2} = \frac{1}{T_p}\int_0^{T_p} x(t)dt = 0$$

$$a_n = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi nt}{T_p}\right)dt = \frac{2}{T_p}\left[\int_{-T_p/2}^{0} -\cos\left(\frac{2\pi nt}{T_p}\right)dt + \int_0^{T_p/2}\cos\left(\frac{2\pi nt}{T_p}\right)dt\right] = 0$$

$$b_n = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi nt}{T_p}\right)dt = \frac{2}{T_p}\left[\int_{-T_p/2}^{0} -\sin\left(\frac{2\pi nt}{T_p}\right)dt + \int_0^{T_p/2}\sin\left(\frac{2\pi nt}{T_p}\right)dt\right] = \frac{2}{n\pi}(1 - cosn\pi)$$

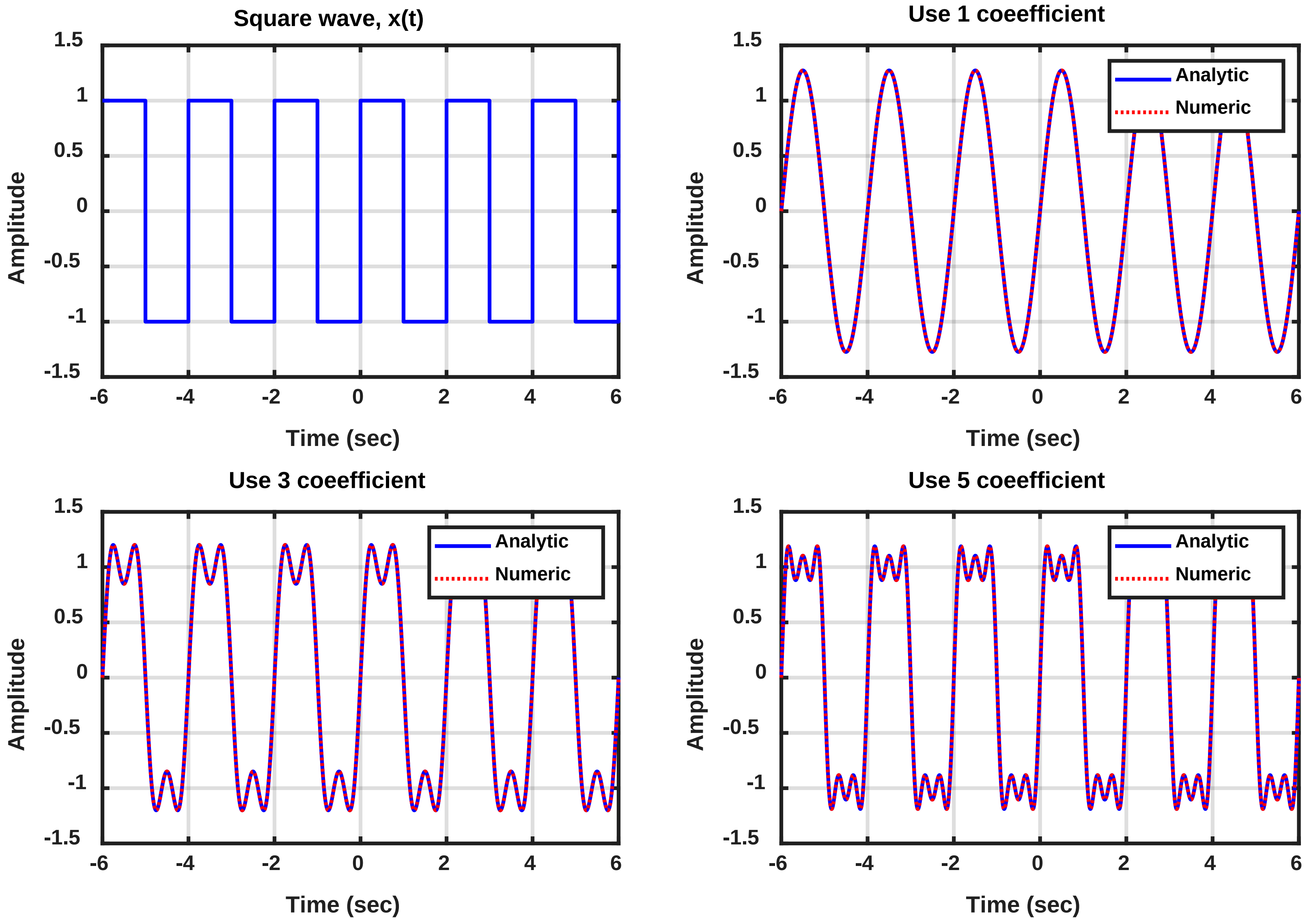
Example: Square Wave (Continue)

$$x(t) = -1 \; if \; -1 < t < 0$$

$$x(t) = \; 1 \; if \; 0 < t < 1$$

$$x(t + 2n) = x(t)$$

$$where \; n = \pm 1, \pm 2, \ldots$$



Complex Form of the Fourier Series

Euler Formula

$$e^{iwt} = coswt + i\ sinwt \qquad e^{-iwt} = coswt - i\ sinwt \qquad coswt = \frac{1}{2}(e^{iwt} + e^{-iwt}) \qquad sinwt = \frac{1}{2}(e^{iwt} - e^{-iwt})$$

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos wnt + b_n \sin wnt = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}\left(e^{iwnt} + e^{-iwnt}\right) + \frac{b_n}{2j}\left(e^{iwnt} - e^{-iwnt}\right) \qquad w = \frac{2\pi}{T_p}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{iwnt} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-iwnt} = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{iwnt} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-iwnt}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{iwnt} + \sum_{n=1}^{\infty} c_n^* e^{-iwnt} \ \ where\ c_0 = \frac{a_0}{2}, \qquad c_n = \frac{a_n - jb_n}{2}, \qquad c_n^* = \frac{a_n + jb_n}{2}$$

$$c_0 = \frac{1}{T_p}\int_0^{T_p} x(t)dt \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad c_n^* = \frac{1}{T_p}\int_0^{T_p} x(t)e^{iwnt}dt = c_{-n}$$

Negative frequency term (c_{-n})

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwnt} \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad w = \frac{2\pi}{T_p}$$

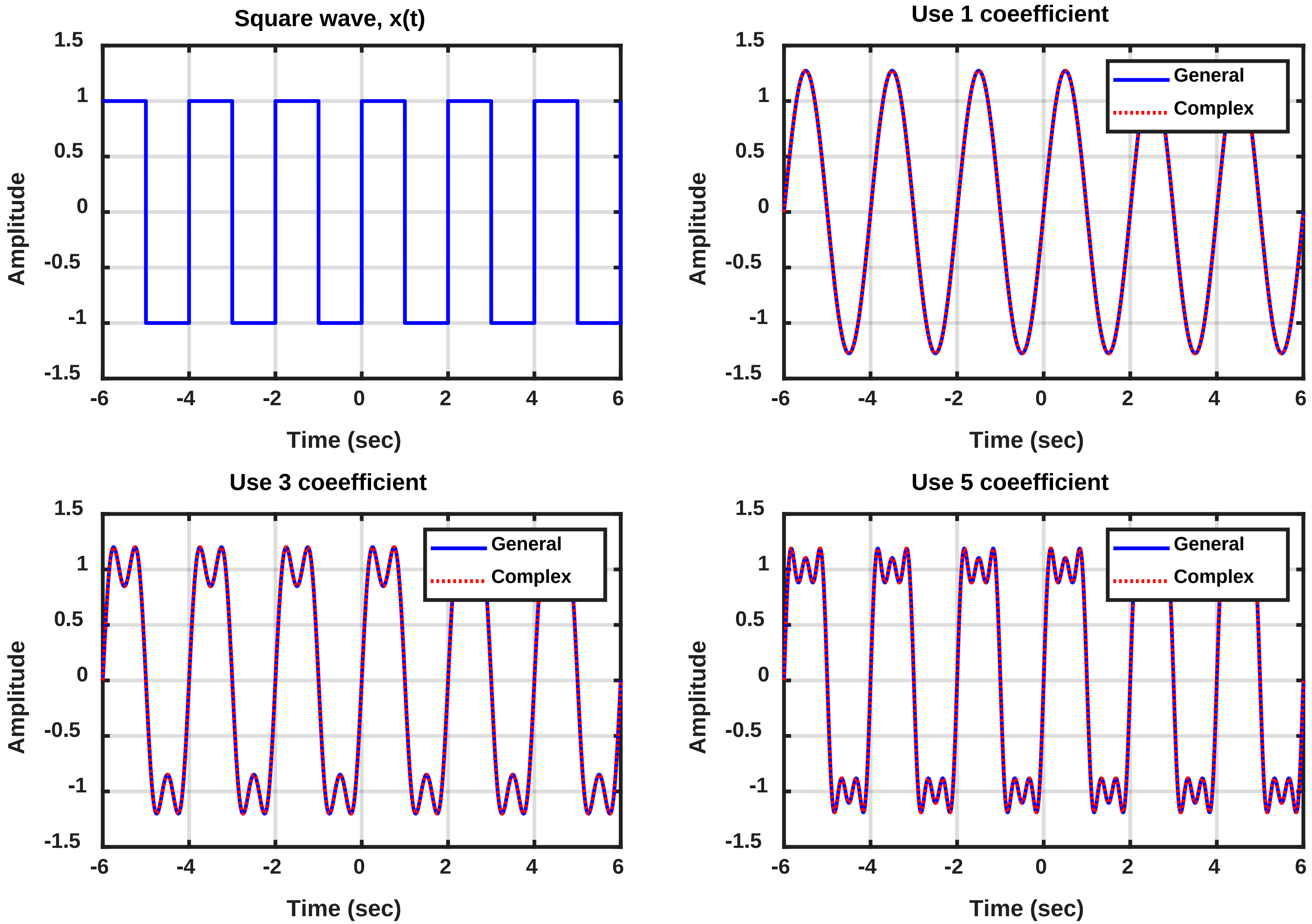
Example: Square Wave (using the Complex Form)

$$x(t) = -1 \; if \; -1 < t < 0$$

$$x(t) = \;\; 1 \; if \; 0 < t < 1$$

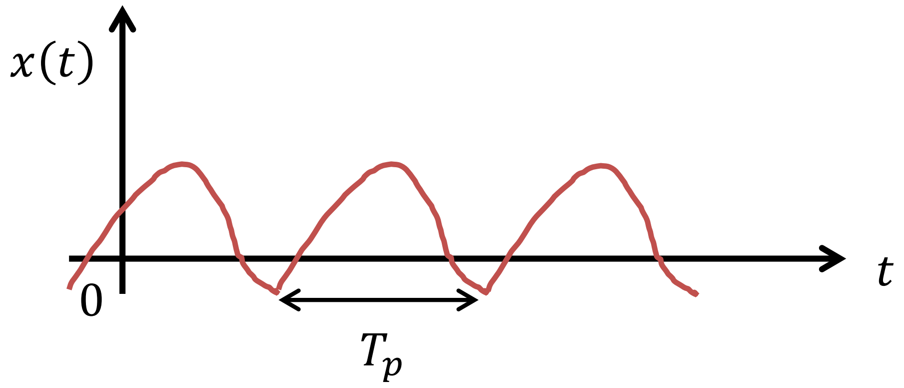
$$x(t + 2n) = x(t)$$

$$where \; n = \pm 1, \pm 2, \ldots$$



Summary and Tutorials

$$x(t) = x(t + nT_p)$$



$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T_p}\right) + b_n \sin\left(\frac{2\pi nt}{T_p}\right)$$

General form

$$\frac{a_0}{2} = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt \qquad a_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\cos\left(\frac{2\pi mt}{T_p}\right)dt \qquad b_m = \frac{2}{T_p}\int_{-T_p/2}^{T_p/2} x(t)\sin\left(\frac{2\pi mt}{T_p}\right)dt$$

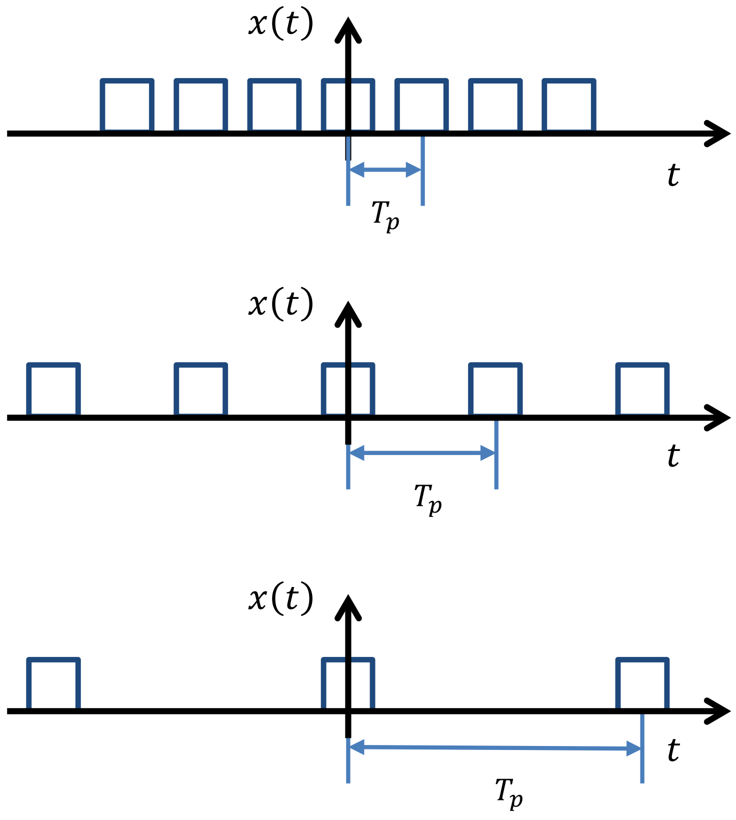
$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwnt} \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-iwnt}dt \qquad w = \frac{2\pi n}{T_p}$$

Complex form

Fourier Integral for Non-periodic Functions

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwnt} \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t) e^{-iwnt} dt \qquad w = \frac{2\pi}{T_p}$$

The fundamental frequency $w = 2\pi/T_p$ becomes smaller and smaller and all other frequencies ($\mathrm{n}w = w_n$), being multiples of the fundamental frequency, are more densely packed on the frequency axis. Their separation is assumed to be $2\pi/T_p = \Delta w$. $\Delta w \to 0$ as $T_p \to \infty$.



$$c_n = \lim_{\substack{T_p\to\infty \\ \Delta w\to 0}} \frac{\Delta w}{2\pi}\int_{-T_p/2}^{T_p/2} x(t) e^{-iw_n t} dt$$

Fourier Integral for Non-periodic Functions (Continue)

$$c_n = \lim_{\substack{T_p \to \infty \\ \Delta w \to 0}} \frac{\Delta w}{2\pi} \int_{-T_p/2}^{T_p/2} x(t) e^{-iw_n t} dt \qquad \lim_{\Delta w \to \infty} \left(\frac{c_n}{\Delta w} \right) = \lim_{\Delta w \to \infty} \left(\frac{1}{2\pi} \int_{-T_p/2}^{T_p/2} x(t) e^{-iw_n t} dt \right)$$

Assuming the limits exist, we write this as

$$X(w_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-iw_n t} dt \qquad X(f_n) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f_n t} dt$$

Since $\Delta f \to \infty$, the frequencies f_n become a continuum, so we write f instead of f_n.

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

Fourier integral or
Fourier transform of $x(t)$

Fourier Integral for Non-periodic Functions (Continue)

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi f_n t/T_p} \qquad \lim_{\Delta f \to \infty} \left(\frac{c_n}{\Delta f}\right) = X(f_n)$$

$x(t)$ can be rewritten as

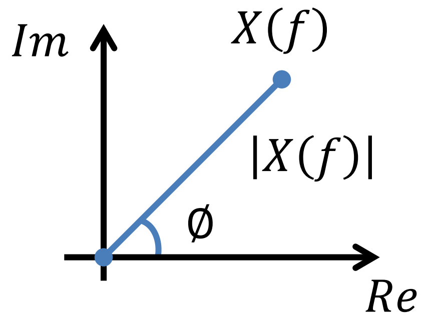
$$x(t) = \lim_{\Delta f \to \infty} \sum_{n=-\infty}^{\infty} \Delta f X(f_n) e^{i2\pi f_n t/T_p}$$

which can be represented in a continuous form as

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi f t} df \qquad X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt$$

Comments on the Fourier Integral

1. $X(f)$ is complex and can be represented as



$$X(f) = X_{Re}(f) + i\,X_{Im}(f) = |X(f)|e^{i\emptyset(f)}$$

2. Fourier transformation using f and w

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{i2\pi ft}df$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{iwt}dw$$

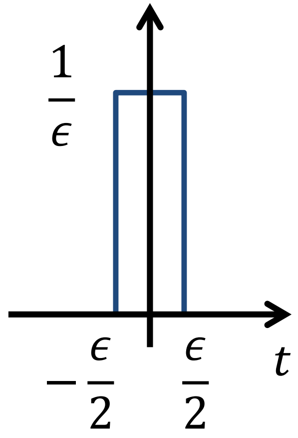
$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt$$

$$X(w) = \int_{-\infty}^{\infty} x(t)e^{-iwt}dt$$

Dirac Delta Function

The Dirac delta function is denoted by $\delta(t)$

$$\delta(t) = 0 \text{ for } t \neq 0, \text{ and } \int_{-\infty}^{\infty} \delta(t)dt = 1$$



$$\delta(t) = \frac{1}{\epsilon} \text{ for } -\frac{\epsilon}{2} < t < \frac{\epsilon}{2}$$
$$= 0 \text{ otherwise}$$

$$\delta(t) = \infty \text{ for } t = 0$$
$$= 0 \text{ otherwise}$$

Properties

$$\int_{-\infty}^{\infty} x(t)\delta(t-a)dt = x(a)$$

$$\int_{-\infty}^{\infty} e^{\pm i2\pi at} dt = \delta(a)$$

Proof:

$$\int_{-\infty}^{\infty} e^{\pm i2\pi at} dt = \lim_{M\to\infty}\left(\int_{-M}^{M} (cos2\pi at \pm i\ sin2\pi at)dt\right)$$

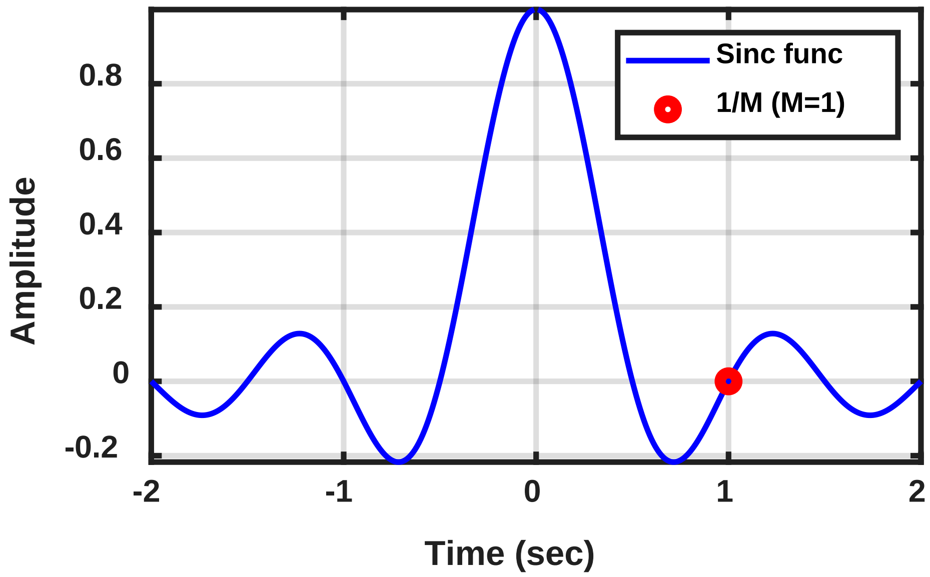
$$= \lim_{M\to\infty}\left(\int_{-M}^{M} (cos2\pi at)dt\right) = \lim_{M\to\infty} 2\left.\frac{sin2\pi at}{2\pi a}\right|_0^M$$

$$= \lim_{M\to\infty} 2M\frac{sin2\pi aM}{2\pi aM} = \delta(a)$$

Sinc function

Sinc Function

$$sinc(x) \equiv \begin{cases} 1 & for\ x = 0 \\ \dfrac{\sin Mx}{Mx} & otherwise \end{cases}$$



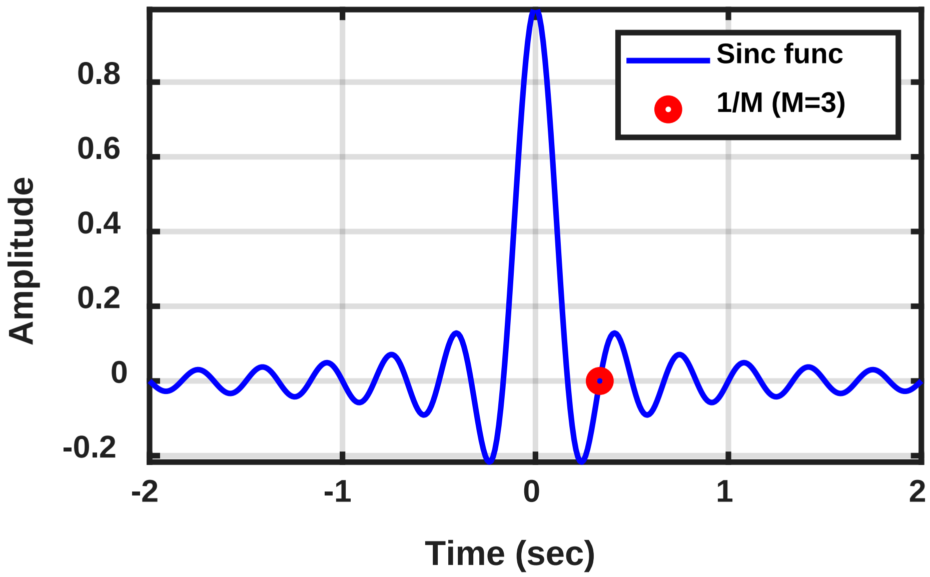


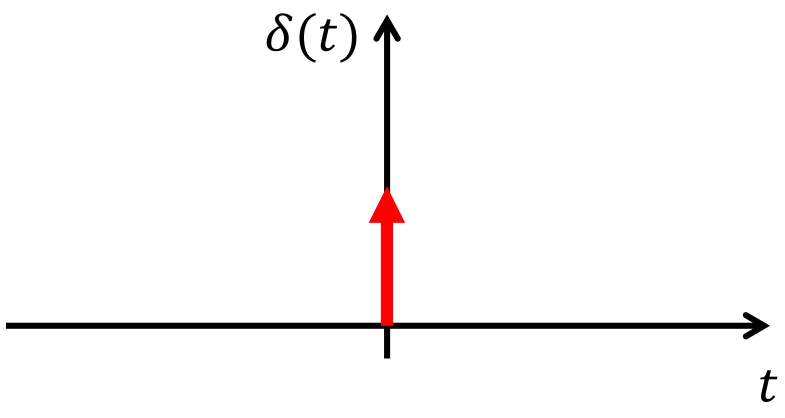
Table of Fourier Transform Pairs

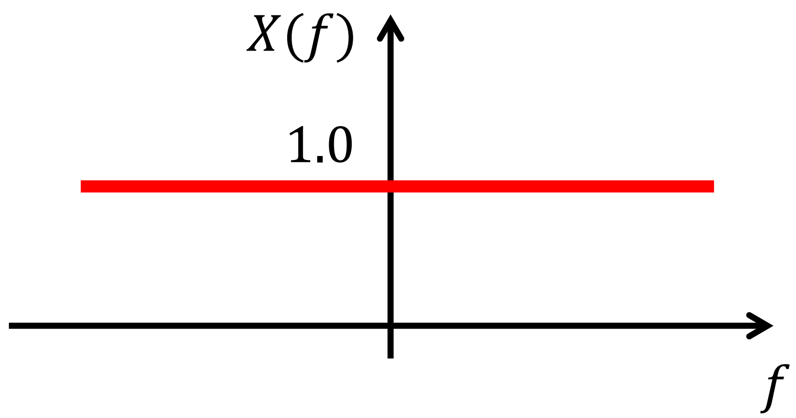
	Time function	Fourier transform	
No.	$x(t)$	$X(f)$	$X(\omega)$
1	$\delta(t)$	1	1
2	1	$\delta(f)$	$2\pi\delta(\omega)$
3	A	$A\delta(f)$	$2\pi A\delta(\omega)$
4	$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$	$\pi\delta(\omega)+\frac{1}{j\omega}$
5	$\delta(t-t_0)$	$e^{-j2\pi f t_0}$	$e^{-j\omega t_0}$
6	$e^{j2\pi f_0 t}$ or $e^{j\omega_0 t}$	$\delta(f-f_0)$	$2\pi\delta(\omega-\omega_0)$
7	$\cos(2\pi f_0 t)$ or $\cos(\omega_0 t)$	$\frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
8	$\sin(2\pi f_0 t)$ or $\sin(\omega_0 t)$	$\frac{1}{2j}[\delta(f-f_0)-\delta(f+f_0)]$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
9	$e^{-\alpha\|t\|}$	$\frac{2\alpha}{\alpha^2+4\pi^2 f^2}$	$\frac{2\alpha}{\alpha^2+\omega^2}$
10	$\frac{1}{\alpha^2+t^2}$	$\frac{\pi}{\alpha}e^{-\alpha 2\pi\|f\|}$	$\frac{\pi}{\alpha}e^{-\alpha\|\omega\|}$
11	$x(t)=e^{-\alpha t}u(t)$	$\frac{1}{\alpha+j2\pi f}$	$\frac{1}{\alpha+j\omega}$
12	$x(t)=A \quad \|t\|<T$ $=0 \quad \|t\|>T$	$2AT\frac{\sin(2\pi fT)}{2\pi fT}$	$2AT\frac{\sin(\omega T)}{\omega T}$
13	$2Af_0\frac{\sin(2\pi f_0 t)}{2\pi f_0 t}$ or $A\frac{\sin(\omega_0 t)}{\pi t}$	$X(f)=A \quad \|f\|<f_0$ $=0 \quad \|f\|>f_0$	$X(\omega)=A \quad \|\omega\|<\omega_0$ $=0 \quad \|\omega\|>\omega_0$
14	$\sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$ or $\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$	$\sum_{n=-\infty}^{\infty} c_n\delta(f-nf_0)$	$2\pi\sum_{n=-\infty}^{\infty} c_n\delta(\omega-n\omega_0)$
15	$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$	$\frac{2}{j\omega}$
16	$\frac{1}{t}$	$-j\pi\,\mathrm{sgn}(f)$	$-j\pi\,\mathrm{sgn}(\omega)$

Example: Dirac Delta Function

$$\delta(t) = \infty \ \text{ for } t = 0$$
$$\quad = 0 \quad \text{otherwise}$$

$$X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-i2\pi f t} dt = e^{-i2\pi f \cdot 0} = 1$$



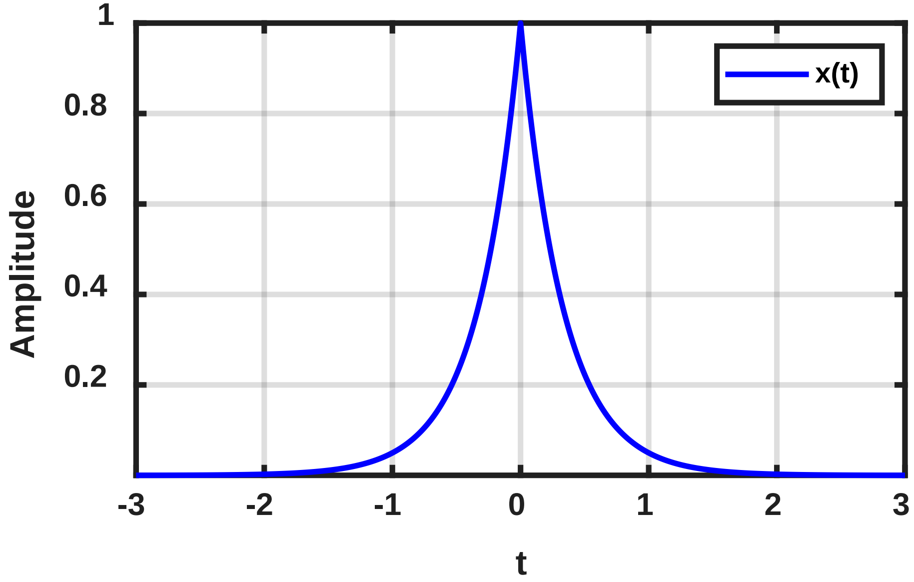


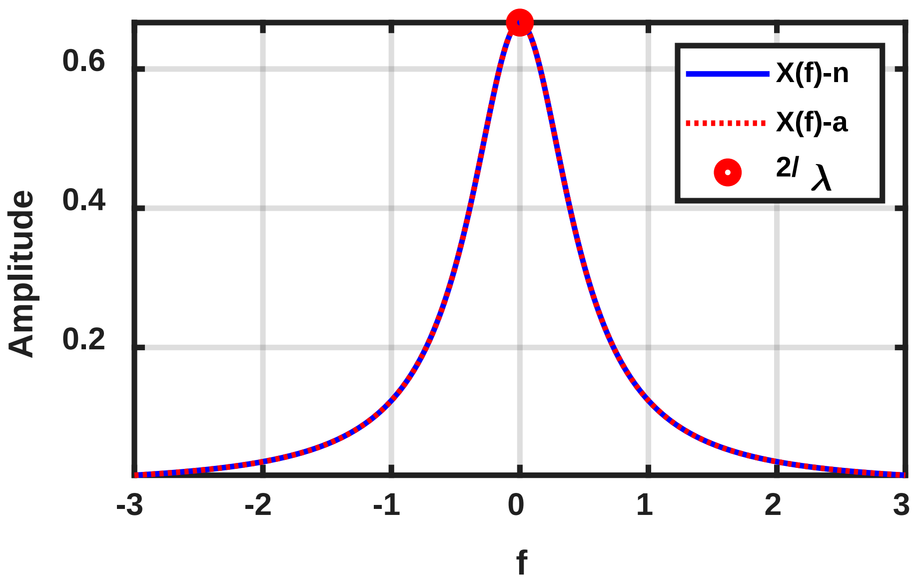
Example: Exponentially Decaying Symmetric Function

$$x(t) = e^{-\lambda|t|}, \qquad \lambda > 0$$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt = \int_{-\infty}^{\infty} e^{-\lambda|t|}e^{-i2\pi ft}dt$$

$$= \int_{-\infty}^{0} e^{\lambda t}e^{-i2\pi ft}dt + \int_{0}^{\infty} e^{-\lambda t}e^{-i2\pi ft}dt = \frac{2\lambda}{\lambda^2 + 4\pi^2 f^2}$$



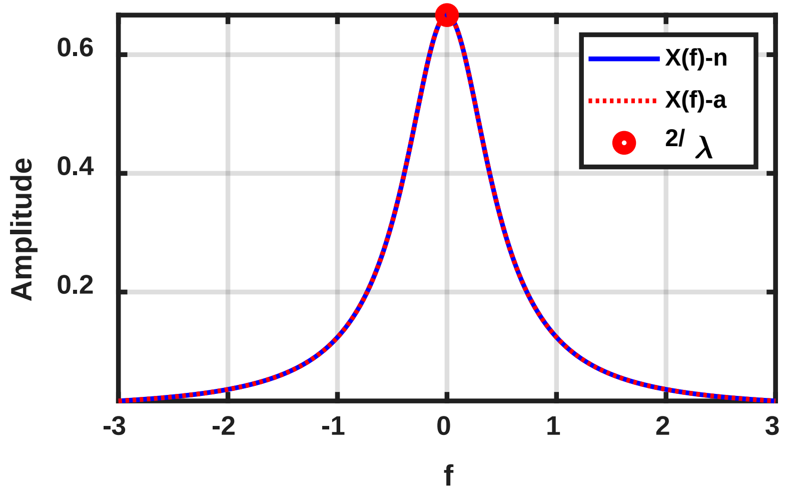


Example: Exponentially Decaying Symmetric Function (Continue)

$$x(t) = e^{-\lambda|t|}, \qquad \lambda > 0 \qquad\qquad X(f) = \frac{2\lambda}{\lambda^2 + 4\pi^2 f^2}$$

$$X(0) = \frac{2\lambda}{\lambda^2} = \frac{2}{\lambda}$$

$$X\left(\frac{\lambda}{2\pi}\right) = \frac{2\lambda}{\lambda^2 + \lambda^2} = \frac{1}{\lambda} = \frac{X(0)}{2}$$



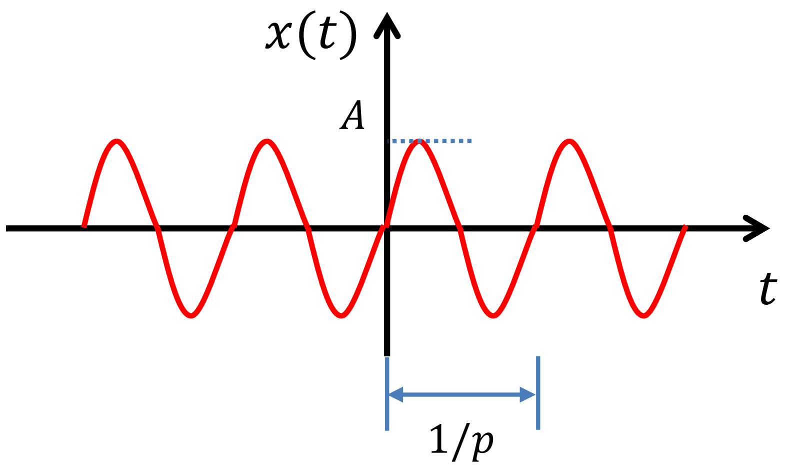
if λ is large then $X(f)$ is narrow in the time domain, but wide in the frequency domain and vice versa. This is an example of the so-called inverse spreading property of the Fourier transform, i.e. the wider in one domain, then the narrower in the other.

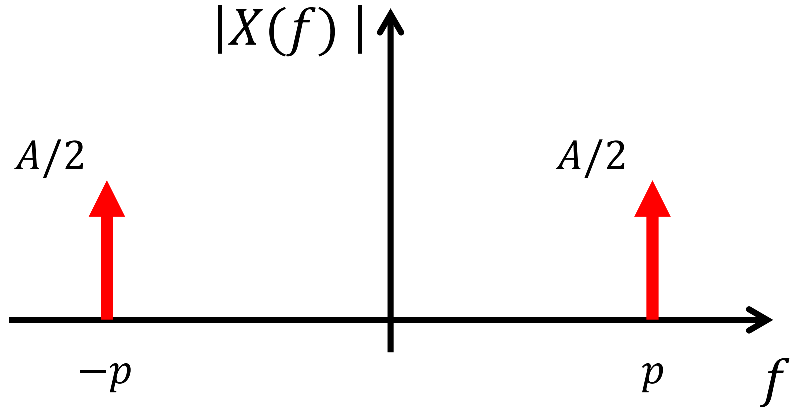
Example: Sine Function

$$x(t) = A\sin 2\pi pt$$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt = \int_{-\infty}^{\infty} A\sin 2\pi pt\, e^{-i2\pi ft}dt = \int_{-\infty}^{\infty} \frac{A}{2i}(e^{i2\pi pt} - e^{-i2\pi pt})e^{-i2\pi ft}dt$$

$$= \frac{A}{2i}\int_{-\infty}^{\infty} (e^{-i2\pi(f-p)t} - e^{--i2\pi(f+p)t})\, dt = \frac{A}{2i}[\delta(f-p) - \delta(f+p)]$$



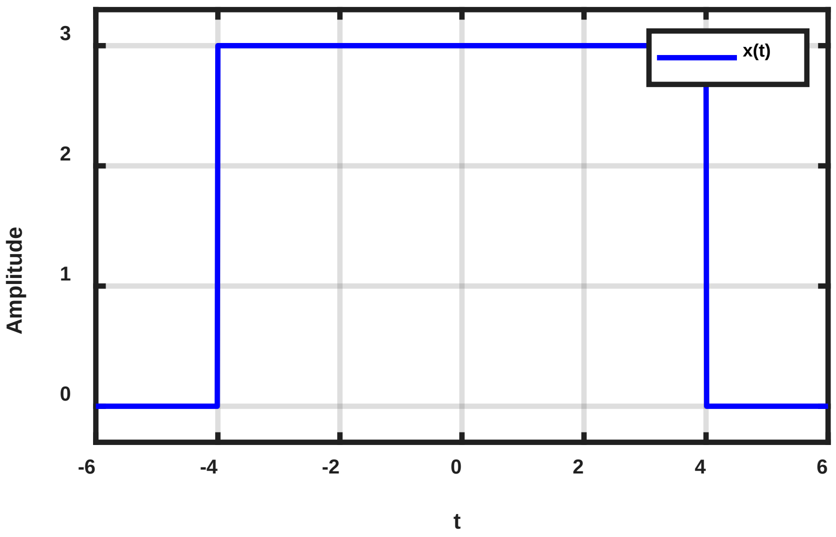


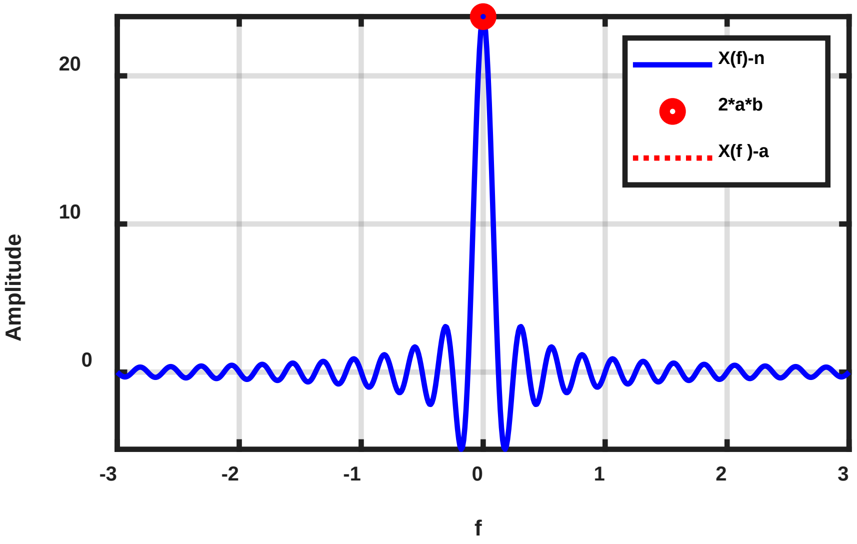
Example: Rectangular Function

$$x(t) = a \text{ for } |t| < \text{b}$$
$$= 0 \text{ for } |t| > \text{b}$$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-i2\pi ft}dt = \int_{-b}^{b} ae^{-i2\pi ft}dt = \frac{2ab\sin 2\pi fb}{2\pi fb}$$

See Sinc function





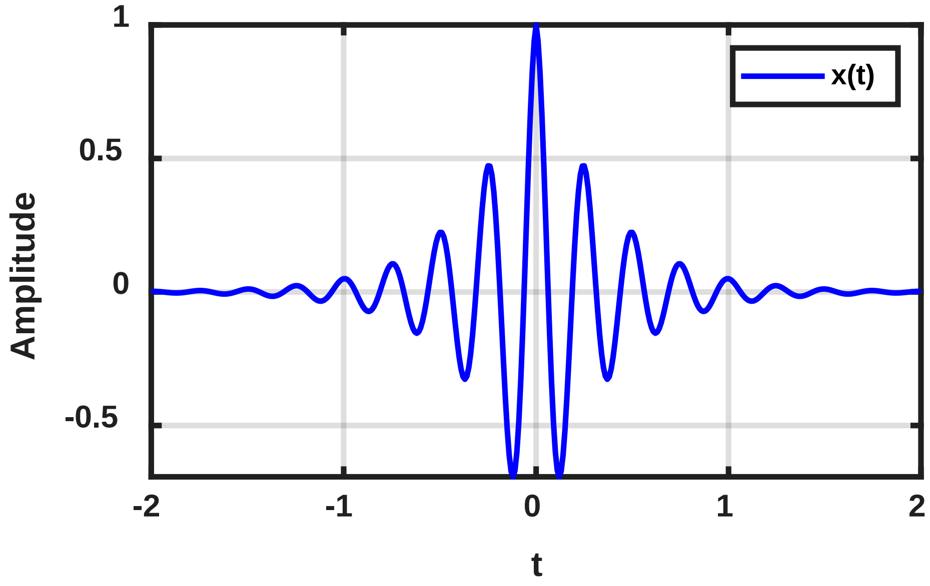
Example: Damped Symmetrically Oscillating Function

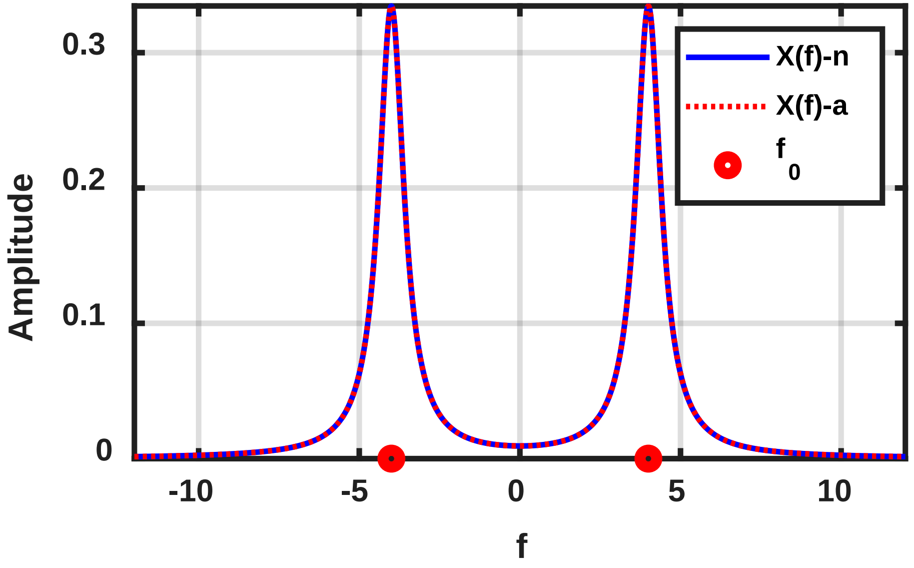
$$x(t) = e^{-a|t|} cos2\pi f_0 t$$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} dt = \int_{-\infty}^{\infty} e^{-a|t|} cos2\pi f_0 t \, e^{-i2\pi f t} dt = \int_{-\infty}^{\infty} e^{-a|t|} \frac{1}{2} (e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}) e^{-i2\pi f t} dt$$

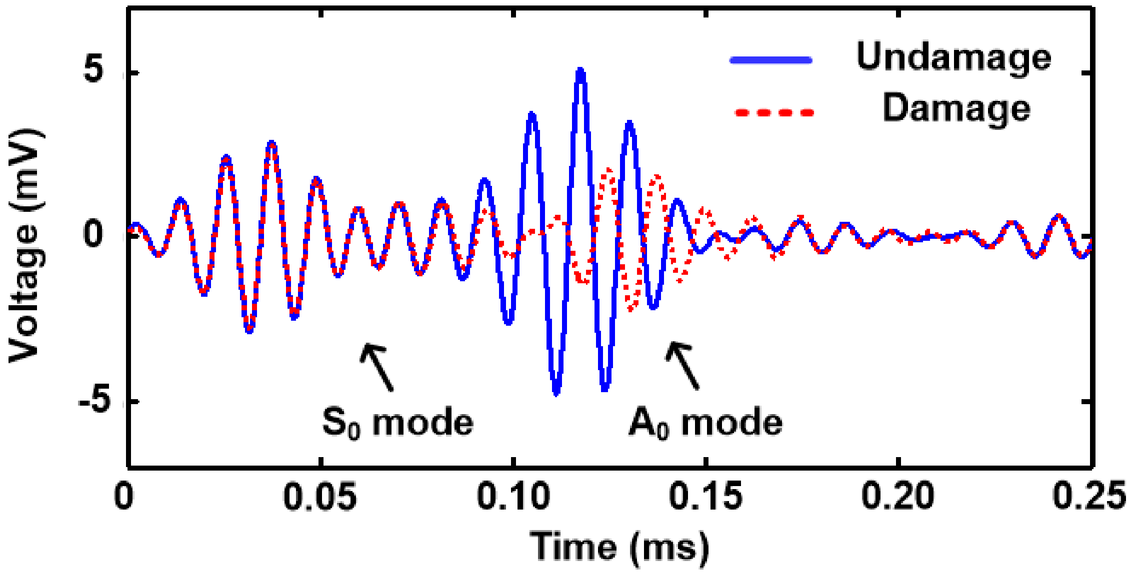
$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-a|t|} (e^{-i2\pi (f - f_0) t} + e^{-i2\pi (f + f_0) t}) \, dt = \frac{a}{a^2 + [2\pi (f - f_0)]^2} + \frac{a}{a^2 + [2\pi (f + f_0)]^2}$$

convolution

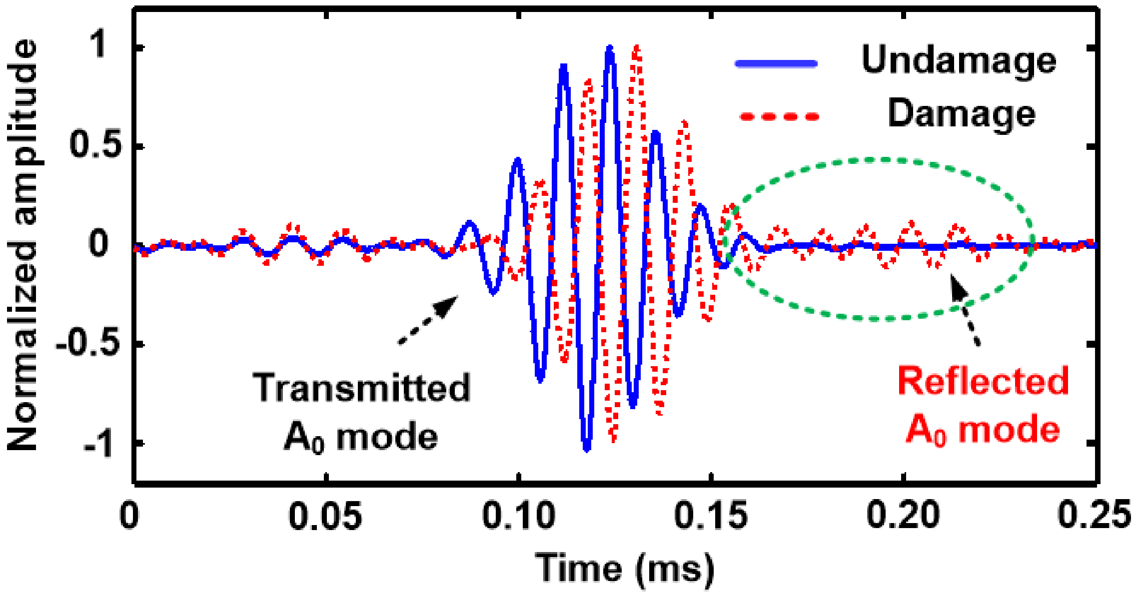




Example: Ultrasonic-based Damage Detection



Pitch-catch Lamb waves obtained from undamaged and damaged conditions

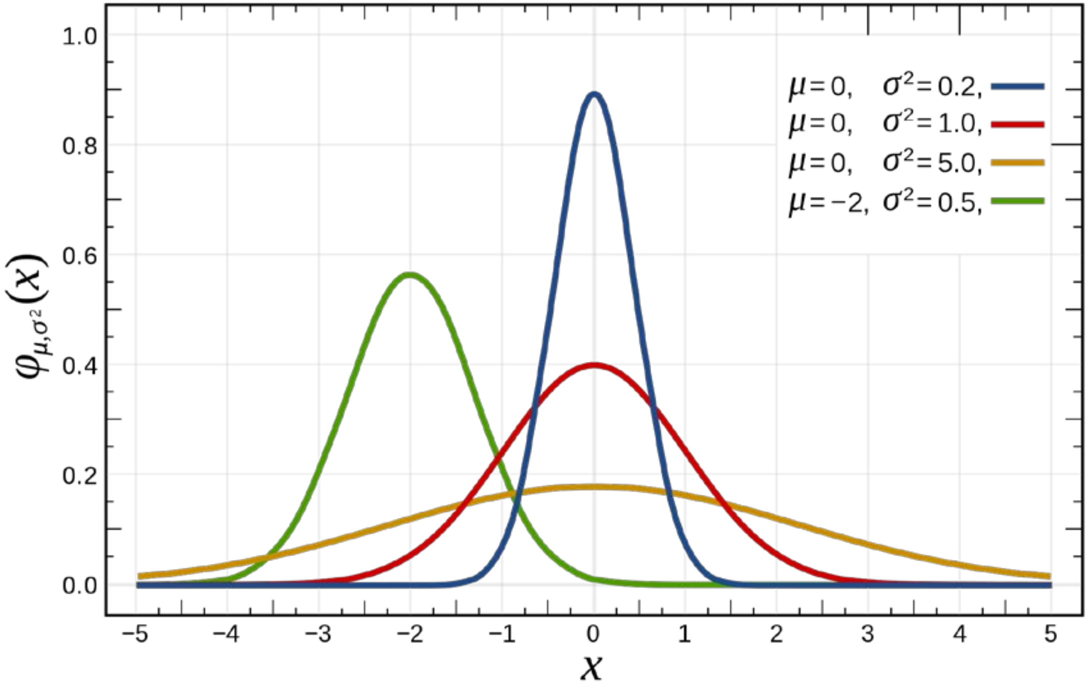


Extracted A_0 mode obtained from undamaged and damaged conditions

https://www.sciencedirect.com/science/article/pii/S0165212511000102

https://www.sciencedirect.com/science/article/pii/S0886779809001266

Gaussian Function



$$f(x) = ae^{-\frac{(x-b)^2}{2c^2}}$$

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-u}{\sigma}\right)^2}$$

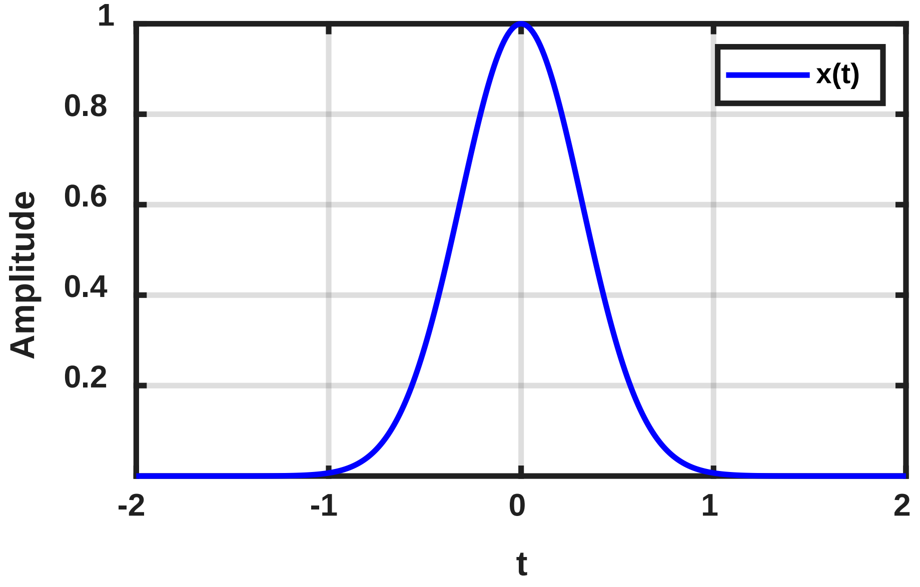
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} ae^{-\frac{(x-b)^2}{2c^2}} dx = \sqrt{2\pi}a|c|$$

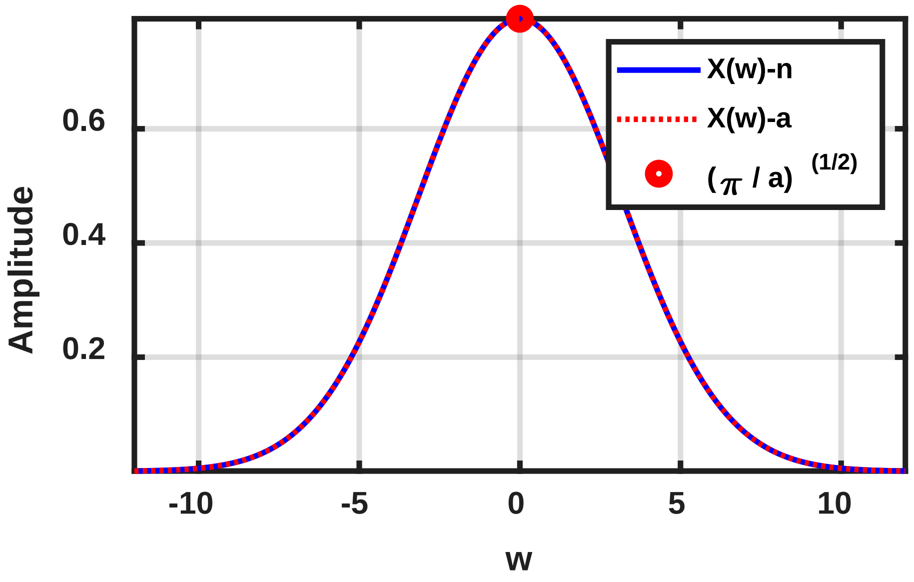
Example: Gaussian Function

$$x(t) = \boxed{e^{-at^2}}$$

$$X(w) = \int_{-\infty}^{\infty} x(t)e^{-iwt}dt = \int_{-\infty}^{\infty} e^{-at^2}e^{-iwt}dt = \int_{-\infty}^{\infty} e^{-a(t^2+\frac{iwt}{a})}dt = e^{-\frac{w^2}{4a}}\int_{-\infty}^{\infty} e^{-a(t^2+\frac{iwt}{a}-\frac{w^2}{4a})}dt$$

$$= e^{-\frac{w^2}{4a}}\int_{-\infty}^{\infty} e^{-a\left(t+i\left(\frac{w}{2a}\right)\right)^2}dt = e^{-\frac{w^2}{4a}}\int_{-\infty}^{\infty} e^{-ay^2}dy = \boxed{\sqrt{\pi/a}\cdot e^{-\frac{w^2}{4a}}}$$





Example: Fourier Transform of a Periodic Function

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p}$$

$$X(f) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p}\, e^{-i2\pi ft} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} c_n e^{i2\pi\left(f-\frac{n}{T_p}\right)t} dt = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} c_n \delta(f - \frac{n}{T_p}) dt$$

Fourier transform of a periodic function is a series of delta functions scaled by c_n, and located at multiples of the fundamental frequency, $1/T_p$.

Properties of Fourier Transforms

1. Time scaling

$$F\big(x(at)\big) = \frac{1}{|a|} X(\frac{f}{a})$$

Inverse spreading relationship

2. Time reversal

$$F\big(x(-t)\big) = X(-f)$$

3. Differentiation

$$F\big(\dot{x}(t)\big) = i2\pi f X(f)$$

4. Time shifting

$$F\big(x(t - t_0)\big) = e^{-i2\pi f t_0} X(f)$$

Only phase shift !

Sine wave

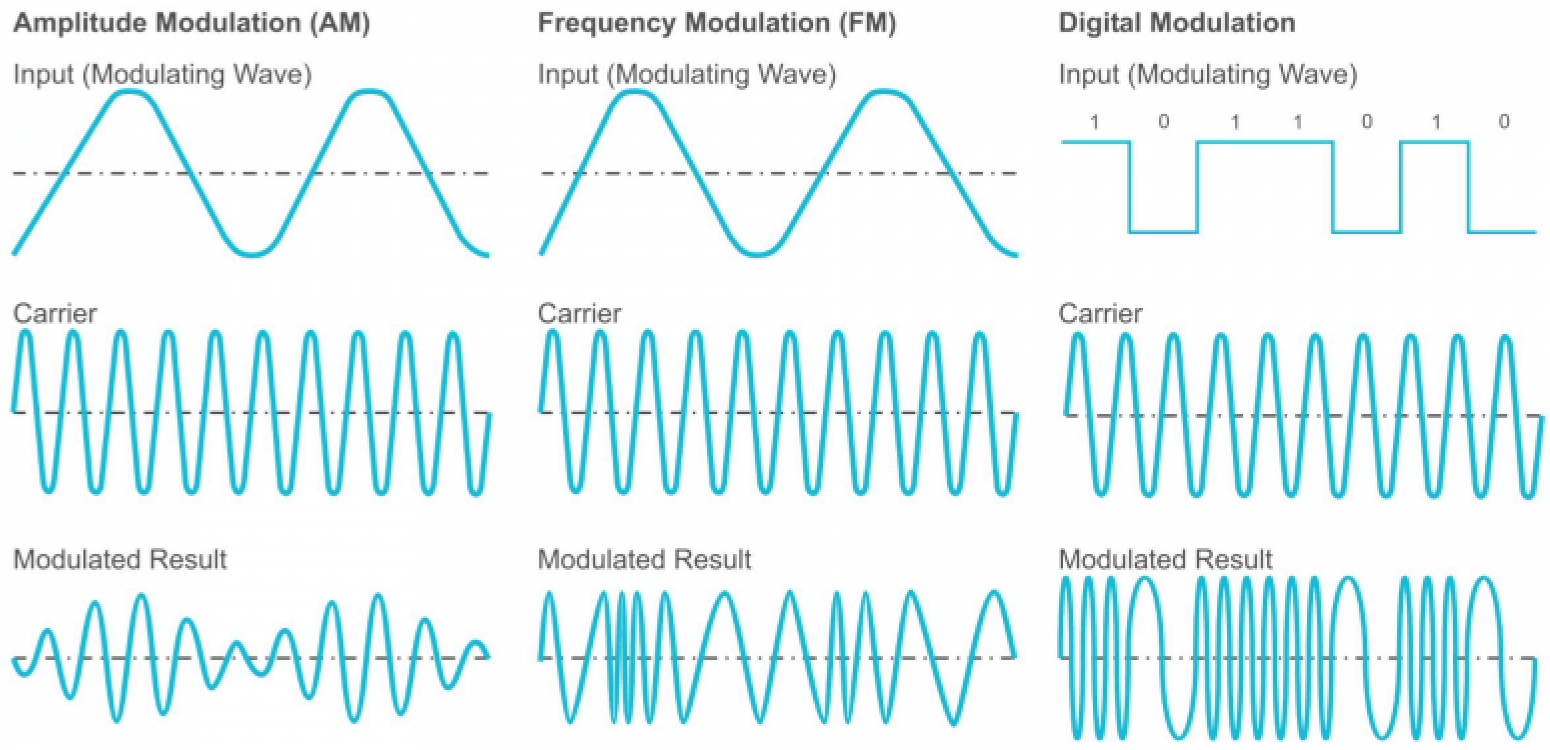
5. Modulation

$$F\big(x(t)e^{i2\pi f_0 t}\big) = X(f - f_0)$$

$$F(x(t)\cos(2\pi f_0 t)\,) = \frac{1}{2}[X(f - f_0)\ + X(f + f_0)\,]$$

wifi

(Off-Topic) How Dose Modulation Work?



https://www.taitradioacademy.com/topic/how-does-modulation-work-1-1/

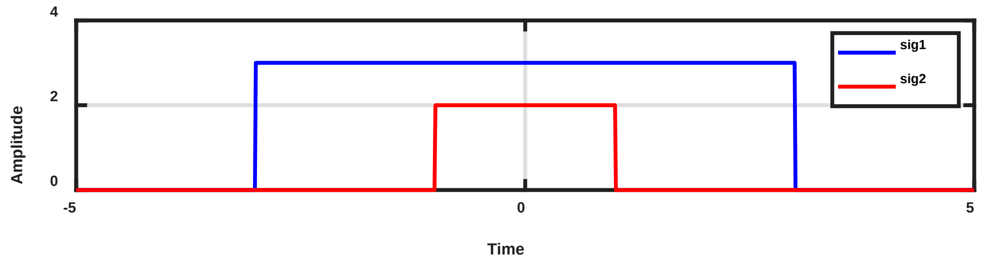
Properties of Fourier Transforms (Convolution)

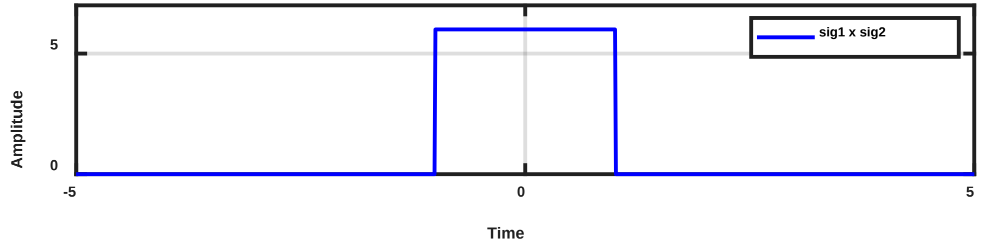
$$F(h(t) * x(t)) = H(f)X(f)$$

Where the convolution of the two functions h(t) and x(t) is defined as

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

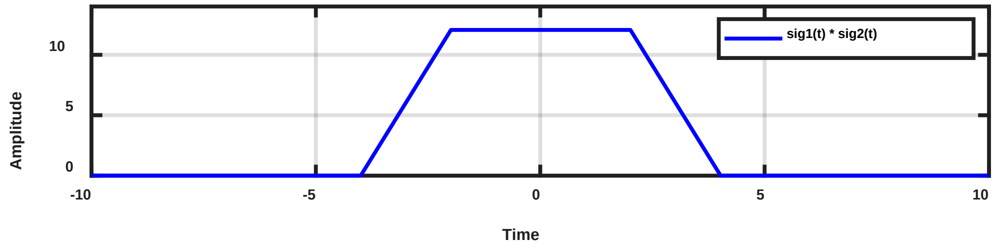
Example: Convolution 1

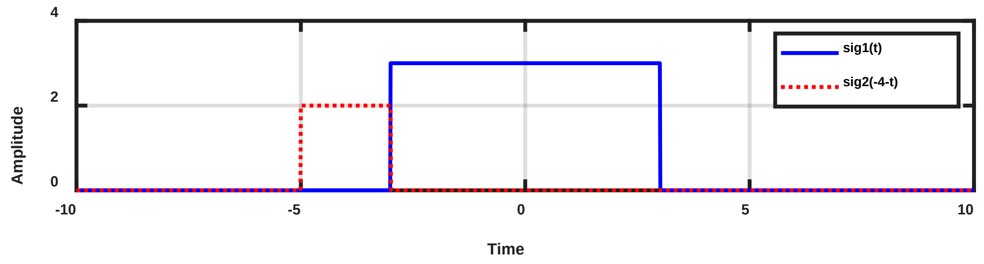


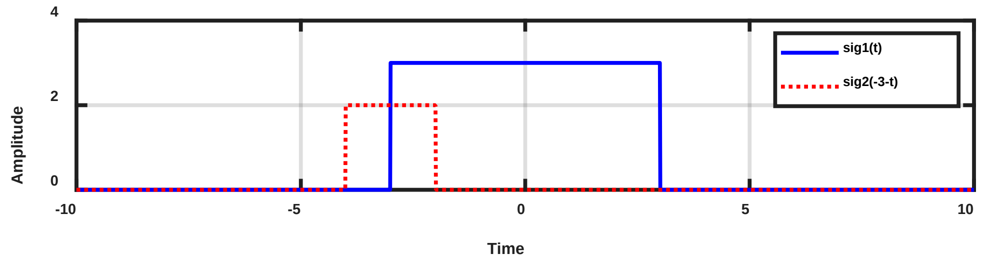


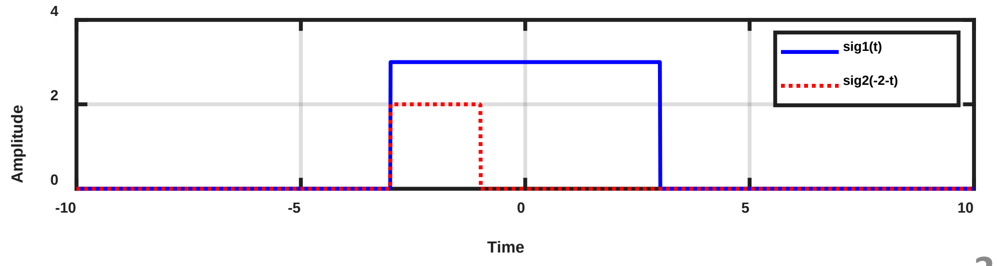
$$h(t)x(t)$$

$$h(t) * x(t)) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$



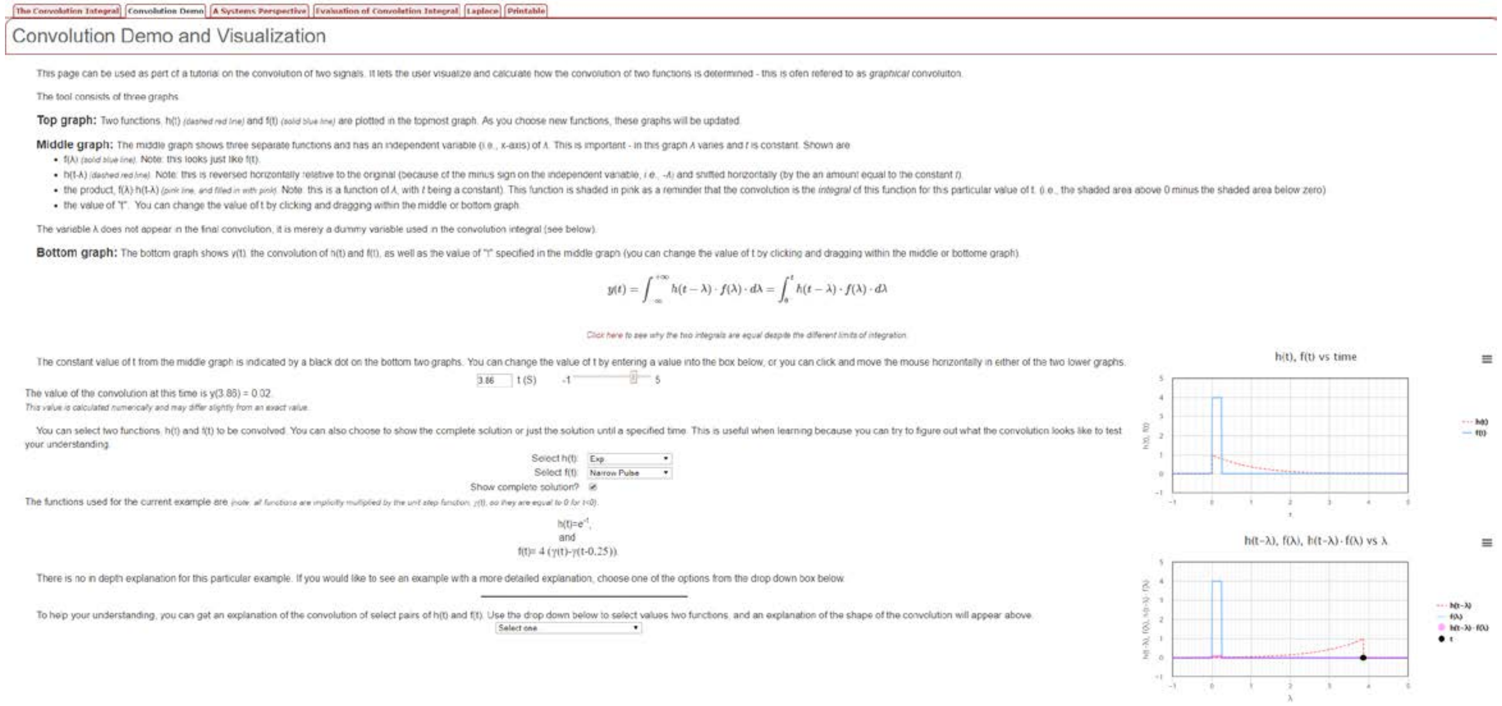






Convolution (Visualization)

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$



The Convolution Integral | Convolution Demo | A Systems Perspective | Evaluation of Convolution Integral | Laplace | Printable

Convolution Demo and Visualization

This page can be used as part of a tutorial on the convolution of two signals. It lets the user visualize and calculate how the convolution of two functions is determined - this is ofen refered to as *graphical* convoluiton.

The tool consists of three graphs.

Top graph: Two functions, h(t) *(dashed red line)* and f(t) *(solid blue line)* are plotted in the topmost graph. As you choose new functions, these graphs will be updated.

Middle graph: The middle graph shows three separate functions and has an independent variable (i.e., x-axis) of λ. This is important - in this graph λ varies and *t* is constant. Shown are

- f(λ) *(solid blue line)*. Note: this looks just like f(t).
- h(t-λ) *(dashed red line)*. Note: this is reversed horizontally relative to the original (because of the minus sign on the independent variable, *i.e.*, -λ) and shifted horizontally (by the an amount equal to the constant *t*).
- the product, f(λ)·h(t-λ) *(pink line, and filled in with pink)*. Note: this is a function of λ, with *t* being a constant). This function is shaded in pink as a reminder that the convolution is the *integral* of this function for this particular value of t. (i.e., the shaded area above 0 minus the shaded area below zero)
- the value of "t". You can change the value of t by clicking and dragging within the middle or bottom graph.

The variable λ does not appear in the final convolution, it is merely a dummy variable used in the convolution integral (see below).

Bottom graph: The bottom graph shows y(t), the convolution of h(t) and f(t), as well as the value of "t" specified in the middle graph (you can change the value of t by clicking and dragging within the middle or bottome graph).

$$y(t) = \int_{-\infty}^{+\infty} h(t-\lambda)\cdot f(\lambda)\cdot d\lambda = \int_{0}^{t} h(t-\lambda)\cdot f(\lambda)\cdot d\lambda$$

Click here to see why the two integrals are equal despite the different limits of integration.

The constant value of t from the middle graph is indicated by a black dot on the bottom two graphs. You can change the value of t by entering a value into the box below, or you can click and move the mouse horizontally in either of the two lower graphs.

3.86 t (S) -1 5

The value of the convolution at this time is y(3.86) = 0.02

This value is calculated numerically and may differ slightly from an exact value.

You can select two functions, h(t) and f(t) to be convolved. You can also choose to show the complete solution or just the solution until a specified time. This is useful when learning because you can try to figure out what the convolution looks like to test your understanding.

Select h(t): Exp

Select f(t): Narrow Pulse

Show complete solution? ☑

The functions used for the current example are *(note: all functions are implicitly multiplied by the unit step function, γ(t), so they are equal to 0 for t<0)*.

$h(t)=e^{-t}$,

and

f(t)= 4 (γ(t)-γ(t-0.25))

There is no in depth explanation for this particular example. If you would like to see an example with a more detailed explanation, choose one of the options from the drop down box below.

To help your understanding, you can get an explanation of the convolution of select pairs of h(t) and f(t). Use the drop down below to select values two functions, and an explanation of the shape of the convolution will appear above.

Select one

http://lpsa.swarthmore.edu/Convolution/CI.html

Proof of Convolution Relationship

$$F\big(h(t) * x(t)\big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\exp^{-i2\pi f t}\,d\tau\,dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau)x(v)\exp^{-i2\pi f(\tau+v)}\,d\tau\,dv$$

$$= \int_{-\infty}^{\infty} h(\tau)\exp^{-i2\pi f(\tau)}\,d\tau \int_{-\infty}^{\infty} x(v)\exp^{-i2\pi f(v)}\,dv = H(f)X(f)$$

Properties of Fourier Transforms (Convolution) (Continue)

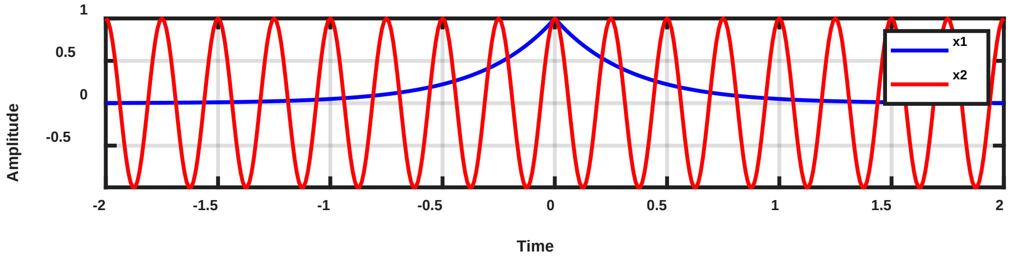
$$F(x(t)w(t)) = \int_{-\infty}^{\infty} X(g)W(f-g)dg = X(f) * W(f)$$

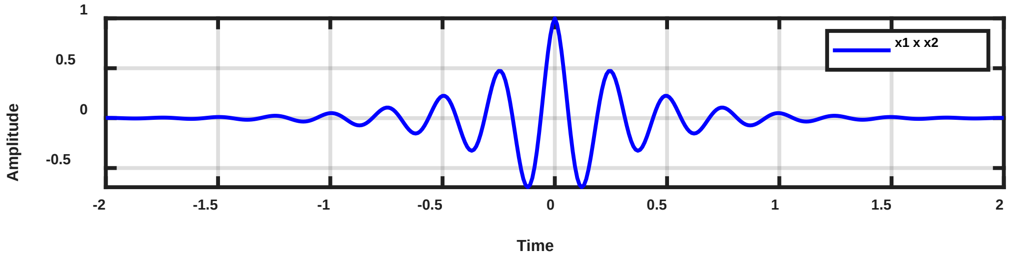
$$F(x(t)w(t)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(f_1)\exp^{i2\pi f_1 t}\, W(f_2)\exp^{i2\pi f_2 t}\exp^{-i2\pi f t}\, df_1\, df_2\, dt$$

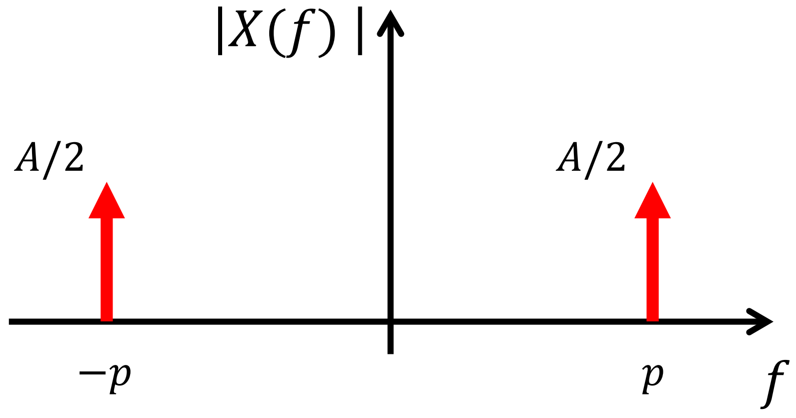
$$= \int_{-\infty}^{\infty} X(f_1)\int_{-\infty}^{\infty} W(f_2)\int_{-\infty}^{\infty} \exp^{-i2\pi(f-f_1-f_2)t}\, dt\, df_2\, df_1$$

$$= \int_{-\infty}^{\infty} X(f_1)\int_{-\infty}^{\infty} W(f_2)\delta(f-f_1-f_2)df_2 df_1 = \int_{-\infty}^{\infty} X(f_1)W(f-f_1)df_1 = X(f) * W(f)$$

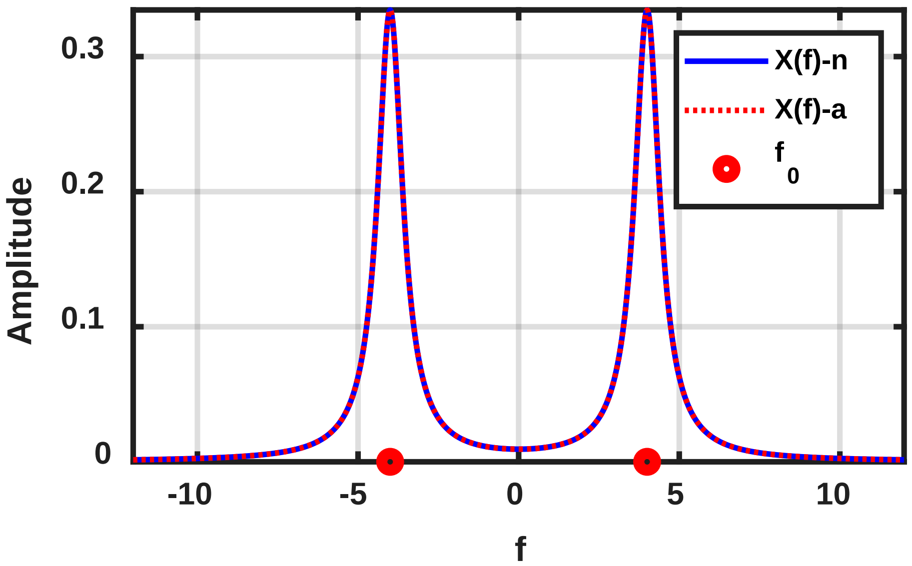
Example: Convolution 2



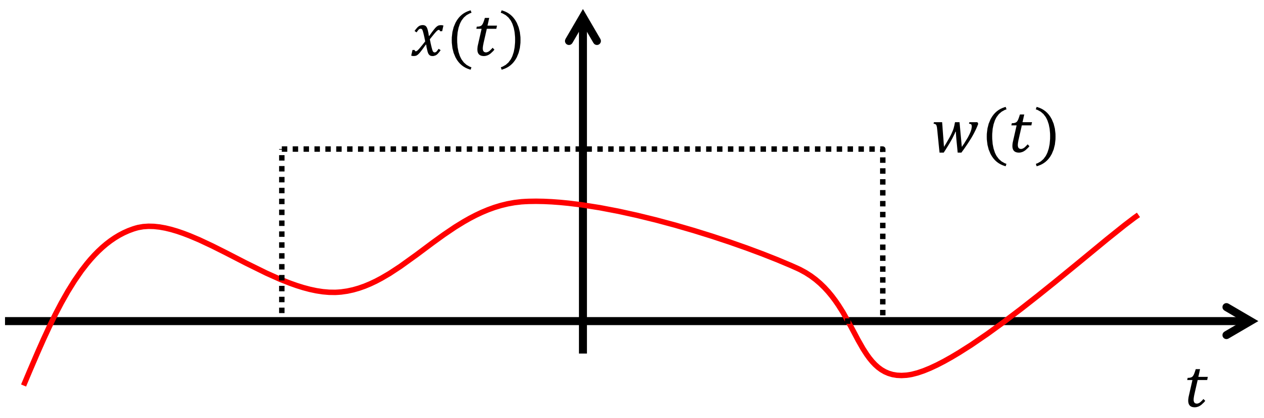




$$x(t) = e^{-a|t|} cos 2\pi f_0 t$$



Effect of Data Truncation (Windowing)



$w(t) = 1$ for $|t|$ <T/2
$= 0$ otherwise

$x(t)$ is known (or recorded) only for $-\frac{T}{2} < t < \frac{T}{2}$, denoted as $x_T(t)$

$$x_T(t) = x(t)w(t) = X(f) * W(f)$$

Fourier transform of the product of two time signals is the convolution of their Fourier transforms.

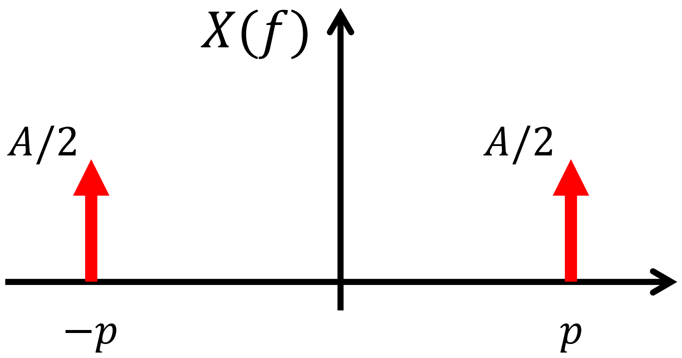
Effect of Data Truncation (Windowing) (Continue)

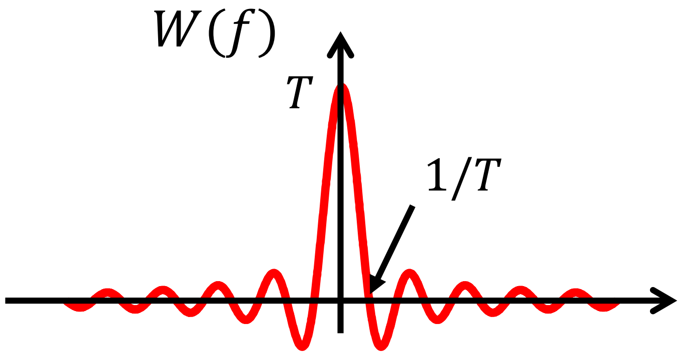
$$x(t) = A\cos 2\pi pt$$

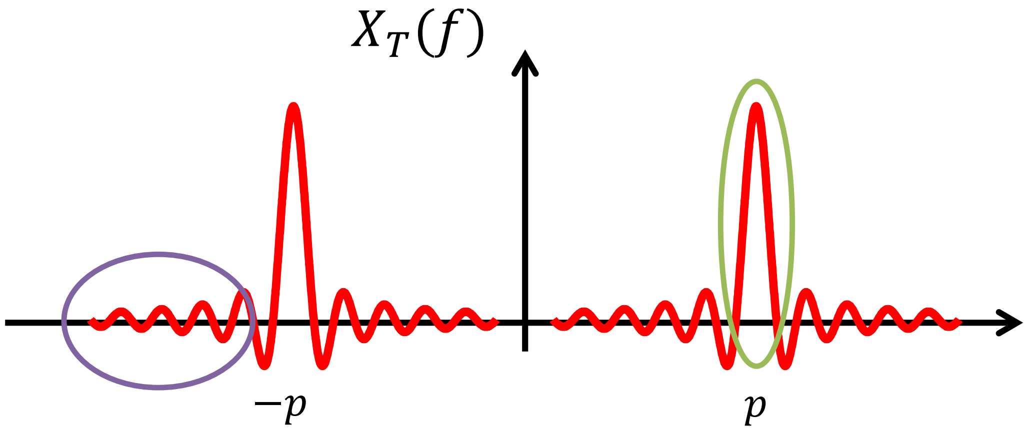
$$X(f) = \frac{A}{2}[\delta(f-p) - \delta(f+p)]$$

$$w(t) = 1$$

$$W(f) = \frac{T\sin \pi fT}{\pi fT}$$



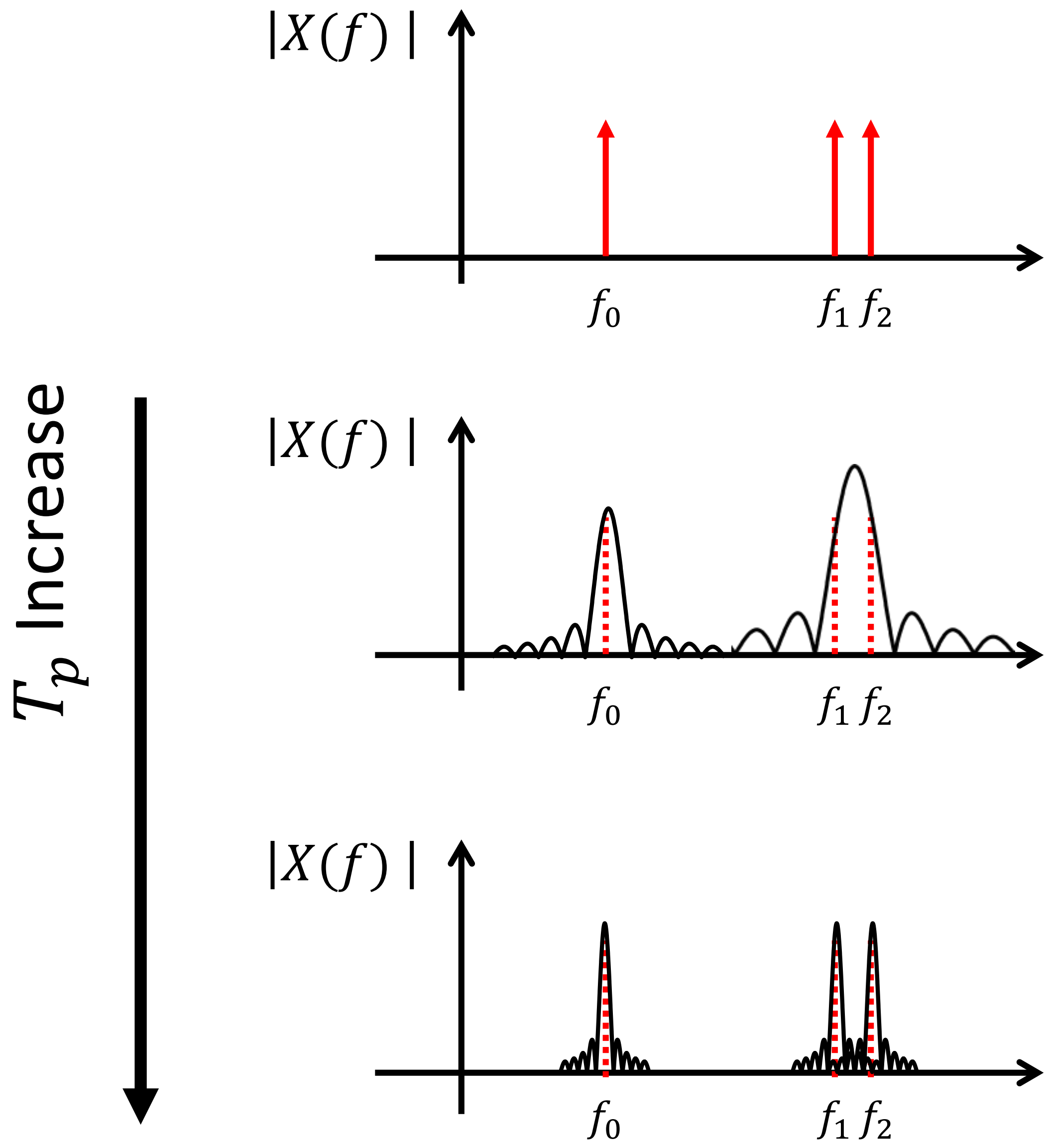




Smearing: Distortion due to the main lobe

Leakage: Distortion due to the side lobe

Effect of Data Truncation (Windowing) (Continue)



$x(t)$ is the sum of three sine (or cosine) waves

Considerable smearing due to the spectral window

Three components are resolved but with considerable leakage at other frequencies

Window Function

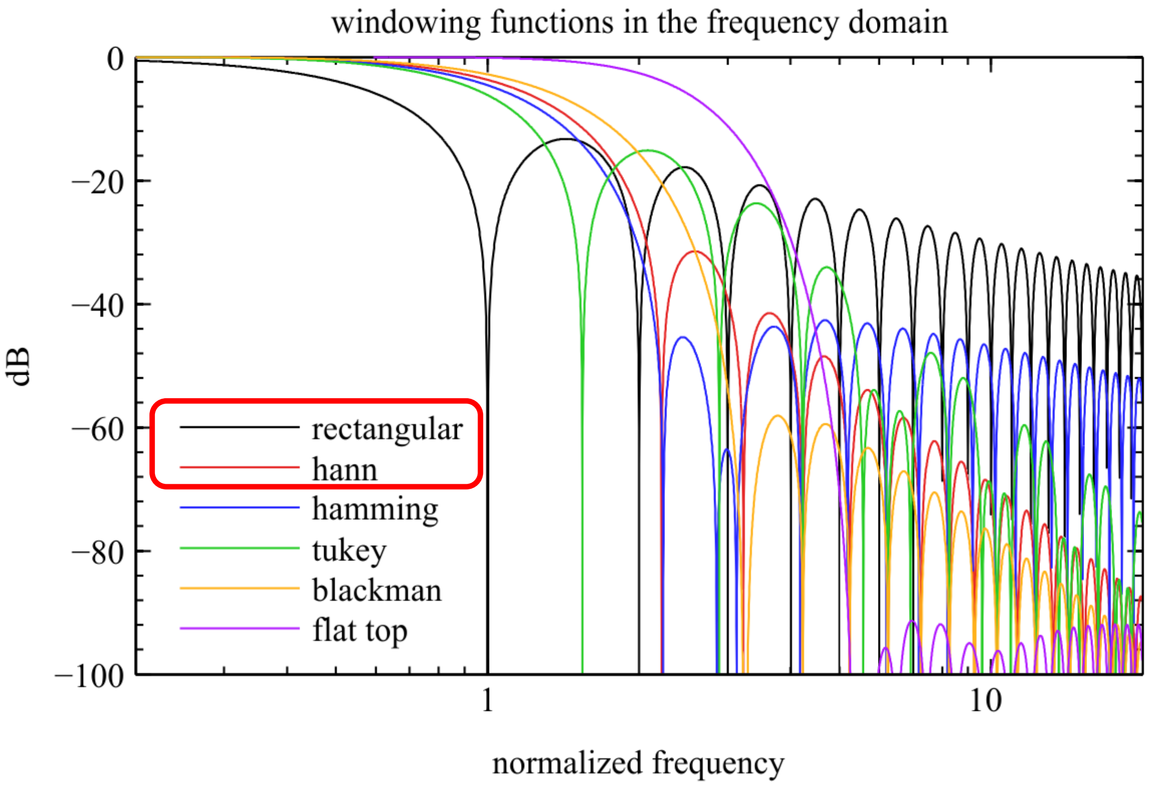
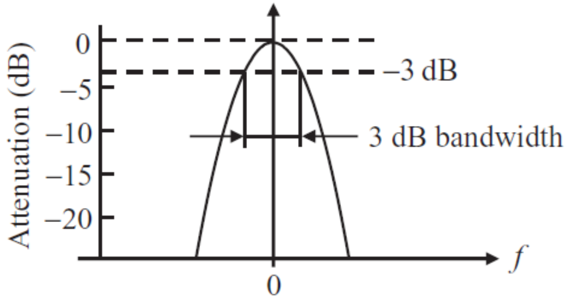


Table 4.2 Properties of some window functions

Window (length T)	Highest side lobe (dB)	Asymptotic roll-off (dB/octave)	3 dB bandwidth	Noise bandwidth	First zero crossing (freq.)
Rectangular	−13.3	6	$0.89\frac{1}{T}$	$1.00\frac{1}{T}$	$\frac{1}{T}$
Bartlett (triangle)	−26.5	12	$1.28\frac{1}{T}$	$1.33\frac{1}{T}$	$\frac{2}{T}$
Hann(ing) (Tukey or cosine squared)	−31.5	18	$1.44\frac{1}{T}$	$1.50\frac{1}{T}$	$\frac{2}{T}$
Hamming	−43	6	$1.30\frac{1}{T}$	$1.36\frac{1}{T}$	$\frac{2}{T}$
Parzen	−53	24	$1.82\frac{1}{T}$	$1.92\frac{1}{T}$	$\frac{4}{T}$



Signal Smoothing Example

https://www.mathworks.com/help/signal/examples/signal-smoothing.html

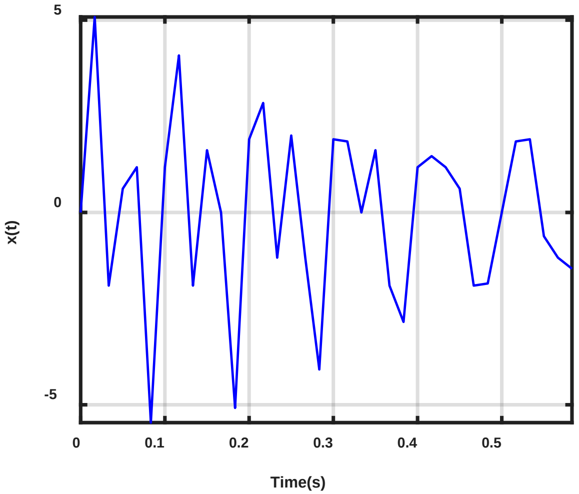
Hanning Window vs Rectangular Window

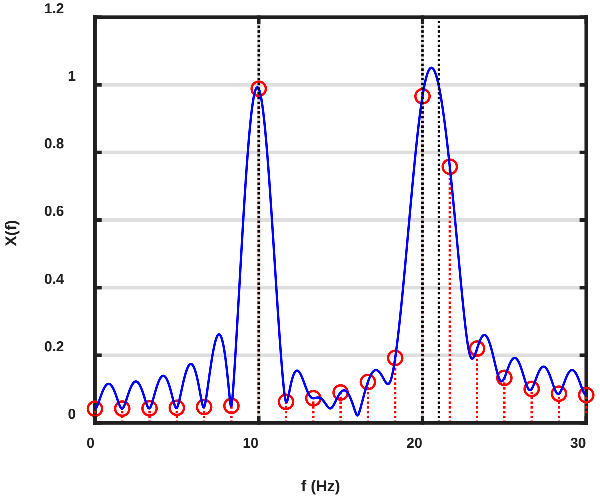
$$x(t) = A_1 \sin 2\pi f_1 t + A_2 sin2\pi f_2 t + A_3 sin2\pi f_3 t$$

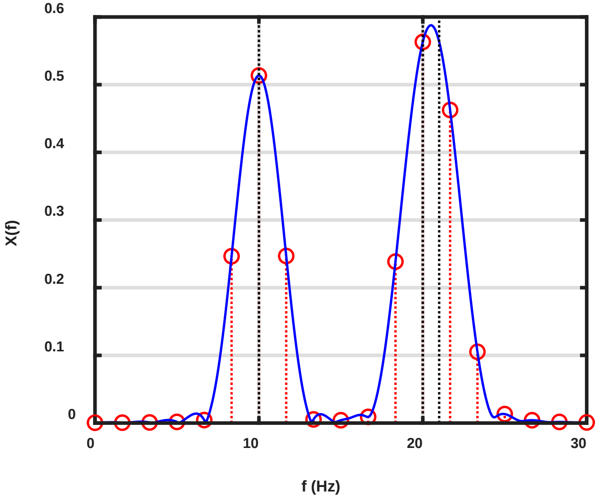
Amplitudes are $A_1 = A_2 = A_3 = 2$, which gives the magnitude '1' for each sinusoidal component in the frequency domain. The frequencies are chosen as $f_1 = 10$, $f_2 = 20$ and $f_3 = 21$.

Hanning Window vs Rectangular Window (Continue)

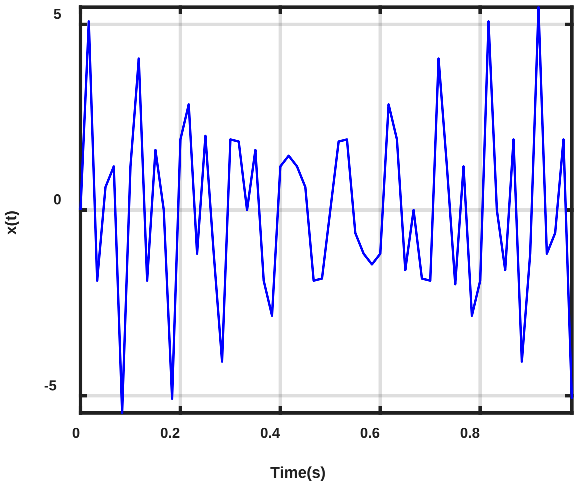
$T_p = 0.6$

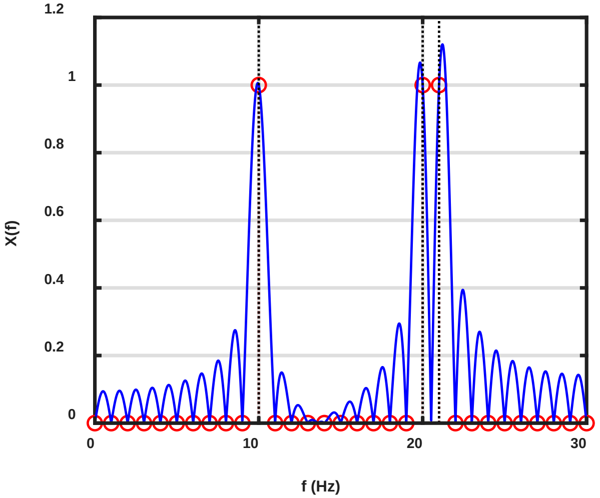


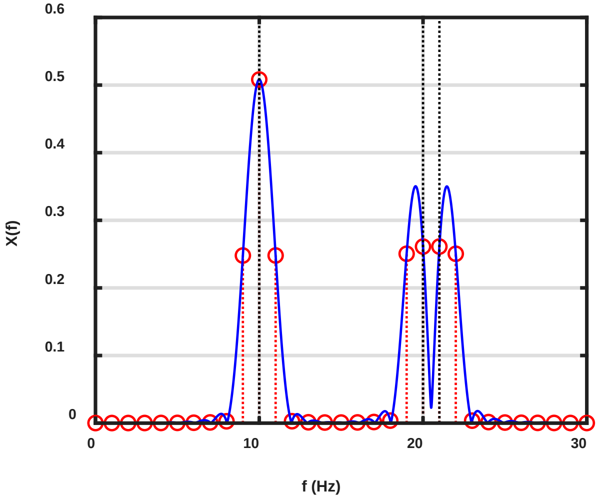




$T_p = 1$

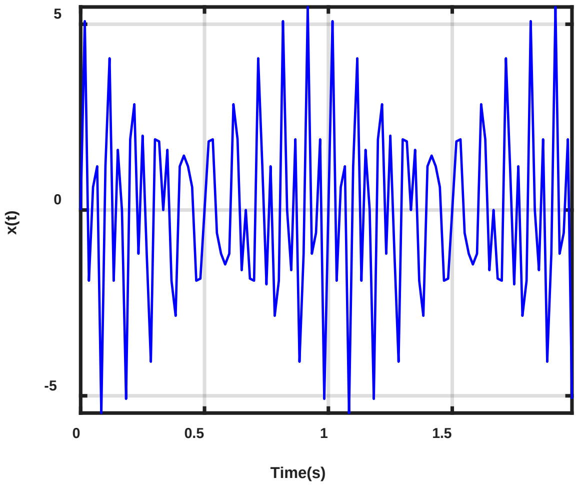


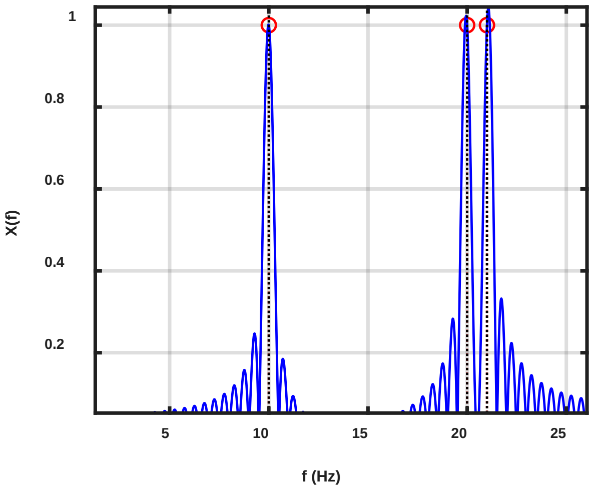


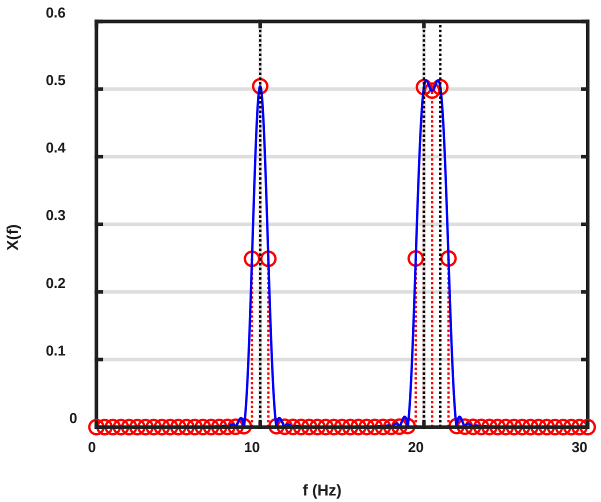


Hanning Window vs Rectangular Window (Continue)

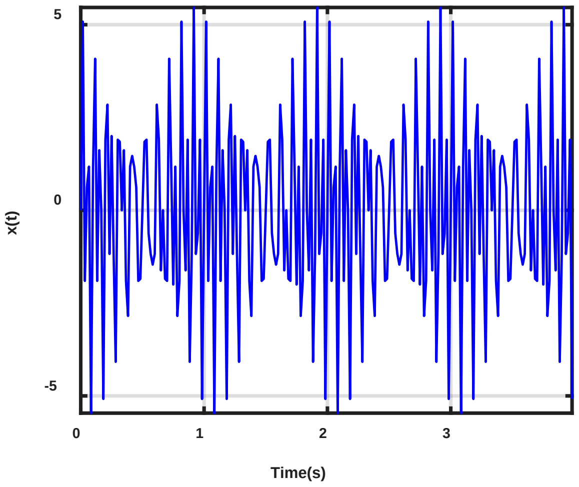
$T_p = 2$

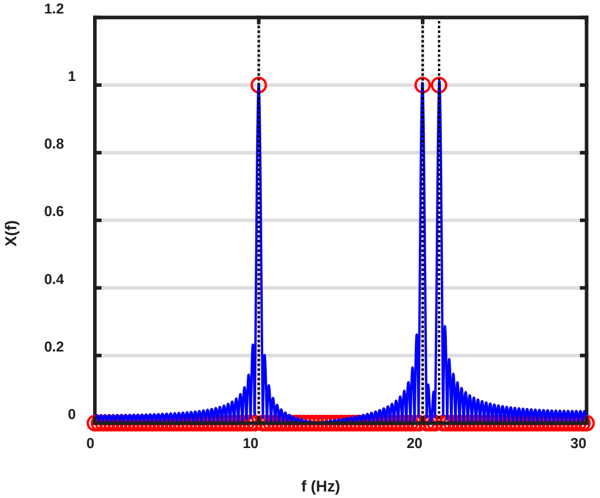


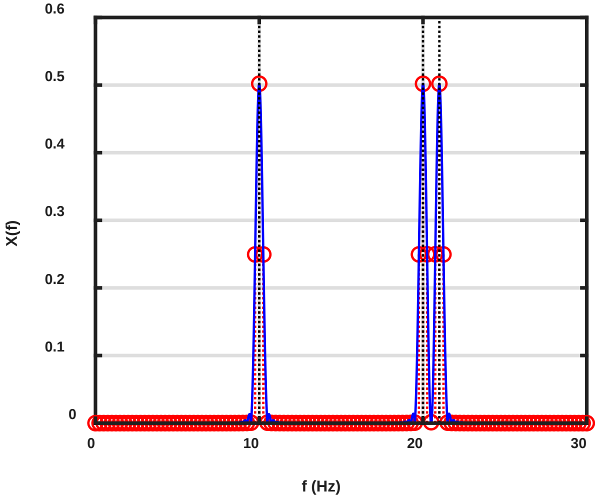




$T_p = 4$



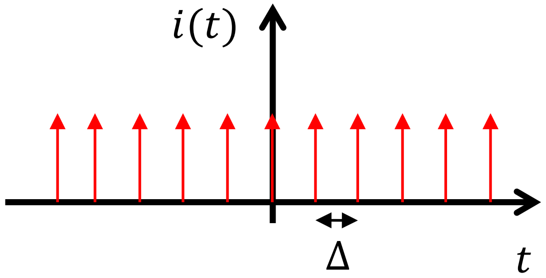




Impluse Train Modulation

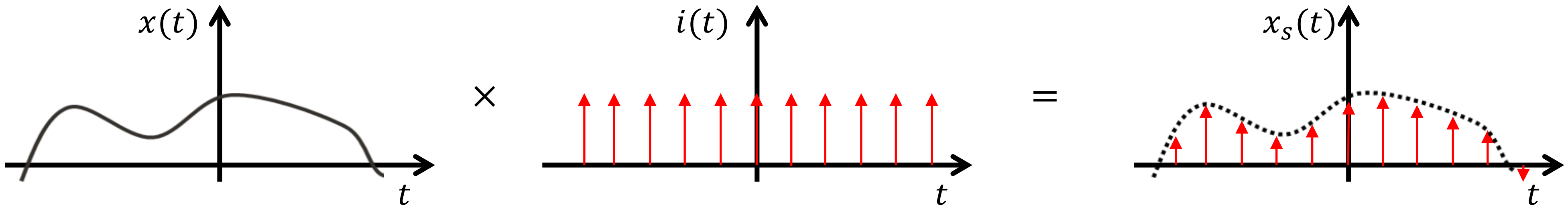
A ‘train’ of delta functions $i(t)$ is expressed as

$$i(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta)$$



The sampling procedure can be illustrated as

$$x_s(t) = x(t)i(t)$$



$$\boxed{X_s(f)} = \int_{-\infty}^{\infty} \left[x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\Delta) \right] e^{-i2\pi f t} dt = \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) e^{-i2\pi f t} \delta(t - n\Delta) dt \right] = \sum_{n=-\infty}^{\infty} x(n\Delta) e^{-i2\pi f n\Delta}$$

Impluse Train Modulation (Continue)

$$X_s(f+r/\Delta) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi(f+r/\Delta)n\Delta} = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta - i2\pi rn} = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta} = X_s(f)$$

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta}$$

Multiplying both sides of equation by $e^{-i2\pi fr\Delta}$ and integrating w.r.t f

$$\int_{-1/2\Delta}^{1/2\Delta} X_s(f)e^{-i2\pi fr\Delta}df = \int_{-1/2\Delta}^{1/2\Delta}\left[\sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta}\right]e^{-i2\pi fr\Delta}df = \sum_{n=-\infty}^{\infty}\int_{-1/2\Delta}^{1/2\Delta}\left[x(n\Delta)e^{-i2\pi fn\Delta}\right]e^{-i2\pi fr\Delta}df$$

$$= \sum_{n=-\infty}^{\infty} x(n\Delta)\int_{-1/2\Delta}^{1/2\Delta}\left[e^{-i2\pi f(n-r)\Delta}\right]df = x(r\Delta)\frac{1}{\Delta}$$

Use L'Hôpital's rule

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi fn\Delta}$$

$$x(n\Delta) = \Delta\int_{-1/2\Delta}^{1/2\Delta} X_s(f)e^{i2\pi fn\Delta}df$$

Link Between Fourier Transform of a Discrete Sequence and Continuous Signal

Fourier coefficients

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nt/T_p} \qquad c_n = \frac{1}{T_p}\int_0^{T_p} x(t)e^{-i2\pi nt/T_p}dt$$

Impluse train

$$i(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta) \qquad c_n = \frac{1}{\Delta}\int_{-1/\Delta}^{1/\Delta} \sum_{n=-\infty}^{\infty} \delta(t - n\Delta)\, e^{-i2\pi nt/\Delta}dt = \frac{1}{\Delta}\int_{-1/\Delta}^{1/\Delta} e^{-i2\pi nt/\Delta}dt = \frac{1}{\Delta}$$

Fourier Transform of the impulse train

$$I(f) = F\big(i(t)\big) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{\Delta} e^{\frac{i2\pi nt}{\Delta}} \cdot e^{-i2\pi ft}dt = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{i2\pi nt}{\Delta}} \cdot e^{-i2\pi ft}\, dt = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i2\pi\left(f-\frac{n}{\Delta}\right)t}dt$$

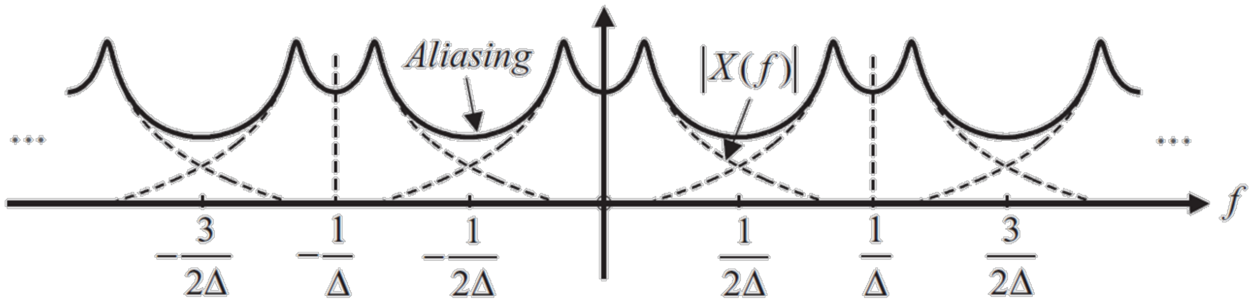
$$= \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{\Delta})$$

Link Between Fourier Transform of a Discrete Sequence and Continuous Signal (Continue)

Fourier Transform of a discrete sequence, $\boldsymbol{x_s(t)}$

$$X_s(f) = I(f) * X(f) = \int_{-\infty}^{\infty} I(g)X(f-g)dg = \int_{-\infty}^{\infty} \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} \delta\left(g-\frac{n}{\Delta}\right)X(f-g)dg = \frac{1}{\Delta}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} \delta\left(g-\frac{n}{\Delta}\right)X(f-g)dg$$

$$= \frac{1}{\Delta}\sum_{n=-\infty}^{\infty} X\left(f-\frac{n}{\Delta}\right) = \frac{1}{\Delta}\left(\dots + X\left(f-\frac{2}{\Delta}\right) + X\left(f-\frac{1}{\Delta}\right) + X(f) + X\left(f+\frac{1}{\Delta}\right) + \dots\right)$$



Discrete Fourier Transform

So far we have considered sequences that run over the range $-\infty < n < \infty$(n integer). For the special case where the sequence is of finite length (i.e. non-zero for a finite number of values) an alternative Fourier representation is possible called the **discrete Fourier transform (DFT)**.

It turns out that the DFT is a Fourier representation of a finite length sequence and is itself a sequence rather than a continuous function of frequency, and it corresponds to samples, **equally spaced in frequency**, of the Fourier transform of the signal. The DFT is fundamental to many digital signal processing algorithms.

$$X_s(f) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-i2\pi f n\Delta}$$

Continuous in frequency

Repeated every $1/\Delta$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-i\frac{2\pi}{N}nk}$$

DFT of a finite (sampled) sequence $x(n\Delta)$ $\quad f = \frac{k}{N\Delta}$

$X(k)$ is $X_s(f)$ evaluated at $f = \frac{k}{N\Delta}$ Hz (k integer)

$$x(n) = \frac{1}{N}\sum_{n=0}^{N-1} X(k)e^{i\frac{2\pi}{N}nk}$$

Inverse DFT

Fast Fourier Transform

A fast Fourier transform (FFT) is an algorithm that computes the discrete Fourier transform (DFT) of a sequence, or its inverse (IDFT). Fourier analysis converts a signal from its original domain (often time or space) to a representation in the frequency domain and vice versa. It manages to reduce the complexity of computing the DFT from $O(n^2)$, which arises if one simply applies the definition of DFT, to $O(n\ log\ n)$, where n is the data size.

FFT in Matlab

fft

Fast Fourier transform

Syntax

```
Y = fft(X)
Y = fft(X,n)
Y = fft(X,n,dim)
```

Y = fft(X) computes the discrete Fourier transform (DFT) of X using a fast Fourier transform (FFT) algorithm

Y = fft(X,n) returns the n-point DFT. If no value is specified, Y is the same size as X.

Zero-Padding

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-i\frac{2\pi}{N}nk}$$

Y = fft(X, n) returns the n-point DFT. If no value is specified, Y is the same size as X.

- If X is a vector and the length of X is less than n, then X is padded with trailing zeros to length n.
- If X is a vector and the length of X is greater than n, then X is truncated to length n.

$X(k)$ is $X_s(f)$ evaluated at $f = \frac{k}{N\Delta}$ Hz

frequency spacing $\frac{1}{N\Delta}$ Hz

Zero-Padding (Continue)

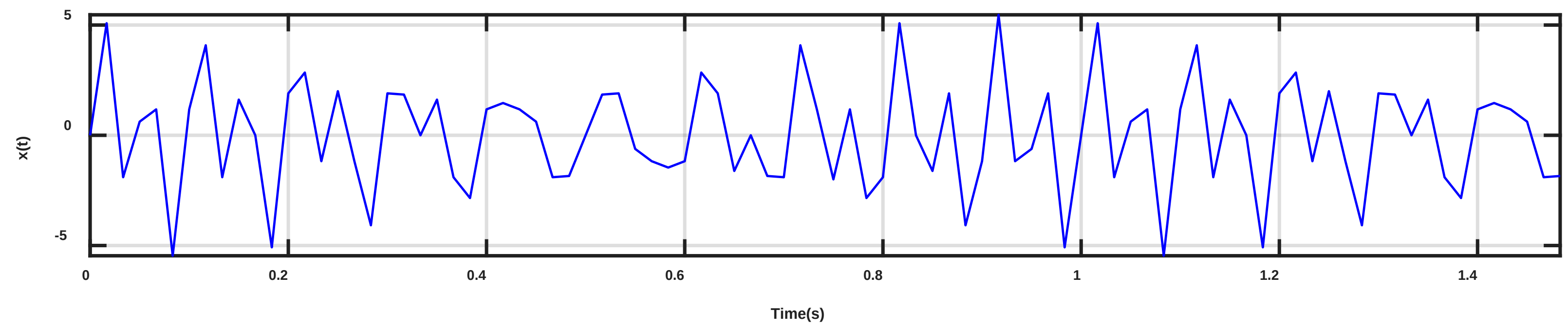
$$\hat{x}(t) = x(n) \quad \text{for } 0 \le n \le N-1$$
$$= 0 \quad \text{for } N \le n \le L-1$$

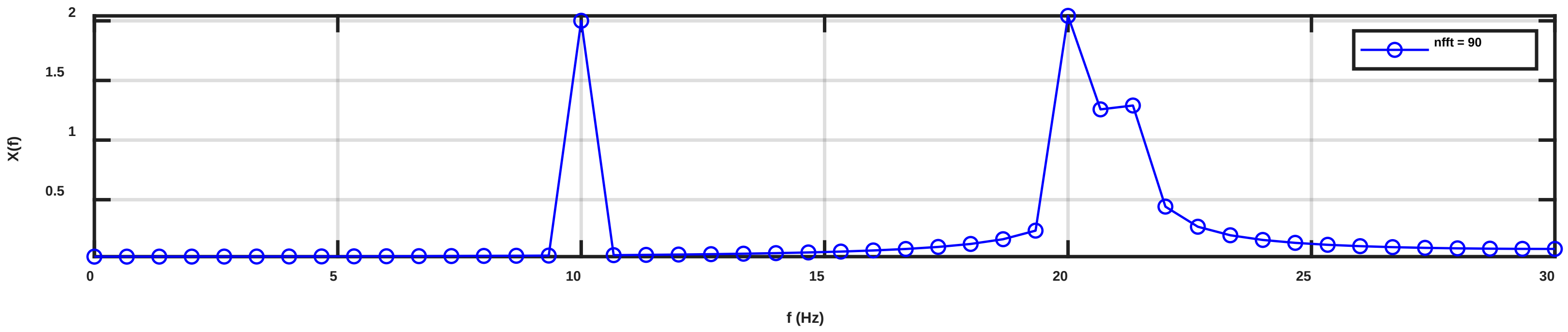
$$\hat{X}(k) = \sum_{n=0}^{L-1} \hat{x}(n) e^{-i\frac{2\pi}{L}nk} = \sum_{n=0}^{N-1} x(n) e^{-i\frac{2\pi}{L}nk}$$

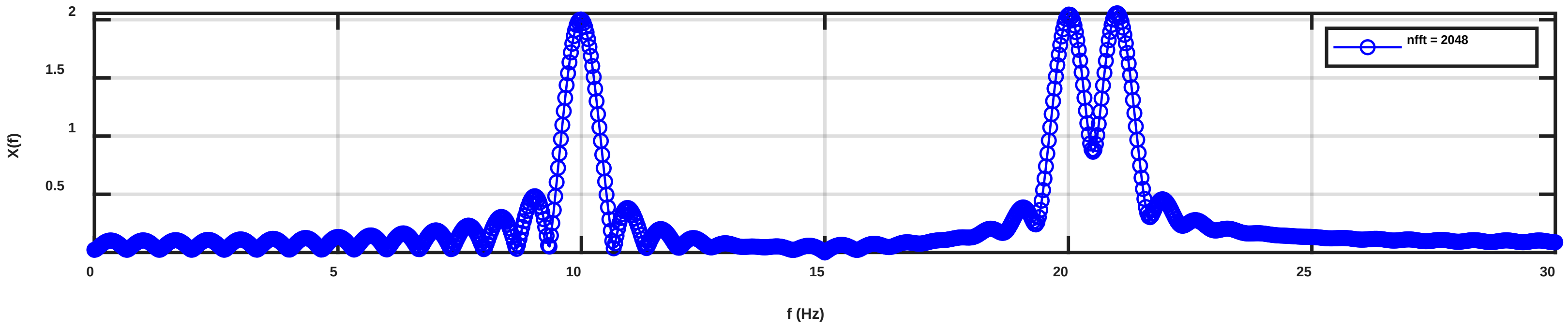
- "Finer" spacing in the frequency domain. However, the zero padding does not increase the "true" resolution.
- Vibration problems, this can be used to obtain the fine detail near resonances

See the slide of "effect of data truncation"

Example: Zero-Padding







Example: FFT

https://www.mathworks.com/help/matlab/ref/fft.html

FFTOfNoisySignalExample_course.mlx

Repeated every 1/ Δ

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i\frac{2\pi}{N}nk}$$

DFT of a finite (sampled) sequence $x(n\Delta)$

$$f = \frac{k}{N\Delta}$$

$X(k)$ is $X_s(f)$ evaluated at $f = \frac{k}{N\Delta}$ Hz (k integer)

$$\hat{X}(k) = \sum_{n=0}^{L-1} \hat{x}(n) e^{-i\frac{2\pi}{L}nk} = \sum_{n=0}^{N-1} x(n) e^{-i\frac{2\pi}{L}nk}$$

$$\begin{aligned} \hat{x}(t) &= x(n) \quad \text{for } 0 \le n \le N-1 \\ &= 0 \quad\quad\; \text{for } N \le n \le L-1 \end{aligned}$$

$$f_1 = \frac{k}{L\Delta}$$

Understanding of Zero-Padding

Zero-padding data for a longer FFT is equivalent to interpolation by a (periodic) Sinc kernel. Interpolation by a (periodic) Sinc kernel can reconstruct points between samples of a signal that was strictly bandlimited (to below the Nyquist frequency) prior to sampling.