Unconstrained Linear Model Predictive Control

EBO780

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Optimal Control and MPC

- MPC is a form of optimal control.
  - MPC uses a model of the system,
  - to predict the future behavior of the system,
  - to calculate optimal control moves,
  - to minimizes a cost function.

- Uses a receding horizon approach.

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Terminology

- Sampling time (τ_S):
  - Time between discrete samples.
- Prediction horizon (P):
  - Length of time the controller predicts into the future.
- Prediction samples (N_P):
  - Number of discrete samples predicted: $P = N_P \times \tau_S$

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Terminology

- Control Horizon (C):
  - Length of time the controller makes control moves.
- Control Moves (N_C):
  - The number of control moves.
- Blocking (N_B):
  - The number of consecutive samples to keep a control move constant.

$$C = N_C \times N_B \times \tau_S$$

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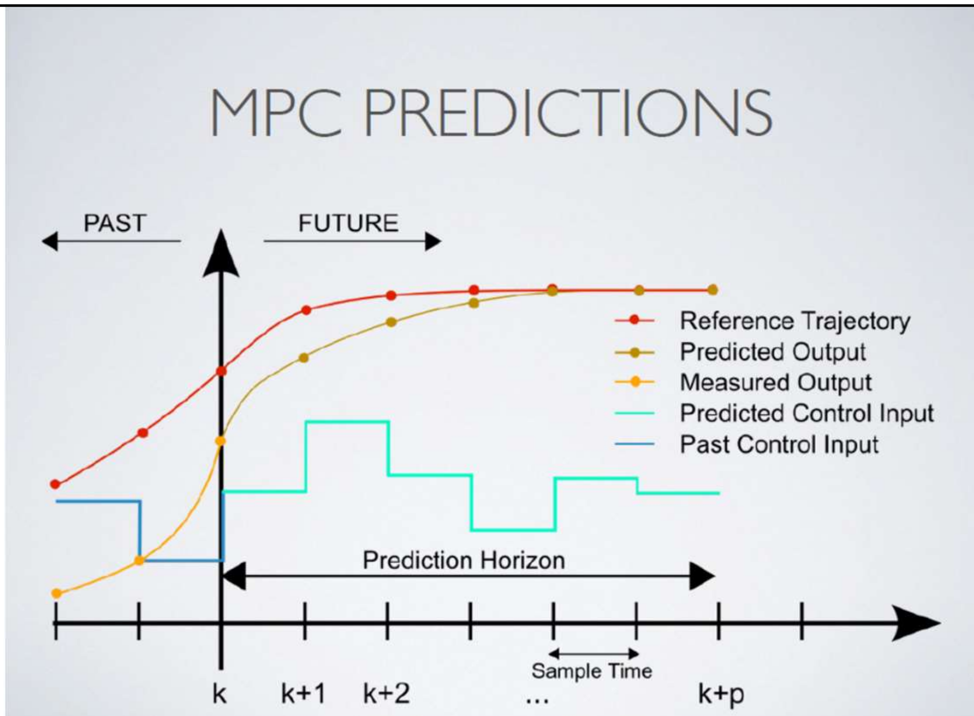

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Problem description (1)

Optimization problem:

$$\min_{u(k+N_C|k),\ldots,u(k|k)} V\big(u, x(k|k)\big)$$

where:

$$u \in \mathbb{R}^{N_C \times N_U}$$
$$x \in \mathbb{R}^{N_P \times N_X}$$
$$x(k|k) \triangleq \text{Initial state}$$

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Problem description (1)

Objective function:

$$V(x,u) = \sum_{i=1}^{N_P} x(k+i|k)^T Q x(k+i|k) + \sum_{i=1}^{N_C} \Delta u(k+i|k)^T R \Delta u(k+i|k)$$

Subject to:

$$x(k+i+1|k) = f\big(x(k+i|k), u(k+i|k)\big) \qquad \forall i = 1, \dots, N_C$$
$$x(k+i+1|k) = f\big(x(k+i|k), u(k+N_C|k)\big) \qquad \forall i = N_C+1, \dots, N_P$$

Problem description (2)

Optimization problem:

$$\min_{u(k+N_C|k),\dots,u(k|k)} V\big(u, x(k|k)\big)$$

where:

$$u \in \mathbb{R}^{N_C \times N_U}$$
$$x \in \mathbb{R}^{N_P \times N_X}$$
$$y \in \mathbb{R}^{N_P \times N_Y}$$
$x(k|k) \triangleq$ Initial state

Problem description (2)

Objective function:

$$V(x,u) = \sum_{i=1}^{N_C} \left(Y_{sp} - h\big(x(k+i|k), u(k+i|k)\big)\right)^T Q \left(Y_{sp} - h\big(x(k+i|k), u(k+i|k)\big)\right) + \sum_{i=N_C+1}^{N_P} \left(Y_{sp} - h\big(x(k+i|k), u(k+N_C|k)\big)\right)^T Q \left(Y_{sp} - h\big(x(k+i|k), u(k+N_C|k)\big)\right) + \sum_{i=1}^{N_C} \Delta u(k+i|k)^T R \Delta u(k+i|k)$$

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Problem description (2)

Subject to:

$$x(k+i+1|k) = f\big(x(k+i|k), u(k+i|k)\big) \qquad \forall i = 1, \dots, N_C$$
$$x(k+i+1|k) = f\big(x(k+i|k), u(k+N_C|k)\big) \qquad \forall i = N_C+1, \dots, N_P$$

$$y(k+i|k) = h\big(x(k+i|k), u(k+i|k)\big) \qquad \forall i = 1, \dots, N_P$$

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Advantages and Disadvantages

- Advantages:
  - Multivariable controller – can handle interactions between variables
  - Able to handle constraints on manipulated variables (MV's) and controlled variables (CV's) explicitly.
  - Good performance for systems with dead-time.

- Disadvantages
  - High computational demand to solve optimisation problem at each interval.
  - An accurate model of the process is necessary.
  - State estimation (e.g. Kalman Filter) is necessary for practical implementation.

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Tuning Q and R

- The absolute values of Q and R are not important. The relative relationship between Q and R determine the MPC response.
  - $Q = 100$, $R = 10$ and $Q = 1000$, $R = 100$ will behave the same.
- Relative sizes of weights in Q will bias the controller to follow more tightly the y value with the highest weight.
- Relative sizes of weights in R will bias the controller to move the MV with the lowest weight more to steer the system to steady state, unless there is not a MV-CV relationship between the highest weighted CV and the lowest weighted MV.

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Tuning Q and R

- Q and R are positive definite diagonal matrices.

$$Q = \begin{bmatrix} Q_{11} & 0 & 0 \\ 0 & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix} \qquad R = \begin{bmatrix} R_{11} & 0 & 0 \\ 0 & R_{22} & 0 \\ 0 & 0 & R_{33} \end{bmatrix}$$

- Examples:
  - Assume steady-state value of $y_1 = 10$.
    - What should Q_{11} be if a 10% deviation from steady-state in y_1 results in $V = 100$?

$$100 = 1 \times Q_{11} \times 1$$
$$\therefore\ Q_{11} = 100$$

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Tuning Q and R

- Examples:
  - Assume steady-state value of $y_1 = 10$.
    - What should Q_{11} be if a 1% deviation from steady-state in y_1 results in $V = 100$?

$$100 = 0.1 \times Q_{11} \times 0.1$$
$$\therefore\ Q_{11} = 10000$$

    - What should R_{11} be if a 1% deviation from steady-state in y_1 should have 100 times the contribution to V as $\Delta u_1 = 10$?

$$0.1 \times Q_{11} \times 0.1 = 100 \times 10 \times R_{11} \times 10$$

If $R_{11} = 1$, then $Q_{11} = 1000000$

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Tuning Q and R

- Effect of scaling:
  - To improve results and numerical stability of optimizer, scale model so that input and output falls in range 0 – 1.
  - If model is not scaled, the weightings must be scaled:
    - If u_1 has range 0-400, and u_2 has range 0-10, what should R_{11} and R_{22} be in order for 10% range change in both MV's to contribute the same to the objective function?

$$40 \times R_{11} \times 40 = 1 \times R_{22} \times 1$$

If $R_{11} = 1$, then $R_{22} = 1600$

See Section 5 of following paper for an example of tuning Q and R:

Le Roux, J. D.; Padhi, R.; Craig, I. K. Optimal control of grinding mill circuit using model predictive static programming: A new nonlinear MPC paradigm. J. Process Control 2014, 24, 29–40.

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N_C and N_P

- Control Horizon:
  - If there are no constraints on Δu, choose N_C long enough to allow at least three control moves with blocking, e.g.,

$$N_B = 3, \qquad N_C = 3, \qquad \tau_s = 2\ s, \qquad C = 3 \times 3 \times 2 = 18$$

- Prediction Horizon:
  - Choose N_P long enough to cover control horizon + longest plant dynamics.
- Longest plant dynamic:
  - Dead-time + time to 99% of steady state. (99% of steady state for first order model is typically 5 x time constant.)

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N_C and N_P

- If $N_P > N_C$, the control be biased towards outputs compared to the inputs. More terms of the output compared to the input will appear in the objective function.
  - Eliminate the biasing effect by scaling Q with $\frac{N_C}{N_P}$

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References

- Rawlings, J.B. (2000). Tutorial overview of Model Predictive Control. IEEE Control Systems Magazine, Volume 20, Issue 3, Pages 38--52.
- Qin, S. J. and T. A. Badgwell (2003). A survey of industrial model predictive control technology. Control Engineering Practice. Volume 11, Pages 733--764.
- Mayne, D. Q., J. B. Rawlings, C. V. Rao and P. O. M. Scokaert. (2000). Constrained model predictive control: Stability and optimality. Automatica. Volume 36, Pages 789--814.

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