

1. Представить ротор, дивергенцию и градиент в сферических координатах.

Решение. В сферических координатах x , y и z выражается следующим образом:

$$\begin{cases} z = r \cos \theta \\ x = r \sin \theta \cos \varphi \\ y = r \sin \theta \sin \varphi \end{cases}$$

Коэффициенты Ламе находятся по формуле:

$$H_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2}.$$

То есть,

$$\begin{aligned} H_r &= \sqrt{\left(\frac{\partial(r \sin \theta \cos \varphi)}{\partial r}\right)^2 + \left(\frac{\partial(r \sin \theta \sin \varphi)}{\partial r}\right)^2 + \left(\frac{\partial(r \cos \theta)}{\partial r}\right)^2} = \\ &= \sqrt{\sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi + \cos^2 \theta} = \sqrt{\sin^2 \theta (\cos^2 \varphi + \sin^2 \varphi) + \cos^2 \theta} = \\ &= \sqrt{\sin^2 \theta + \cos^2 \theta} = \sqrt{1} = 1; \end{aligned}$$

$$\begin{aligned} H_\theta &= \sqrt{\left(\frac{\partial(r \sin \theta \cos \varphi)}{\partial \theta}\right)^2 + \left(\frac{\partial(r \sin \theta \sin \varphi)}{\partial \theta}\right)^2 + \left(\frac{\partial(r \cos \theta)}{\partial \theta}\right)^2} = \\ &= \sqrt{r^2 \cos^2 \theta \cos^2 \varphi + r^2 \cos^2 \theta \sin^2 \varphi + r^2 \sin^2 \theta} = \\ &= \sqrt{r^2 (\sin^2 \theta + (\cos^2 \varphi + \sin^2 \varphi) \cos^2 \theta)} = \sqrt{r^2 (\sin^2 \theta + \cos^2 \theta)} = \\ &= \sqrt{r^2} = r; \end{aligned}$$

$$\begin{aligned} H_\varphi &= \sqrt{\left(\frac{\partial(r \sin \theta \cos \varphi)}{\partial \varphi}\right)^2 + \left(\frac{\partial(r \sin \theta \sin \varphi)}{\partial \varphi}\right)^2 + \left(\frac{\partial(r \cos \theta)}{\partial \varphi}\right)^2} = \\ &= \sqrt{r^2 \sin^2 \theta \sin^2 \varphi + r^2 \sin^2 \theta \cos^2 \varphi} = \end{aligned}$$

$$= \sqrt{(\sin^2 \varphi + \cos^2 \varphi) r^2 \sin^2 \theta} = \sqrt{r^2 \sin^2 \theta} = r \sin \theta;$$

Градиент в криволинейных координатах определяется таким образом:

$$\text{grad } \mathbf{U} = \frac{1}{H_1} \frac{\partial \mathbf{U}}{\partial q_1} \mathbf{e}_1 + \frac{1}{H_2} \frac{\partial \mathbf{U}}{\partial q_2} \mathbf{e}_2 + \frac{1}{H_3} \frac{\partial \mathbf{U}}{\partial q_3} \mathbf{e}_3.$$

Получаем:

$$\begin{aligned} \text{grad } \mathbf{U} &= \frac{1}{1} \frac{\partial \mathbf{U}}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \mathbf{U}}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \mathbf{U}}{\partial \varphi} \mathbf{e}_\varphi = \\ &= \frac{\partial \mathbf{U}}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial \mathbf{U}}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \mathbf{U}}{\partial \varphi} \mathbf{e}_\varphi. \end{aligned}$$

Дивергенция в криволинейных координатах находится по формуле

$$\text{div } \mathbf{U} = \frac{1}{H_1 H_2 H_3} \left(\frac{\partial}{\partial q_1} (H_2 H_3 \mathbf{U}_1) + \frac{\partial}{\partial q_2} (H_1 H_3 \mathbf{U}_2) + \frac{\partial}{\partial q_3} (H_1 H_2 \mathbf{U}_3) \right).$$

Подставляем значения:

$$\begin{aligned} \text{div } \mathbf{U} &= \frac{1}{r^2 \sin \theta} \left(\frac{\partial}{\partial r} (r^2 \mathbf{U}_r \sin \theta) + \frac{\partial}{\partial \theta} (r \mathbf{U}_\theta \sin \theta) + \frac{\partial}{\partial \varphi} (r \mathbf{U}_\varphi) \right) = \\ &= \frac{1}{r^2 \sin \theta} \left(\sin \theta \cdot \frac{\partial}{\partial r} (r^2 \mathbf{U}_r) + r \frac{\partial}{\partial \theta} (\mathbf{U}_\theta \sin \theta) + r \frac{\partial}{\partial \varphi} (\mathbf{U}_\varphi) \right) = \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \mathbf{U}_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\mathbf{U}_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \mathbf{U}_\varphi}{\partial \varphi}. \end{aligned}$$

Формула для ротора в криволинейных координатах:

$$\text{rot } \mathbf{U} = \frac{1}{H_1 H_2 H_3} \begin{vmatrix} H_1 \mathbf{e}_1 & H_2 \mathbf{e}_2 & H_3 \mathbf{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ H_1 \mathbf{U}_1 & H_2 \mathbf{U}_2 & H_3 \mathbf{U}_3 \end{vmatrix}$$

Подставляем коэффициенты:

$$\begin{aligned} \text{rot } \mathbf{U} &= \frac{1}{r^2 \sin \theta} \begin{vmatrix} 1 \mathbf{e}_r & r \mathbf{e}_\theta & r \mathbf{e}_\varphi \sin \theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ 1 \mathbf{U}_r & r \mathbf{U}_r & r \mathbf{U}_\varphi \sin \theta \end{vmatrix} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \mathbf{e}_r & r \mathbf{e}_\theta & r \mathbf{e}_\varphi \sin \theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ \mathbf{U}_r & r \mathbf{U}_r & r \mathbf{U}_\varphi \sin \theta \end{vmatrix} = \\ &= \frac{1}{r^2 \sin \theta} \left(\mathbf{e}_r \frac{\partial}{\partial \theta} (r \mathbf{U}_\varphi \sin \theta) + r \mathbf{e}_\theta \frac{\partial \mathbf{U}_r}{\partial \varphi} + r \mathbf{e}_\varphi \sin \theta \cdot \frac{\partial}{\partial r} (r \mathbf{U}_\theta) - \right. \\ &\quad \left. - r \mathbf{e}_\theta \frac{\partial}{\partial r} (r \mathbf{U}_\varphi \sin \theta) - \mathbf{e}_r \frac{\partial}{\partial \varphi} (r \mathbf{U}_\theta) - r \mathbf{e}_\varphi \sin \theta \cdot \frac{\partial \mathbf{U}_r}{\partial \theta} \right) = \end{aligned}$$

$$\begin{aligned}
&= \frac{r}{r^2 \sin \theta} \left(\mathbf{e}_r \left(\frac{\partial}{\partial \theta} (\mathbf{U}_\varphi \sin \theta) - \frac{\partial \mathbf{U}_\theta}{\partial \varphi} \right) + \mathbf{e}_\theta \left(\frac{\partial \mathbf{U}_r}{\partial \varphi} - \sin \theta \cdot \frac{\partial}{\partial r} (r \mathbf{U}_\varphi) \right) + \right. \\
&\quad \left. + \mathbf{e}_\varphi \sin \theta \cdot \left(\frac{\partial}{\partial r} (r \mathbf{U}_\theta) - \frac{\partial \mathbf{U}_r}{\partial \theta} \right) \right) = \\
&= \frac{1}{r \sin \theta} \left(\mathbf{e}_r \left(\frac{\partial}{\partial \theta} (\mathbf{U}_\varphi \sin \theta) - \frac{\partial \mathbf{U}_\theta}{\partial \varphi} \right) + \mathbf{e}_\theta \left(\frac{\partial \mathbf{U}_r}{\partial \varphi} - \sin \theta \cdot \frac{\partial}{\partial r} (r \mathbf{U}_\varphi) \right) + \right. \\
&\quad \left. + \mathbf{e}_\varphi \sin \theta \cdot \left(\frac{\partial}{\partial r} (r \mathbf{U}_\theta) - \frac{\partial \mathbf{U}_r}{\partial \theta} \right) \right) = \\
&= \frac{\mathbf{e}_r}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (\mathbf{U}_\varphi \sin \theta) - \frac{\partial \mathbf{U}_\theta}{\partial \varphi} \right) + \frac{\mathbf{e}_\theta}{r \sin \theta} \left(\frac{\partial \mathbf{U}_r}{\partial \varphi} - \sin \theta \cdot \frac{\partial}{\partial r} (r \mathbf{U}_\varphi) \right) + \\
&\quad + \frac{\mathbf{e}_\varphi}{r} \left(\frac{\partial}{\partial r} (r \mathbf{U}_\theta) - \frac{\partial \mathbf{U}_r}{\partial \theta} \right)
\end{aligned}$$