1. Представить ротор, дивергенцию и градиент в сферических координатах.

   Решение. В сферических координатах x, y и z выражается следующим образом:

$$\begin{cases} z = r\cos\theta \\ x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \end{cases}$$

   Коэффициенты Ламе находятся по формуле:

$$H_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2}.$$

   То есть,

$$H_r = \sqrt{\left(\frac{\partial(r\sin\theta\cos\varphi)}{\partial r}\right)^2 + \left(\frac{\partial(r\sin\theta\sin\varphi)}{\partial r}\right)^2 + \left(\frac{\partial(r\cos\theta)}{\partial r}\right)^2} =$$

$$= \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta} = \sqrt{\sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta} =$$

$$= \sqrt{\sin^2\theta + \cos^2\theta} = \sqrt{1} = 1;$$

$$H_\theta = \sqrt{\left(\frac{\partial(r\sin\theta\cos\varphi)}{\partial \theta}\right)^2 + \left(\frac{\partial(r\sin\theta\sin\varphi)}{\partial \theta}\right)^2 + \left(\frac{\partial(r\cos\theta)}{\partial \theta}\right)^2} =$$

$$= \sqrt{r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta} =$$

$$= \sqrt{r^2(\sin^2\theta + (\cos^2\varphi + \sin^2\varphi)\cos^2\theta)} = \sqrt{r^2(\sin^2\theta + \cos^2\theta)} =$$

$$= \sqrt{r^2} = r;$$

$$H_\varphi = \sqrt{\left(\frac{\partial(r\sin\theta\cos\varphi)}{\partial \varphi}\right)^2 + \left(\frac{\partial(r\sin\theta\sin\varphi)}{\partial \varphi}\right)^2 + \left(\frac{\partial(r\cos\theta)}{\partial \varphi}\right)^2} =$$

$$= \sqrt{r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi} =$$

$$= \sqrt{(\sin^2\varphi + \cos^2\varphi) r^2 \sin^2\theta} = \sqrt{r^2\sin^2\theta} = r\sin\theta;$$

Градиент в криволенейных координатах определяется таким образом:

$$\operatorname{grad}\mathbf{U} = \frac{1}{H_1}\frac{\partial \mathbf{U}}{\partial q_1}\mathbf{e}_1 + \frac{1}{H_2}\frac{\partial \mathbf{U}}{\partial q_2}\mathbf{e}_2 + \frac{1}{H_3}\frac{\partial \mathbf{U}}{\partial q_3}\mathbf{e}_3.$$

Получаем:

$$\operatorname{grad}\mathbf{U} = \frac{1}{1}\frac{\partial \mathbf{U}}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \mathbf{U}}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial \mathbf{U}}{\partial \varphi}\mathbf{e}_\varphi =$$
$$= \frac{\partial \mathbf{U}}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \mathbf{U}}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial \mathbf{U}}{\partial \varphi}\mathbf{e}_\varphi.$$

Дивергенция в криволинейных координатах находится по формуле

$$\operatorname{div}\mathbf{U} = \frac{1}{H_1H_2H_3}\left(\frac{\partial}{\partial q_1}\left(H_2H_3\mathbf{U}_1\right) + \frac{\partial}{\partial q_2}\left(H_1H_3\mathbf{U}_2\right) + \frac{\partial}{\partial q_3}\left(H_1H_2\mathbf{U}_3\right)\right).$$

Подставляем значения:

$$\operatorname{div}\mathbf{U} = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(r^2\mathbf{U}_r\sin\theta\right) + \frac{\partial}{\partial \theta}\left(r\mathbf{U}_\theta\sin\theta\right) + \frac{\partial}{\partial \varphi}\left(r\mathbf{U}_\varphi\right)\right) =$$
$$= \frac{1}{r^2\sin\theta}\left(\sin\theta\cdot\frac{\partial}{\partial r}\left(r^2\mathbf{U}_r\right) + r\frac{\partial}{\partial \theta}\left(\mathbf{U}_\theta\sin\theta\right) + r\frac{\partial}{\partial \varphi}\left(\mathbf{U}_\varphi\right)\right) =$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\mathbf{U}_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\mathbf{U}_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \mathbf{U}_\varphi}{\partial \varphi}.$$

Формула для ротора в криволинейных координатах:

$$\operatorname{rot}\mathbf{U} = \frac{1}{H_1H_2H_3}\begin{vmatrix} H_1\mathbf{e}_1 & H_2\mathbf{e}_2 & H_3\mathbf{e}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ H_1\mathbf{U}_1 & H_2\mathbf{U}_2 & H_3\mathbf{U}_3 \end{vmatrix}$$

Подставляем коэффициенты:

$$\operatorname{rot}\mathbf{U} = \frac{1}{r^2\sin\theta}\begin{vmatrix} 1\mathbf{e}_r & r\mathbf{e}_\theta & r\mathbf{e}_\varphi\sin\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \varphi} \\ 1\mathbf{U}_r & r\mathbf{U}_r & r\mathbf{U}_\varphi\sin\theta \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\mathbf{e}_\varphi\sin\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \varphi} \\ \mathbf{U}_r & r\mathbf{U}_r & r\mathbf{U}_\varphi\sin\theta \end{vmatrix} =$$

$$= \frac{1}{r^2\sin\theta}\left(\mathbf{e}_r\frac{\partial}{\partial \theta}\left(r\mathbf{U}_\varphi\sin\theta\right) + r\mathbf{e}_\theta\frac{\partial \mathbf{U}_r}{\partial \varphi} + r\mathbf{e}_\varphi\sin\theta\cdot\frac{\partial}{\partial r}\left(r\mathbf{U}_\theta\right) -\right.$$
$$\left. - r\mathbf{e}_\theta\frac{\partial}{\partial r}\left(r\mathbf{U}_\varphi\sin\theta\right) - \mathbf{e}_r\frac{\partial}{\partial \varphi}\left(r\mathbf{U}_\theta\right) - r\mathbf{e}_\varphi\sin\theta\cdot\frac{\partial \mathbf{U}_r}{\partial \theta}\right) =$$

$$= \frac{r}{r^2 \sin\theta} \left(\mathbf{e}_r \left(\frac{\partial}{\partial\theta} \left(\mathbf{U}_\varphi \sin\theta \right) - \frac{\partial \mathbf{U}_\theta}{\partial\varphi} \right) + \mathbf{e}_\theta \left(\frac{\partial \mathbf{U}_r}{\partial\varphi} - \sin\theta \cdot \frac{\partial}{\partial r} \left(r\mathbf{U}_\varphi \right) \right) + \right.$$

$$\left. + \mathbf{e}_\varphi \sin\theta \cdot \left(\frac{\partial}{\partial r} \left(r\mathbf{U}_\theta \right) - \frac{\partial \mathbf{U}_r}{\partial\theta} \right) \right) =$$

$$= \frac{1}{r \sin\theta} \left(\mathbf{e}_r \left(\frac{\partial}{\partial\theta} \left(\mathbf{U}_\varphi \sin\theta \right) - \frac{\partial \mathbf{U}_\theta}{\partial\varphi} \right) + \mathbf{e}_\theta \left(\frac{\partial \mathbf{U}_r}{\partial\varphi} - \sin\theta \cdot \frac{\partial}{\partial r} \left(r\mathbf{U}_\varphi \right) \right) + \right.$$

$$\left. + \mathbf{e}_\varphi \sin\theta \cdot \left(\frac{\partial}{\partial r} \left(r\mathbf{U}_\theta \right) - \frac{\partial \mathbf{U}_r}{\partial\theta} \right) \right) =$$

$$= \frac{\mathbf{e}_r}{r \sin\theta} \left(\frac{\partial}{\partial\theta} \left(\mathbf{U}_\varphi \sin\theta \right) - \frac{\partial \mathbf{U}_\theta}{\partial\varphi} \right) + \frac{\mathbf{e}_\theta}{r \sin\theta} \left(\frac{\partial \mathbf{U}_r}{\partial\varphi} - \sin\theta \cdot \frac{\partial}{\partial r} \left(r\mathbf{U}_\varphi \right) \right) +$$

$$+ \frac{\mathbf{e}_\varphi}{r} \left(\frac{\partial}{\partial r} \left(r\mathbf{U}_\theta \right) - \frac{\partial \mathbf{U}_r}{\partial\theta} \right)$$