Numerical Technique

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AE121: Computational Method

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Numeric Technique

- Algorithms that are used to obtain numerical solutions of a mathematical problem
- It is useful when no analytical solution exists or analytical solution is difficult to obtain.

$$f(x) = 2x + 3$$

$$f'(3)$$

$$f(x) = \log(x) * 2x + e^{3x} * x^{2/3}$$

$$f'(3)$$

Theory: Differentiation

The derivative is a measure of the rate at which a function is changing

The derivative of $f(x)$ at $x = a$ can be defined using limits as:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

With a small change in notation, we can write that:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

For one point

Instead of finding a derivate at every point in a function, we can find the derivative for a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \textit{ can be written as } f'(x) = \frac{d}{dx} f(x) = \frac{dy}{dx}$$

For a function

Theory: Tangent Lines

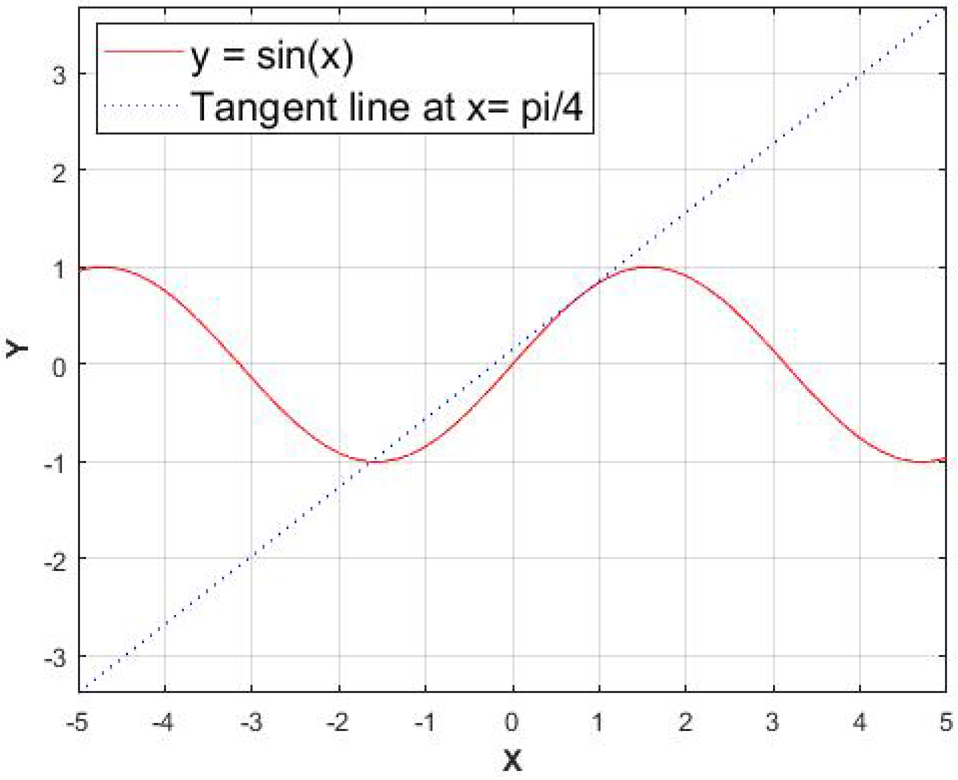
We can find the equation of a tangent line to $f(x)$ at point (a, $f(x)$).
Recall the general equation of a line $y = mx + b$
where m is the slope and b is the y-intercept. The slope of the tangent line found by $m = f'(a)$

Given two points on a line, the slope can be found using:

$$Slope = \frac{y_2 - y_1}{x_2 - x_1}$$

From there, the equation of a line can be found using:

$$y - y_1 = m(x - x_1)$$



Example: Differentiation 1 – Symbolic Way

Example: Find the derivative of:

$$f(x) = \sqrt{x^2 + 1}$$

- $f(x) = \sqrt{u} \rightarrow u = x^2 + 1$
- $f'(x) = \frac{1}{2} u^{\frac{1}{2}} * (2x)$
- $f'(x) = \frac{2x}{2\sqrt{u}} = \frac{x}{\sqrt{x^2+1}}$

```
% symbolic method

syms x
fx = sqrt(x^2 + 1);
fxp = diff(fx);

fxp
fxp_2 = subs(fxp, x, 2)
double(fxp_2)
```

fxp =

$$\frac{x}{\sqrt{x^2+1}}$$

fxp_2 =

$$\frac{2\ \sqrt{5}}{5}$$

ans = 0.8944

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Example: Differentiation 1 – Numerical Way

Example: Find the derivative of:

$$f(x) = \sqrt{x^2 + 1}$$

- $f(x) = \sqrt{u} \rightarrow u = x^2 + 1$
- $f'(x) = \frac{1}{2} u^{\frac{1}{2}} * (2x)$
- $f'(x) = \frac{2x}{2\sqrt{u}} = \frac{x}{\sqrt{x^2+1}}$

```
% numerical method

h = 0.000000001;
x = 2;
(diff_ex1(x+h) - diff_ex1(x))/h
```

```
ans = 0.8944
```

```
function fx = diff_ex1(x)

fx = sqrt(x^2 + 1);
end
```

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Example: Differentiation 2

Derivatives of Exponents and Logarithms

- $\frac{d}{dx}\ln(x) = \frac{1}{x}$
- $\frac{d}{dx}log_a(x) = \frac{1}{x}log_a e$
- $\frac{d}{dx}e^x = e^x$
- $\frac{d}{dx}a^x = a^x \ln a$

$$y = x^2 e^{-2x}$$

$$\frac{dy}{dx} = 2xe^{-2x} + x^2(-2)e^{-2x}$$

$$= 2xe^{-2x} - 2x^2 e^{-2x}$$

$$= 2xe^{-2x}(1 - x)$$

Example: Differentiation 2

```
% symbolic method

syms x
fx = x^2 * exp(-2*x);
fxp = diff(fx);

fxp
fxp_3 = subs(fxp, x, 3)
double(fxp_3)
```

fxp = $2\,x\,e^{-2\,x} - 2\,x^2\,e^{-2\,x}$

fxp_3 = $-12\,e^{-6}$

ans = -0.0297

```
% numerical method

h = 0.000000001;
x = 3;
(diff_ex2(x+h) - diff_ex2(x))/h
```

ans = -0.0297

```
function fx = diff_ex2(x)

fx = x^2 * exp(-2*x);

end
```

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Theory: L'Hôpital's Rule

L'Hôpital's Rule is a way of solving certain limits of an **indeterminant** form.

In order to apply L'Hôpitals rule the limit must be:

1. A ratio, like $\frac{f(x)}{g(x)}$
2. Indeterminate

So, L'Hôpitals rule can be applied to limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

If $f(x)$ and $g(x)$ are different functions and if $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ is indeterminate, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example: Find the following using L’Hôpital’s rule

1. $\lim\limits_{x \to 1} \frac{\ln(x)}{x-1} \Rightarrow \frac{0}{0}$

Using L’Hôpital’s rule $= \lim\limits_{x \to 1} \frac{\frac{1}{x}}{1} = 1$

2. $\lim\limits_{x \to \infty} \frac{e^x}{x^2} \Rightarrow \frac{0}{0}$

Using L’Hôpitlals rule $= \lim\limits_{x \to \infty} \frac{e^x}{2x} = \lim\limits_{x \to \infty} \frac{e^x}{2} = \infty$

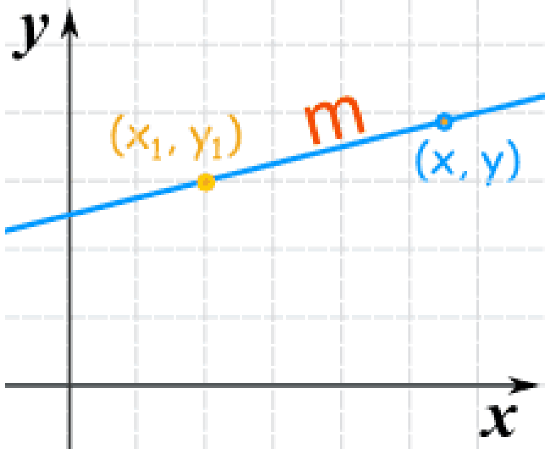
Example: L'Hopitals Rule (Script)

```
tol = 10^-4;
x = 10;
val1 = sin(x)/x

x = 10;
if abs(x)<tol
    x = tol;
end
val2 = sin(x)/x

x = 0;
if abs(x)<tol
    x = tol;
end
val3 = sin(x)/x

x = 0;
val4 = cos(x)
```

```
val1 = -0.0544

val2 = -0.0544

val3 = 1.0000

val4 = 1
```

Theory: Point-Slope Equation of a Line

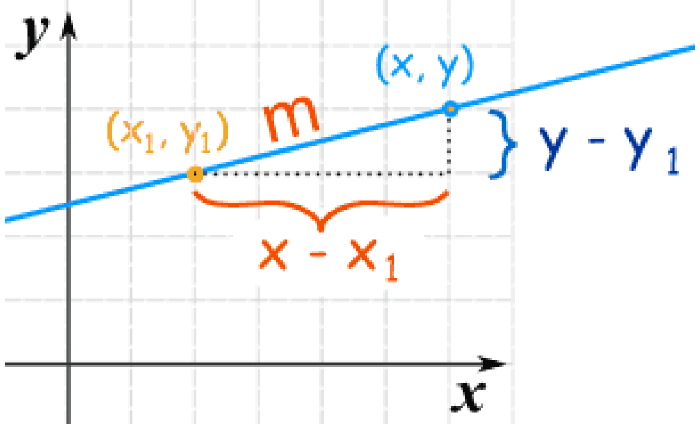
$$y - y_1 = m(x - x_1)$$



(x_1, y_1) is a **known** point

m is the **slope** of the line

(x, y) is any other point on the line



Slope $m = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{y - y_1}{x - x_1}$

Starting with the slope: $\dfrac{y - y_1}{x - x_1} = m$

we rearrange it like this: $\dfrac{y - y_1}{\boxed{x - x_1}} = m(x - x_1)$

to get this: $y - y_1 = m(x - x_1)$

https://www.mathsisfun.com/algebra/line-equation-point-slope.html

Theory: Newton's Method

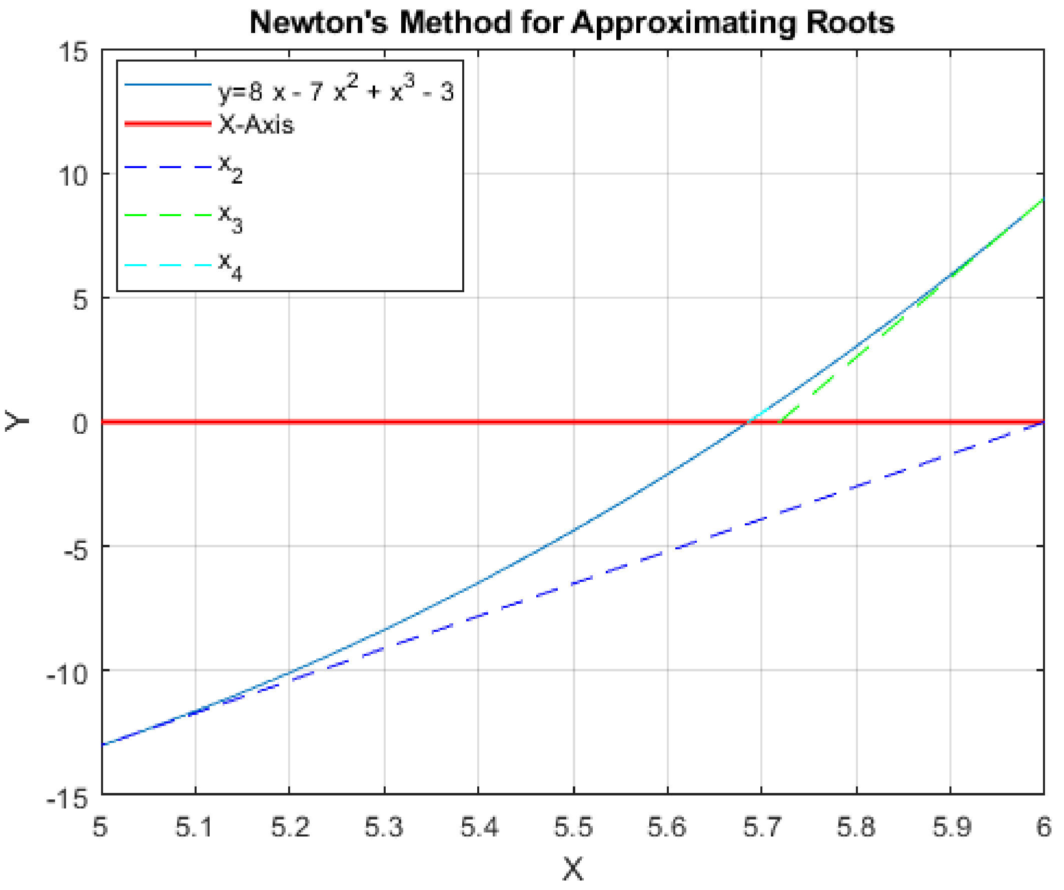
Newton's Method is a way successively finding better and better approximations to the roots of a function.
The slope of tangent line L is $f'(x)$so its equation is:

$$y - f(x_1) = f'(x_1)(x - x_1)$$

To find the roots, we need to find the x-intercepts where $y = 0$, so we assume x_L is where $y = 0$,
$0 - f(x_1) = f'(x_1)(x_2 - x_1)$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

We call x_2 the second approximation to r, but what if we want x_2 to be even more accurate?



Initial Estimation: x = 5

This process can be repeated for $x_1, x_2, x_3 \ldots$
In general, if the n^{th} approximation is x_n and $f'(x_n) \neq 0$, then the next approximation x_{n+1} is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Why important?

Example: Newton's Method 1

Example: Use Newton's Method to find a root:

$$f(x) = x^6 - 2$$

$$f'(x) = 6x^5$$

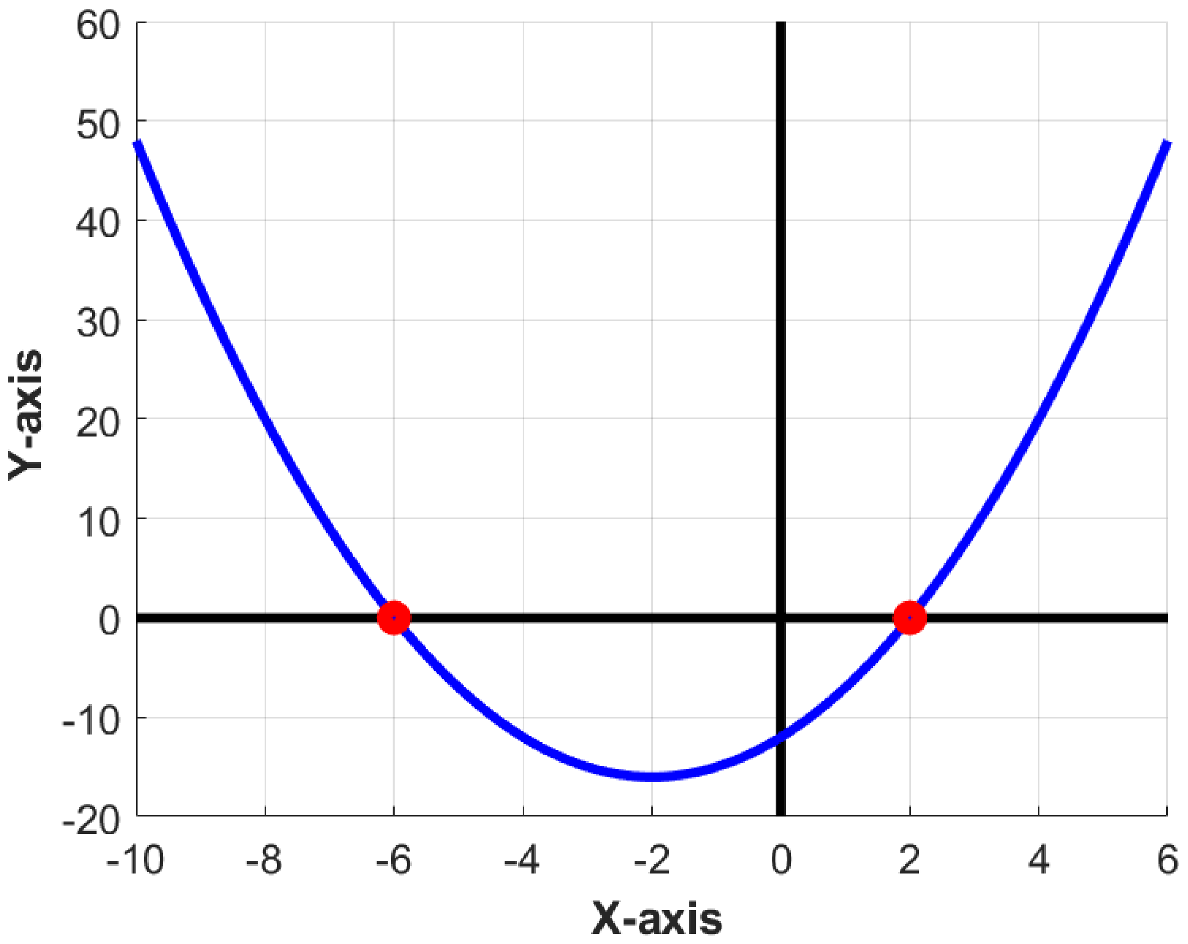
We can apply Newton's method to solve for the root $y = 0$. $x_1 = 1$ (Initial Guess)

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

- $x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1^6-2}{6(1^5)} \approx 1.16666667$
- $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 1.12644368$
- $x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 1.12249707$
- $x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} \approx 1.12246205$
- $x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} \approx 1.12246205$

Example: Root Finding 1

$$f(x) = (x-2)(x-6)$$
$$= x^2 - 4x - 12$$




```
% plot a graph
x = -10:0.01:6;
y = (x-2) .* (x + 6);

figure(1);
line([0 0 ], [-20 60], 'color', ...
    'k', 'Linewidth', 3); hold on; % y-axis
line([min(x) max(x)], [0 0 ], 'color', ...
    'k', 'Linewidth', 3); % x-axis
plot(x, y, 'b', 'LineWidth',3); % graph
plot(2, 0, 'or', 'LineWidth',5);
plot(-6, 0, 'or', 'LineWidth',5); hold off
xlabel('\bf X-axis')
ylabel('\bf Y-axis')
xticks(-10:2:6);
xticklabels({'-10', '-8', '-6', '-4', ...
    '-2','0','2', '4','6'})
set(gca, 'fontsize', 13)
xlim([-10 6])
grid on;
```

Example: Root Finding 1 (Simulation)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

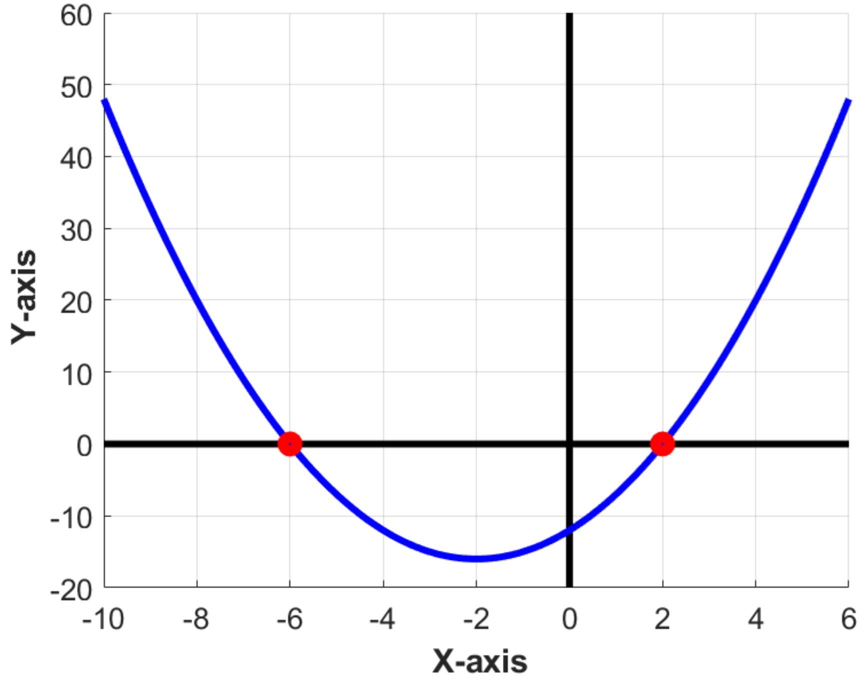
```
x1 = 5;
x2 = x1 - myfun(x1)/myfunp(x1)
x3 = x2 - myfun(x2)/myfunp(x2)
x4 = x3 - myfun(x3)/myfunp(x3)
x5 = x4 - myfun(x4)/myfunp(x4)
```

```
x2 = 2.6429
x3 = 2.0445
x4 = 2.0002
x5 = 2.0000
```

```
function fx = myfun(x)

fx = (x-2) .* (x + 6);

end

function fxp = myfunp(x)

fxp = 2*x + 4;

end
```

$$f(x) = x^2 - 4x - 12$$

$$f'(x) = 2x - 4$$



Example: Root Finding 1 (Simulation)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

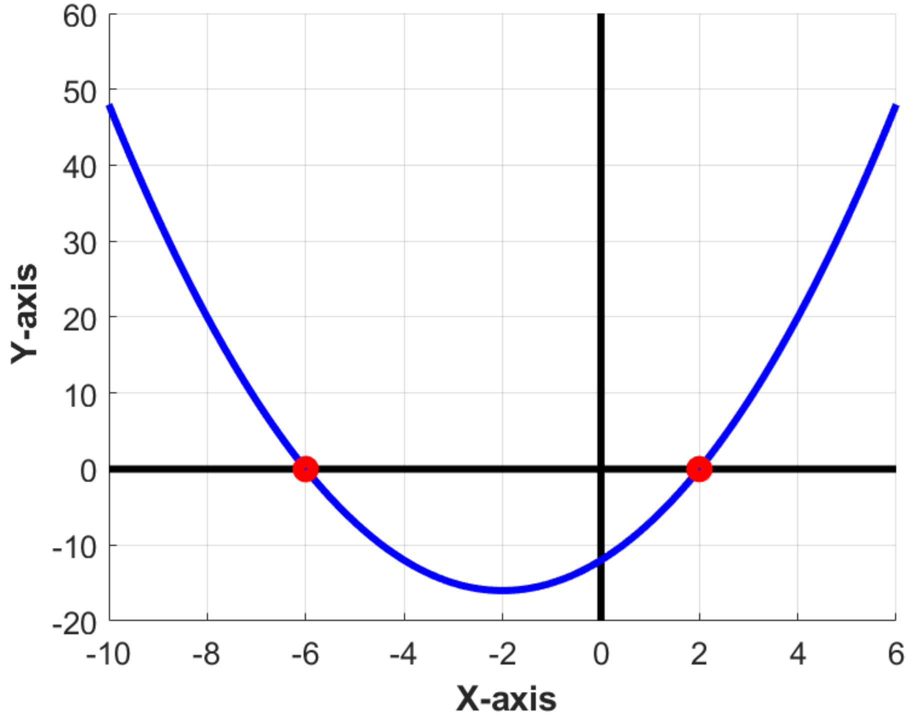
```
x1 = -8;
x2 = x1 - myfun(x1)/myfunp(x1)
x3 = x2 - myfun(x2)/myfunp(x2)
x4 = x3 - myfun(x3)/myfunp(x3)
x5 = x4 - myfun(x4)/myfunp(x4)
```

```
x2 = -6.3333
x3 = -6.0128
x4 = -6.0000
x5 = -6.0000
```

```
function fx = myfun(x)

fx = (x-2) .* (x + 6);

end

function fxp = myfunp(x)

fxp = 2*x + 4;

end
```

$$f(x) = x^2 - 4x - 12$$

$$f'(x) = 2x - 4$$



Quiz: How to Write the Script using a Loop Structure

```
x1 = -8;
x2 = x1 - myfun(x1)/myfunp(x1)
x3 = x2 - myfun(x2)/myfunp(x2)
x4 = x3 - myfun(x3)/myfunp(x3)
x5 = x4 - myfun(x4)/myfunp(x4)
x6 = x5 - myfun(x5)/myfunp(x5)
x7 = x6 - myfun(x6)/myfunp(x6)
x8 = x7 - myfun(x7)/myfunp(x7)
x9 = x8 - myfun(x8)/myfunp(x8)
x10 = x9 - myfun(x9)/myfunp(x9)
```

```
function fx = myfun(x)

fx = (x-2) .* (x + 6);

end

function fxp = myfunp(x)

fxp = 2*x + 4;

end
```

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

```
x1 = -8;
for ii=1:9

end
```

Quiz: How to Write the Script using a Loop Structure (Continue)

```
x1 = -8;
x2 = x1 - myfun(x1)/myfunp(x1)
x3 = x2 - myfun(x2)/myfunp(x2)
x4 = x3 - myfun(x3)/myfunp(x3)
x5 = x4 - myfun(x4)/myfunp(x4)
x6 = x5 - myfun(x5)/myfunp(x5)
x7 = x6 - myfun(x6)/myfunp(x6)
x8 = x7 - myfun(x7)/myfunp(x7)
x9 = x8 - myfun(x8)/myfunp(x8)
x10 = x9 - myfun(x9)/myfunp(x9)
```

```
x1 = -8;
for ii=1:9
    x10 = x1 - myfun(x1)/myfunp(x1)
    x1 = x10;
end
```

ii == 1	x1 == x1	x10 == x2
ii == 2	x1 == x2	x10 == x3
ii == 3	x1 == x3	x10 == x4
ii == 4	x1 == x4	x10 == x5
ii == 9	x1 == x9	x10 == x10

The variables generated from the above code are marked in red.

Example: Root Finding 2

Show that $3x + 2\cos(x) + 5 = 0$ has exactly 1 root, and fine that root exact to 5 decimal places.

Step 1: Show at least 1 root exists

$$f(0) = 7 \qquad f(-10) = -26.28 \qquad f(-10) < f(c) < f(0)$$

∴ By the zero IVT (Intermediate Value Theorem), there exists at least 1 x-value that gives $f(c) = 0$
∴ there is at least 1 root

Step 2: Show 1 root exists using Mean Value Theorem (MVT)

If there were 2 roots, one at $x = a$ and one at $x = b$, then $f(a) = f(b) = 0$.

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0 - 0}{b - a} = 0$$

We know $f'(x) = 3 - 2\sin(x)$, which can never be 0, since $2\sin(x)$ can't be greater than 2. Therefore, we know that there is no case where $f'(c) = 0$ and there cannot be more than 1 root.

Example: Root Finding 2 (Continue)

Step 3: Approximate the root using Newton's Method

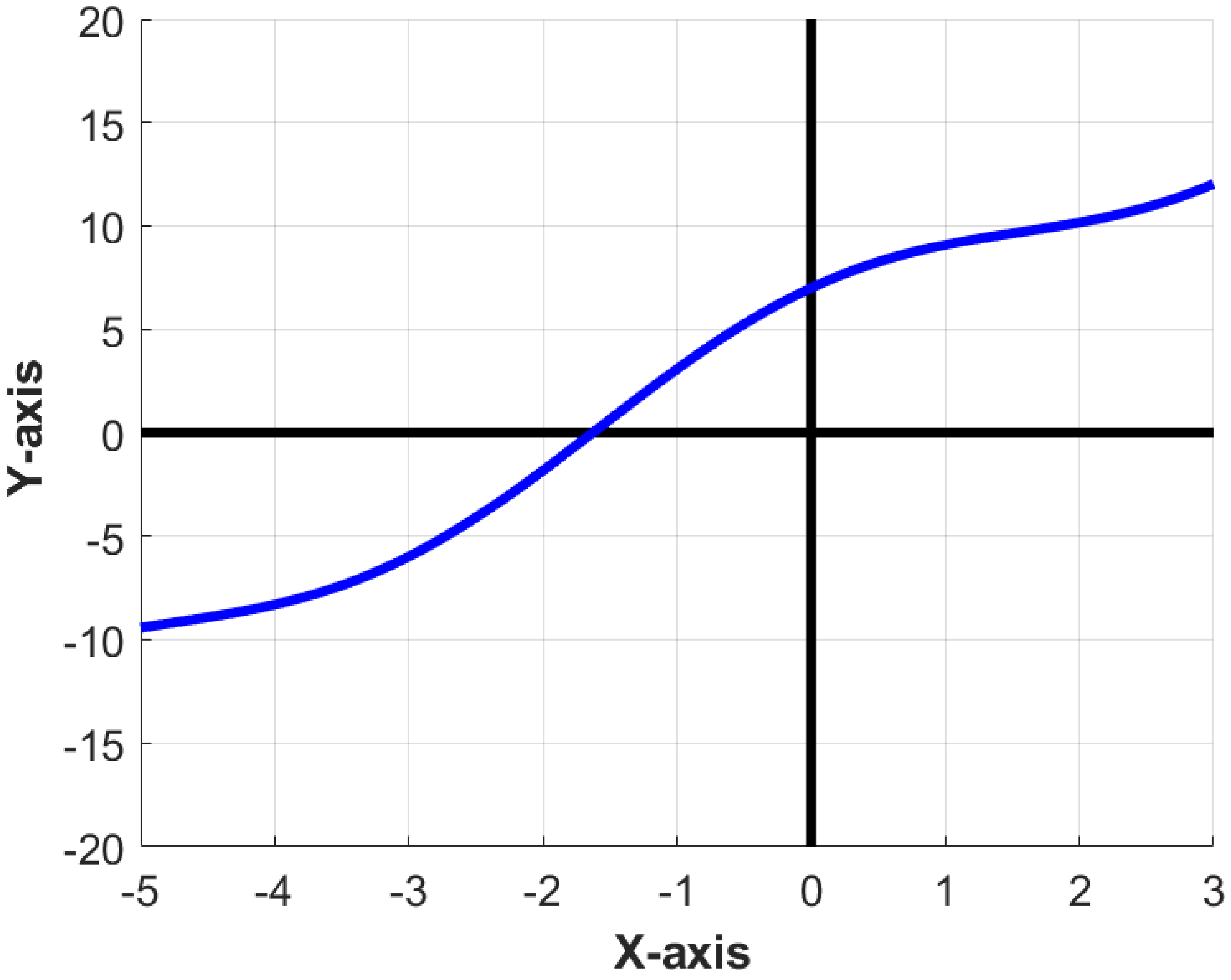
- $x_1 = 0$
- $x_2 = x_1 - \frac{f(0)}{f'(0)} = -2.33333$
- $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -1.572785$

- $x_4 = -1.628318575$
- $x_5 = -1.628331226$
- $x_6 = -1.628331226$

Since x_5 and x_6 are the same to 5 decimal places, -1.62833 is the approximation.

Example: Root Finding 2 (Continue)

$$f(x) = 3x + 2\cos x + 5$$




```
% plot a graph
x = -5:0.01:3;
y = 3*x + 2*cos(x) + 5;

figure(1);
line([0 0 ], [-20 20], 'color', ...
    'k', 'Linewidth', 3); hold on; % y-axis
line([min(x) max(x)], [0 0 ], 'color', ...
    'k', 'Linewidth', 3); % x-axis
plot(x, y, 'b', 'LineWidth',3); % graph
xlabel('\bf X-axis')
ylabel('\bf Y-axis')
xticks(-5:1:3);
xticklabels({'-5', '-4', '-3', '-2', ...
    '-1','0','1','2', '3'})
set(gca, 'fontsize', 13)
xlim([-5 3])
grid on;
```

Example: Root Finding 2 (Continue)

```
function fx = myfun(x)

fx = 3*x + 2*cos(x) + 5;

end

function fxp = myfunp(x)

fxp = 3 - 2*sin(x);

end
```

```
x1 = -4;
x2 = x1 - myfun(x1)/myfunp(x1)
x3 = x2 - myfun(x2)/myfunp(x2)
x4 = x3 - myfun(x3)/myfunp(x3)
x5 = x4 - myfun(x4)/myfunp(x4)
```

```
x2 = 1.5889
x3 = -8.1384
x4 = -4.0779
x5 = 1.9822
```

```
x1 = 3;
x2 = x1 - myfun(x1)/myfunp(x1)
x3 = x2 - myfun(x2)/myfunp(x2)
x4 = x3 - myfun(x3)/myfunp(x3)
x5 = x4 - myfun(x4)/myfunp(x4)
```

```
x2 = -1.4228
x3 = -1.6290
x4 = -1.6283
x5 = -1.6283
```

Example: Root Finding 2 (Continue)

```
x1 = -4;
for ii=1:15
    x10 = x1 - myfun(x1)/myfunp(x1)
    x1 = x10;
end
```

```
function fx = myfun(x)

fx = 3*x + 2*cos(x) + 5;

end

function fxp = myfunp(x)

fxp = 3 - 2*sin(x);

end
```

```
x10 = 1.5889
x10 = -8.1384
x10 = -4.0779
x10 = 1.9822
x10 = -6.7137
x10 = -3.2391
x10 = -0.8479
x10 = -1.6878
x10 = -1.6283
x10 = -1.6283
x10 = -1.6283
x10 = -1.6283
x10 = -1.6283
x10 = -1.6283
x10 = -1.6283
```

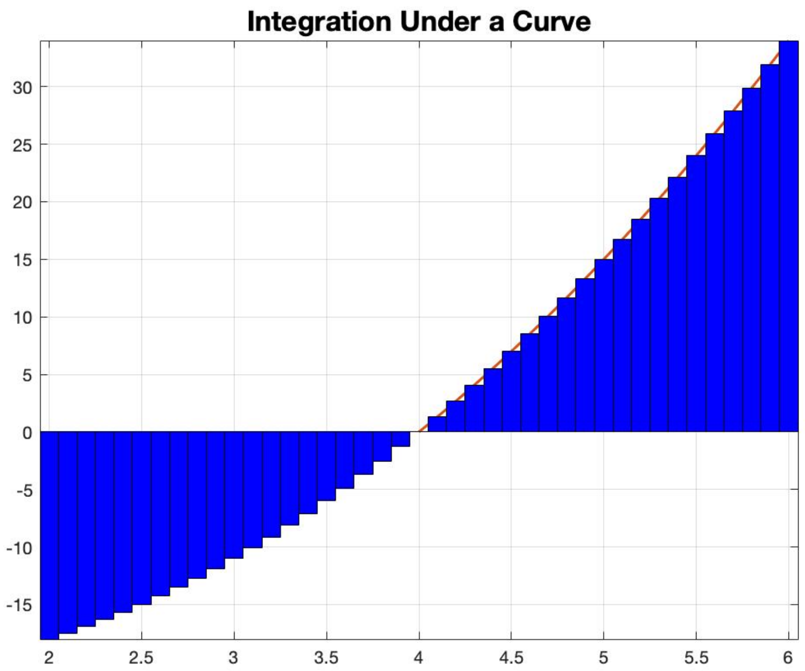
Theory: Integral

Integration is the technique of determining the **area under a curve**. This process is **opposite the process of differentiation**.

To find the area under a curve, we divide the curve into many equal segments of equal width. Each rectangle is multiplied by it's corresponding y-value to get the area of that rectangle. The rectangle's areas are summed for the total area under that curve segment:

- Left endpoints can be used so that the rectangular segments give an underestimation
- Right endpoints can be used so that the rectangular segments give an overestimation
- Center endpoints can be used to try and balance the error from overestimations and underestimations

How can you make your estimation even more accurate?
Take smaller segments!



The Definite Integral

If f is defined for $a \leq x \leq b$, we divide the interval [a,b] into n segments of equal width $\Delta x = \frac{b-a}{n}$.

We let $x_o = a, x_1, x_2, \ldots, x_n = b$ be the endpoints of these segments and let $x_1^*, x_2^*, x_3^*, \ldots, x_n^*$ be sample points in these segments such that x_i^* lies in the i^{th} segment. This means that the definite integral from a to b is:

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

Recall that,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ and } \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{4} \quad and \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Example: Definite Integral

Solve the following definite integral using the definition of the definite integral:

$$\int_0^3 (x^3 - 6x)\, dx$$

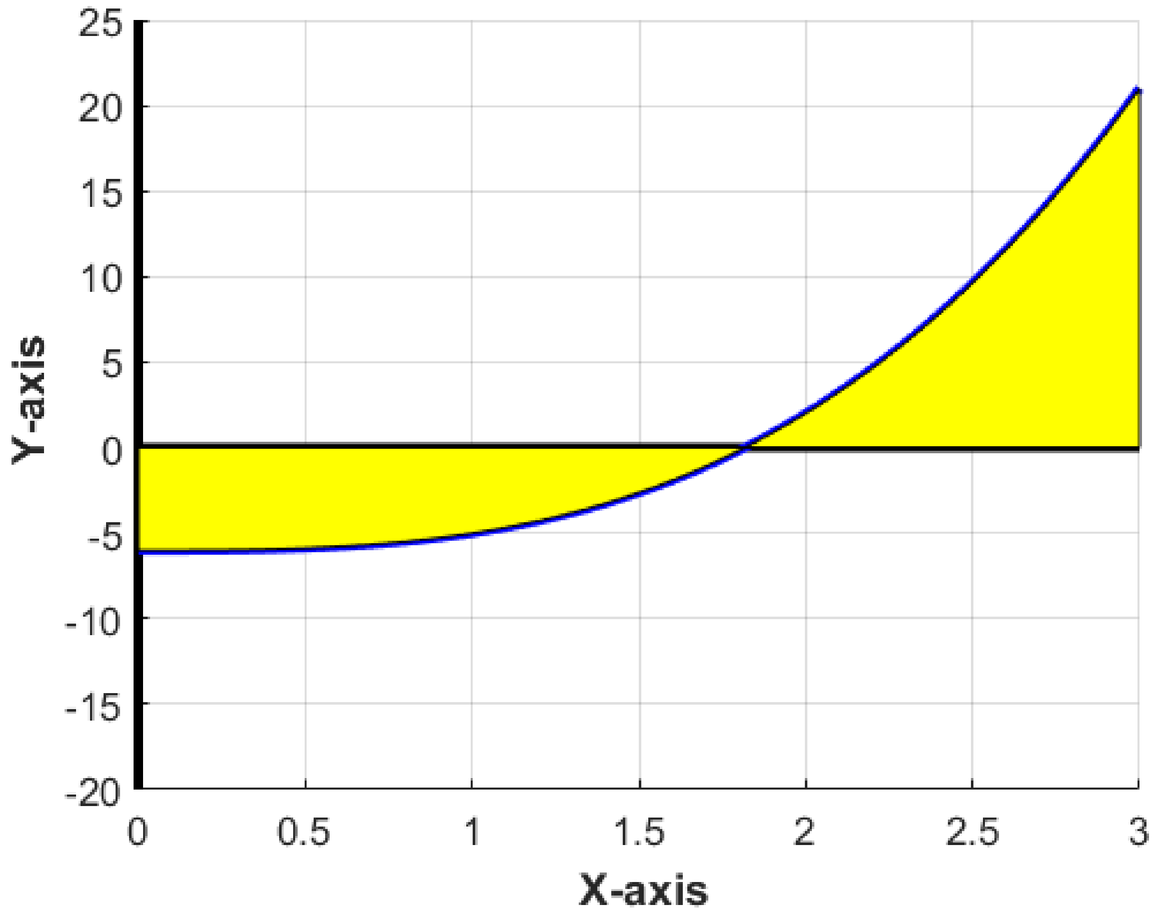
$$\int_0^3 (x^3 - 6x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\Delta x = \frac{3}{n}, x_i = \frac{3i}{n}$$

- $= \lim\limits_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$
- $= \lim\limits_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$
- $= \lim\limits_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} \left[\frac{27i^3}{n^3} - \frac{18i}{n}\right]$
- $= \lim\limits_{n\to\infty} \left[\left(\frac{81}{n^4}\right)\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right]$
- $= \lim\limits_{n\to\infty} \left[\left(\frac{81}{n^4}\right)\left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right]$
- $= \lim\limits_{n\to\infty} \left[\left(\frac{81}{n^4}\right)\left(\frac{n^2+n}{2}\right)^2 - \frac{54}{n^2}\left(\frac{n^2+n}{2}\right)\right]$
- $= \lim\limits_{n\to\infty} \left[\left(\frac{81}{n^4}\right)\left(\frac{n^4+2n^3+n^2}{4}\right) - \frac{54}{n^2}\left(\frac{n^2+n}{2}\right)\right]$
- $= \frac{81}{4} - \frac{54}{2} = \frac{-27}{4}$

Example: Integral (graph)

$$f(x) = x^3 - 6x$$




```
% plot a graph
x = 0:0.01:3;
y = x.^3 - 6;

figure(1);
line([0 0 ], [-20 25], 'color', ...
    'k', 'Linewidth', 3); hold on; % y-axis
line([min(x) max(x)], [0 0 ], 'color', ...
    'k', 'Linewidth', 3); % x-axis
plot(x, y, 'b', 'LineWidth',3); % graph
area(x, y, 'FaceColor','y'); % filled area
xlabel('\bf X-axis')
ylabel('\bf Y-axis')
xticks(0:0.5:3);
xticklabels({'0','0.5','1','1.5', '2', '2.5', '3'})
set(gca, 'fontsize', 13)
xlim([0 3]);
ylim([-20 25])
grid on;
```

Example: Integral (Symbolic)

```
% symbolic method
syms x
y = x^3 -6*x;
int_y_ab = int(y, 0, 3)
double(int_y_ab)
```

```
int_y_ab =
```

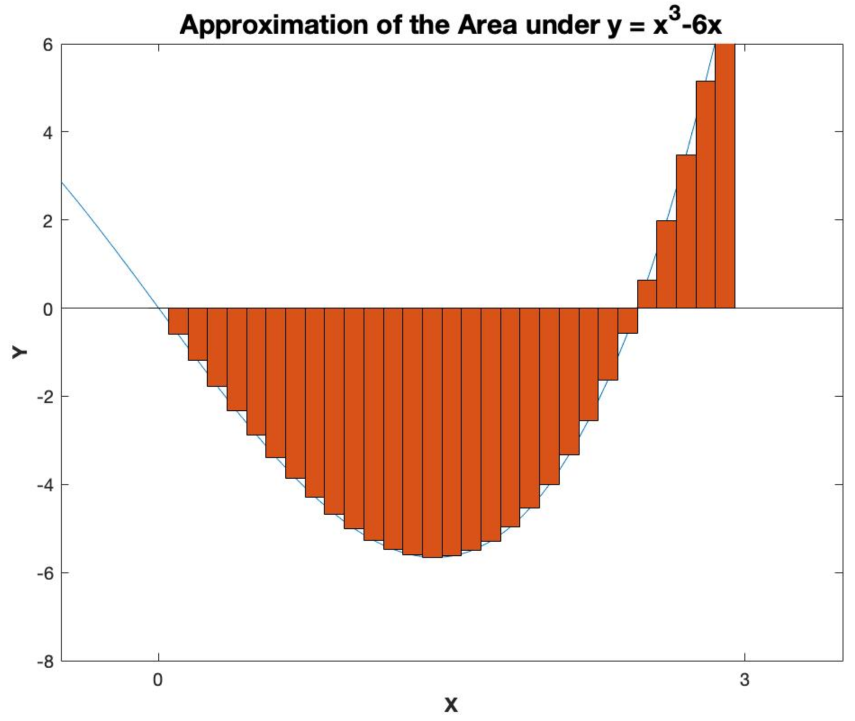
$$-\frac{27}{4}$$

```
ans = -6.7500
```

Example: Integral (Numeric 1)

```
% numerical method 1

n = 10000;
a = 0;
b = 3;
del_x = (b-a)/n;

area_fx = 0;
for ii=1:n
    x_star = a + del_x*ii;
    area_fx = area_fx + myfun(x_star)*del_x;
end
area_fx

error_est = area_fx - (-27/4)
```

```
function fx = myfun(x)

fx = x^3 - 6*x;

end
```



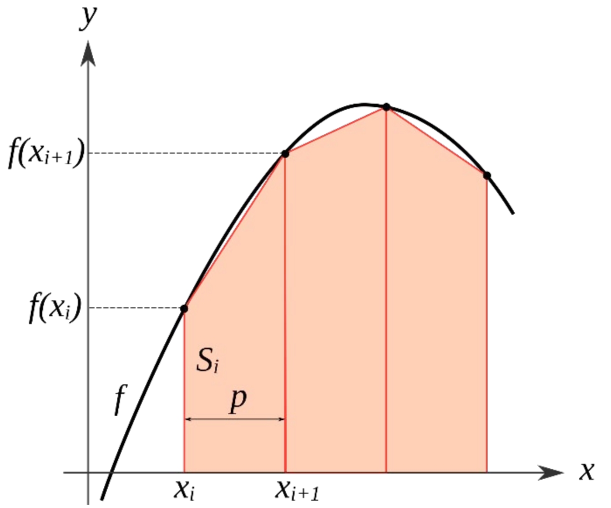

```
area_fx = -6.7486

error_est = 0.0014
```

$$f(x) = x^3 - 6x$$

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}$$

Example: Integral (Numeric 2)

```
% numeric method 2

n = 10000;
a = 0;
b = 3;
del_x = (b-a)/n;

area_fx = 0;
for ii=1:n-1
    x_star1 = a + del_x*ii;
    x_star2 = a + del_x*(ii+1);
    area_fx = area_fx + (myfun(x_star2)+myfun(x_star1))/2*del_x;
end
area_fx

error_est = area_fx - (-27/4)
```




```
area_fx = -6.7500

error_est = 4.7250e-07
```

Example: Integral (Numeric 1 vs Numeric 2)

```
% numeric method 1 and 2

n = 10000;
a = 0;
b = 3;
del_x = (b-a)/n;

area_fx1 = 0;
for ii=1:n
    x_star = a + del_x*ii;
    area_fx1 = area_fx1 + myfun(x_star)*del_x;
end

area_fx2 = 0;
for ii=1:n-1
    x_star1 = a + del_x*ii;
    x_star2 = a + del_x*(ii+1);
    area_fx2 = area_fx2 + (myfun(x_star2)+myfun(x_star1))/2*del_x;
end

area_fx1
area_fx2

error_est1 = area_fx1 - (-27/4)
error_est2 = area_fx2 - (-27/4)
```

```
area_fx1 = -6.7486
area_fx2 = -6.7500

error_est1 = 0.0014
error_est2 = 4.7250e-07
```

Slide Credits and References

- Stormy Attaway, 2018, Matlab: A Practical Introduction to Programming and Problem Solving, 5th edition
- Lecture slides for "Matlab: A Practical Introduction to Programming and Problem Solving"
- Holly Moore, 2018, MATLAB for Engineers, 5th edition