§5.2 平稳过程的自相关函数

一、平稳过程自相关函数的性质

定理5.2.1 复平稳过程$\{X(t), t\in T\}$的自相关函数$R_X(\tau)$，有如下性质：

1) $R(0)=E[|X(t)|^2]\geq 0;$

2) $|R_X(\tau)|\leq R_X(0);\quad (|C_X(\tau)|\leq C_X(0);)$

3) $R_X(-\tau)=\overline{R_X(\tau)};$（共轭对称）

4) **非负定性** 对$\forall n \geq 1, t_1, \cdots, t_n \in T$,

及复数 $\alpha_1, \alpha_2, \ldots, \alpha_n$ **有**

$$\sum_{k,j=1}^{n} \alpha_j \overline{\alpha_k} R_X(t_k - t_j) \geq 0.$$

证明

1) $R_X(0) = E\{X(t)\overline{X(t)}\} = E\{|X(t)|^2\} \geq 0;$

2）**由许瓦兹不等式**

$$|R_X(\tau)|^2 = |R_X(t, t+\tau)|^2 = \left|E(X(t)\overline{X(t+\tau)})\right|^2$$

$$\leq E[|X(t)|^2]E[|X(t+\tau)|^2] = R_X^2(0);$$

3) $$\overline{R_X(\tau)} = \overline{E[X(t)\overline{X(t+\tau)}]} = E[\overline{X(t)}X(t+\tau)]$$

$$= E[X(t+\tau)\overline{X(t+\tau+(-\tau))}] = R_X(-\tau);$$

4) $$\sum_{k,j=1}^{n} \alpha_j \overline{\alpha_k} R_X(t_k - t_j)$$

$$= \sum_{k,j=1}^{n} \alpha_j \overline{\alpha_k} E[X(t_j)\overline{X(t_k)}]$$

$$= E[\sum_{k,j=1}^{n} \alpha_j \overline{\alpha_k} X(t_j)\overline{X(t_k)}] = E[\left|\sum_{k=1}^{n} \alpha_k X(t_k)\right|^2] \geq 0$$

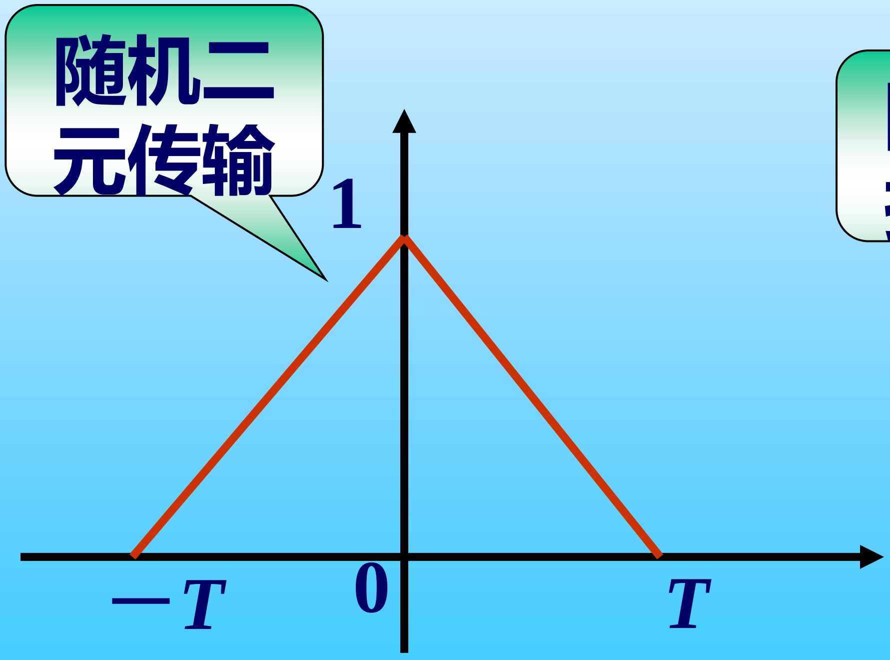
推论1 实平稳过程$\{X(t), t\in T\}$的$R_X(\tau)$有:

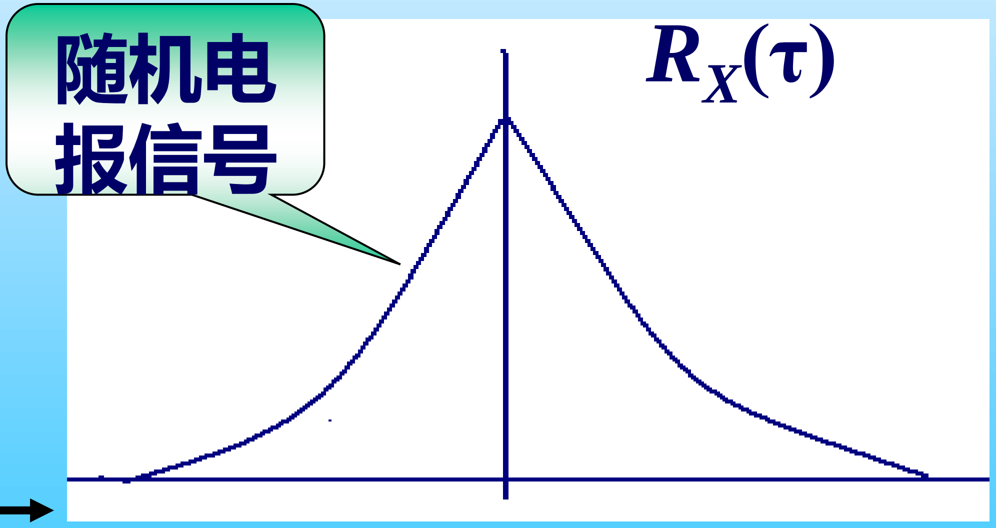
1) $R(0)\geq 0$;

2) $|R_X(\tau)|\leq R_X(0)$;

3) $R_X(-\tau)=R_X(\tau)$;

4) 具有非负定性.





Ex.1 讨论随机过程$\{X(t), t\geq 0\}$是否为平稳过程，其中$X(t)=\sin\omega t$，ω在$[0, 2\pi]$上服从均匀分布.

解 $R(s,t)=E(X(s)X(t))$

$$=\int_0^{2\pi}\frac{1}{2\pi}\sin\omega s\sin\omega t d\omega$$

$$=\frac{1}{4\pi}\left[\frac{\sin 2\pi(t-s)}{t-s}-\frac{\sin 2\pi(t+s)}{t+s}\right]$$

$R(s,t)$不是关于$s-t$的偶函数，故实随机过程$\{X(t),t\geq 0\}$不是平稳过程.

定理5.2.2 如果$\{X(t), t\in T\}$是周期为L的周期平稳过程，即有 $P\{X(t+L)=X(t)\}=1$，

则$R_X(\tau)$也是周期函数,有 $R(t+L)=R(t)$.

证 $$P\{X(0)\overline{X(t+L)}=X(0)\overline{X(t)}\}=1,$$

$$\Rightarrow E\{X(0)\overline{X(t+L)}\}=E\{X(0)\overline{X(t)}\},$$

$$\Rightarrow R_X(t+L)=R_X(t).$$

Ex.2 设平稳过程$X(t)$的相关函数为$R_X(\tau)$，且$R_X(L)=R_X(0)$，L为一个常数，$L>0$，试证：

$X(t+L)=X(t)$ 依概率为1成立；

定理5.2.2的逆

证 因 $R_X(0)-R_X(L)=0,$

由切比雪夫不等式，对 $\forall\varepsilon>0,$

$$P\{|X(t+L)-X(t)|>\varepsilon\}\le E|X(t+L)-X(t)|^2/\varepsilon^2$$

$$=\varepsilon^{-2}[R_X(0)-R_X(L)+R_X(0)-R_X(-L)]$$

$$=2\varepsilon^{-2}\operatorname{Re}(R_X(0)-R_X(L))=0,$$

共轭对称

$$\Rightarrow\quad P\{X(t+L)\ne X(t)\}=0,$$

定理5.2.3 平稳过程$\{X(t), t\in T\}$均方连续的**充要条件**是相关函数$R_X(\tau)$在 $\boxed{\tau = 0}$ 处连续，且此时$R_X(\tau)$一致连续.

证 **充分性** 设$R_X(\tau)$在$\tau=0$ 处连续, 则

对$\forall t_0 \in T$, $\quad \lim\limits_{t\to t_0} R_X(t-t_0) = R_X(0)$,

$$E[|X(t)-X(t_0)|^2] = E\{[X(t)-X(t_0)][\overline{X(t)-X(t_0)}]\}$$

$$= E[X(t)\overline{X(t)}] - E[X(t_0)\overline{X(t)}]$$

$$- E[X(t)\overline{X(t_0)}] + E[X(t_0)\overline{X(t_0)}]$$

$$= R_X(0) - R_X(t-t_0) + R_X(0) - R_X(t_0 - t)$$

$$= 2\,\mathrm{Re}(R_X(0) - R_X(t-t_0)) \to 0, \quad (as \quad t \to t_0).$$

即$X(t)$在T上均方连续.

必要性 **若$X(t)$在 $t=t_0$处均方连续, 即**

$$\lim_{\tau \to 0} E[|X(t_0+\tau) - X(t_0)|^2] = 0.$$

则 $|R_X(\tau) - R_X(0)|^2 = |E\{X(t_0)[\overline{X(t_0+\tau) - X(t_0)}]\}|^2$

$$\le E[|X(t_0)|^2] E[|X(t_0+\tau) - X(t_0)|^2]$$

$$\xrightarrow{\tau \to 0} 0$$

即$R_X(\tau)$在$\tau=0$处连续.

一致连续 对任意τ_0,

$$\left|R_X(\tau)-R_X(\tau_0)\right|^2=\left|E\{X(0)[\overline{X(\tau)-X(\tau_0)}]\}\right|^2$$

$$\le E[\left|X(0)\right|^2]E[\left|X(\tau)-X(\tau_0)\right|^2]$$

$$=R_X(0)*2\operatorname{Re}(R_X(0)-R_X(\tau-\tau_0))$$

$$\le 2R_X(0)\left|R_X(0)-R_X(\tau-\tau_0)\right|$$

即$R_X(\tau)$在任意$\tau=\tau_0$处的连续性可由在$\tau=0$处的连续性决定.

Ex.3 随机电报信号

$$X(t) = X_0(-1)^{N(t)}, \quad t \geq 0,$$

的自相关函数 $R(\tau) = C^2 e^{-2\lambda|\tau|}$

在$\tau = 0$ 处连续,从而$R(\tau)$在$(-\infty,\infty)$上连续.

故随机电报信号过程$\{X(t), t\geq 0\}$是均方连续, 均方可积的.

Ex.4 $\{X(t), t\geq 0\}$为均方可微的实宽平稳过程，试验证

$$Y(t)=\frac{dX(t)}{dt},\ t\in T$$

也是实宽平稳过程.

证 因二阶矩过程的导数过程也是二阶矩过程, 有

$$E[Y^2(t)]=E[|X'(t)|^2]<+\infty,\ t\in T$$

$\{Y(t), t\in T\}$是二阶矩过程.

$$E[Y(t)] = E[X'(t)] = \frac{dE[X(t)]}{dt} = 0$$

$$R_Y(t_1, t_2) = E[X'(t_1)X'(t_2)] = \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_2 - t_1)$$

$$= \frac{\partial^2}{\partial t_1} R'_X(t_2 - t_1) = -R''_X(t_2 - t_1) = R_Y(t_2 - t_1)$$

故$\{Y(t), t \in T\}$也是实宽平稳过程.

定理5.2.4 对于平稳过程$X_T=\{X(t), t\in T\}$

1) X_T均方可微的一个**充分**条件是$R_X(\tau)$在$\tau=0$处二次连续可微;

2）若X_T均方可微, 则其均方导数过程仍为平稳过程,有

均值函数 $m_{X'}(t)=0,$

相关函数 $R_{X'}(\tau)=-R''_X(\tau).$

互相关函数 $R_{XX'}(s,t)=R'_t(s,t)=R'_X(\tau)\quad(\tau=t-s)$

$$R_{X'X}(s,t)=R'_s(s,t)=-R'_X(\tau)$$

证 1) 由均方可微准则, X_T均方可微 ⟺

相关函数$R(s, t)$在(t_0, t_0)处广义二阶可微,即

$$\lim_{\substack{\Delta t\to 0\\ \Delta s\to 0}}\frac{1}{\Delta t\Delta s}[R(t_0+\Delta t,t_0+\Delta s)-R(t_0+\Delta t,t_0)-R(t_0,t_0+\Delta s)+R(t_0,t_0)]$$

$$=\lim_{\substack{\Delta t\to 0\\ \Delta s\to 0}}\frac{1}{\Delta t\Delta s}[R(\Delta s-\Delta t)-R(-\Delta t)-R(\Delta s)+R(0)]$$

平稳性

存在.

2) 设$X_T=\{X(t), t\in T\}$均方可导, 则

$$m_{X'}(t)=E[X'(t)]=\frac{d}{dt}E[X(t)]=\frac{d}{dt}(m_X)=0$$

$$R_{X'}(s,t)=E[X'(s)\overline{X'(t)}]=\frac{\partial^2}{\partial t\partial s}R_X(t-s)$$

$$=\frac{\partial}{\partial s}R'_X(t-s)=-R''_X(t-s)=-R''_X(\tau)$$

$\{X'(t),t\in T\}$是平稳过程.

推论：平稳过程均方可微的必要条件：
自相关函数一阶可微。

续Ex.3 随机电报信号$\{X(t),t\geq 0\}$的自相关函数：

$$R(\tau)=C^2e^{-2\lambda|\tau|}$$

有 $R'_X(0+)=-2\lambda C^2$， $R'_X(0-)=2\lambda C^2$

$R'_X(0)$不存在 $\Rightarrow$ **$X(t)$均方不可导.**

Ex.5 设实平稳过程$\{X(t), t\in T\}$的相关函数为

$$Ae^{-\alpha|\tau|}(1+\alpha|\tau|)$$

其中A、α均为常数，$\alpha>0$，求

$$Y(t)=\frac{dX(t)}{dt}$$

的相关函数.

解 当$\tau\neq 0$,

$$R_Y(\tau)=-R_X''(\tau)=-Ae^{-\alpha|\tau|}(-\alpha^2+\alpha^3|\tau|);$$

$$R_Y(0)=E\{[Y(t)]^2\}=E\{[\underset{\Delta t\to 0}{\mathrm{l.i.m}}\frac{X(t+\Delta t)-X(t)}{\Delta t}]^2\}$$

$$= \lim_{\Delta t \to 0} \frac{2[R_X(0) - R_X(\Delta t)]}{(\Delta t)^2} = A\alpha^2.$$

左右导数存在并相等

推论1 设$\{X(t), t \in T\}$是均方可微的实平稳过程, 则对$\forall t \in T, X(t)$与$X'(t)$不相关.

证 $\{X(t), t \in T\}$是实平稳过程

$$\Rightarrow \quad E[X(t)X'(t)] = \frac{\partial}{\partial t} R_X(t,t) = R'_X(0)$$

又因 $R_X(-\tau) = R_X(\tau) \Rightarrow \quad R'_X(-\tau) = -R'_X(\tau)$

特别 $R'_X(0) = -R'_X(0) \quad \Rightarrow R'_X(0) = 0,$

$\Longrightarrow$ $E\left[X(t)X'(t)\right]=0$, 即$X'(t)$与$X(t)$不相关.

推论2 $\{X(t),t\in T\}$是均方可微的**实正态**平稳过程, 则对$\forall t\in T$, $X(t)$与$X'(t)$相互独立.

定理5.2.5 设$\{X(t),t\in T\}$是均方连续的平稳过程, 则在有限区间上, 均方积分

$$\int_a^b X(t)dt$$

存在, 且有

$$E[\overline{\int_a^b X(s)ds}\int_a^b X(t)dt]=\int_a^b\int_a^b R_X(t-s)dsdt$$

特别若$\{X(t), t\in T\}$是实平稳过程,则

1) $E[\int_a^b X(t)dt] = m_X(b-a)$;

2) $E[\int_a^b X(t)dt]^2 = 2\int_0^{b-a}[(b-a)-|\tau|]R(\tau)d\tau.$

证 由均方可积准则及过程的平稳性可得

$$E[\int_a^b X(s)ds\overline{\int_a^b X(t)dt}] = \int_a^b\int_a^b R_X(t-s)dsdt$$

当$\{X(t), t\in T\}$是实平稳过程

$$\Rightarrow \begin{cases} E[X(t)] = m_X\text{是常数}, \\ R_X(s,t) = R_X(\tau)\text{是偶函数}, \end{cases}$$

故 1) $E[\int_a^b X(t)dt] = \int_a^b m_X dt = m_X(b-a);$

2) $E[\int_a^b X(t)dt]^2 = \int_a^b\int_a^b R(t-s)dsdt$

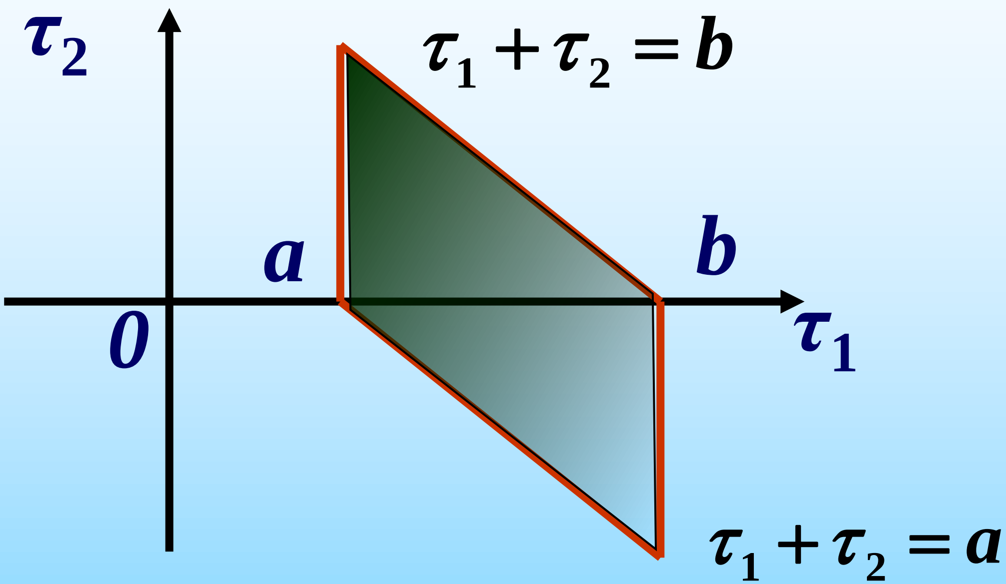
做积分变换，令

$$\begin{cases} \tau_1 = s \\ \tau_2 = t-s \end{cases} \qquad 则\ |J| = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

将 $D = \{(s,t) | a \le s \le b, a \le t \le b\}$ **变换为**

$$G = \{(\tau_1, \tau_2) | a \le \tau_1 \le b, a-\tau_1 \le \tau_2 \le b-\tau_1\}$$

$$G = \{(\tau_1, \tau_2) | a \le \tau_1 \le b, a - \tau_1 \le \tau_2 \le b - \tau_1\}$$

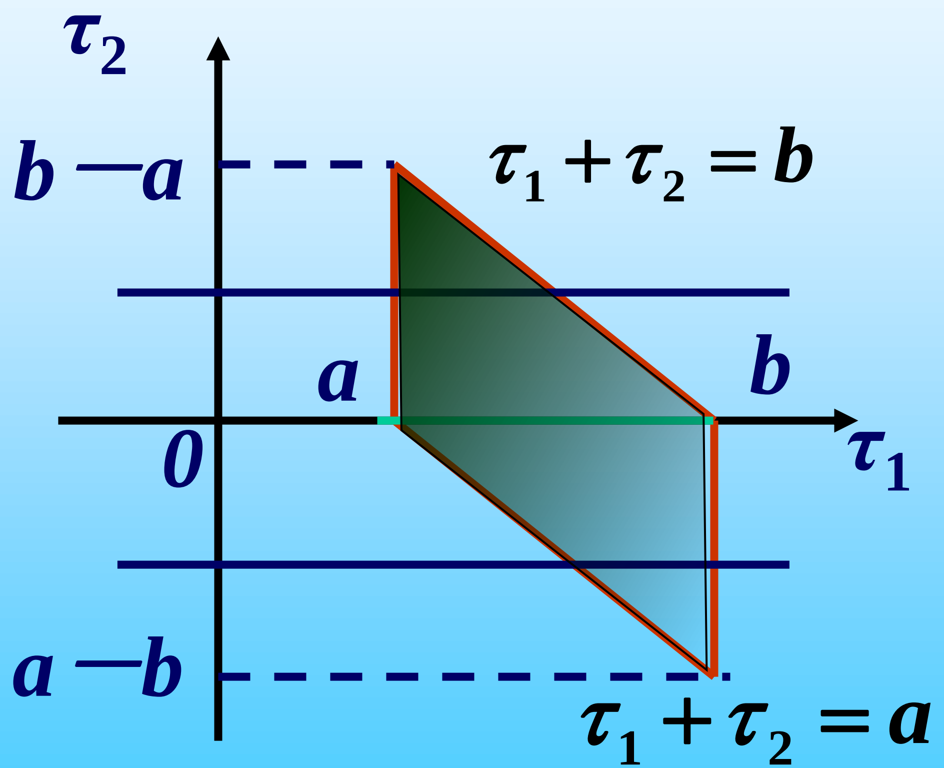


$$E[\int_a^b X(t)dt]^2 = \int_a^b \int_a^b R(t-s)dsdt$$

$$= \iint_G R(\tau_2) d\tau_1 d\tau_2$$

$$= \iint_G R(\tau_2) d\tau_1 d\tau_2$$

$$= \int_{a-b}^{0} R(\tau_2) d\tau_2 \int_{a-\tau_2}^{b} d\tau_1 + \int_{0}^{b-a} R(\tau_2) d\tau_2 \int_{a}^{b-\tau_2} d\tau_1$$



$$= \int_{a-b}^{0} R(\tau_2)d\tau_2 \int_{a-\tau_2}^{b} d\tau_1 + \int_{0}^{b-a} R(\tau_2)d\tau_2 \int_{a}^{b-\tau_2} d\tau_1$$

$$= \int_{a-b}^{0} [(b-a)+\tau_2]R(\tau_2)d\tau_2$$

$$+ \int_{0}^{b-a} [(b-a)-\tau_2]R(\tau_2)d\tau_2$$

$$= 2\int_{0}^{b-a} [(b-a)-|\tau|]R(\tau)d\tau$$

重要公式:

$$E[\int_a^b X(t)dt]^2 = 2\int_0^{b-a} [(b-a)-|\tau|]R(\tau)d\tau.$$

Ex.6 设实平稳过程$\{X(t),t\in T\}$的相关函数为$R_X(\tau)$，均值函数为0，若

$$Y(t)=\int_0^t X(t)dt$$

求$Y(t)$的自协方差函数和方差.

解 $m_Y(t)=\int_0^t m_X dt = 0\cdot(t-0)=0;$

$$C_Y(s,t)=R_Y(s,t)=\int_0^s dv\int_0^t R(u-v)du;$$

$$D[Y(t)]=C_Y(t,t)=R_Y(t,t)$$

$$=\int_0^t dv\int_0^t R(u-v)du=2\int_0^t (t-|\tau|)R_X(\tau)d\tau.$$

二、联合平稳过程的互相关函数

实用中需同时研究多个关联随机过程的统计规律.

要从输出中检测出有用信号，需同时研究输入信号和噪声的联合统计特性.

Ex.6 设$X(t)$是雷达的发射信号，遇到目标后的回波信号是 $aX(t-b)$, $a<<b$, b是信号返回时间,回波信号必然伴有噪音. 记噪音为$N(t)$,则接受机收到的全信号为

$$Y(t)=aX(t-b)+N(t),$$

需考虑$X(t)$与$N(t)$的联合统计特性.

又如 $\{X(t),t\in T\}$和$\{Y(t), t\in T\}$都是平稳过程,问$\{Z(t)= X(t)+ Y(t), t\in T\}$ 是否是平稳过程?

定义5.2.1 称平稳过程$\{X(t), t\in T\}$和平稳过程$\{Y(t), t\in T\}$为**联合平稳**的(平稳相关), 若对任意τ,
$$R_{XY}(s+\tau, t+\tau)=R_{XY}(s,t).$$

即
$$R_{XY}(s+\tau,t+\tau)=E[X(s+\tau)\overline{Y(t+\tau)}]$$
$$=E[X(s)\overline{Y(t)}]=R_{XY}(s,t).$$

互相关函数仅与$(t-s)$的大小有关.

可将联合平稳过程的互相关函数定义为
$$R_{XY}(t,t+\tau)=E[X(t)\overline{Y(t+\tau)}]=R_{XY}(\tau)$$

Ex.7 设$\{X(t),t\in T\}$和$\{Y(t),t\in T\}$是平稳相关的平稳过程,讨论$\{Z(t)=X(t)+Y(t),\ t\in T\}$的平稳性.

解 $m_Z(t)=E[X(t)+Y(t)]=m_X+m_Y;$

$$R_Z(t,t+\tau)=E[Z(t)\overline{Z(t+\tau)}]$$

$$=E\{[X(t)+Y(t)][\overline{X(t+\tau)+Y(t+\tau)}]\}$$

$$=R_X(\tau)+R_{XY}(\tau)+R_{YX}(\tau)+R_Y(\tau)$$

故$\{Z(t),t\in T\}$是平稳过程.

定理5.2.6 平稳相关的平稳过程的互相关函数$R_{XY}(\tau)$有下性质：

1) $R_{XY}(\tau)=\overline{R_{YX}(-\tau)}$,

当$X(t)$与$Y(t)$均为实平稳过程,则 $R_{XY}(\tau)=R_{YX}(-\tau)$.

2) $|R_{XY}(\tau)|^2 \le R_X(0)R_Y(0), |R_{YX}(\tau)|^2 \le R_X(0)R_Y(0)$;

3) 对任意复常数α, β, $\alpha X(t)+\beta Y(t)$也是平稳过程, 且其相关函数满足

$$R_{\alpha X+\beta Y}(\tau)=|\alpha|^2 R_X(\tau)+\alpha\overline{\beta}R_{XY}(\tau)$$
$$+\overline{\alpha}\beta R_{YX}(\tau)+|\beta|^2 R_Y(\tau)$$

Ex.8 设$X(t)$与$Y(t)$均为实平稳随机过程，且二者相互独立，令

$$Z(t)=X(t)Y(t),\qquad W(t)=2X(t)+Y(t),$$

试求$R_Z(\tau)$与$R_W(\tau)$.

解 因$X(t)$与$Y(t)$相互独立，有

$$R_Z(t,t+\tau)=E[X(t)Y(t)\overline{X(t+\tau)Y(t+\tau)}]$$
$$=E[X(t)\overline{X(t+\tau)}]E[Y(t)\overline{Y(t+\tau)}]$$
$$=R_X(\tau)R_Y(\tau)$$

$$R_W(\tau)=4R_X(\tau)+2R_{XY}(\tau)+2R_{YX}(\tau)+R_Y(\tau)$$
$$=4R_X(\tau)+R_Y(\tau)+4m_X m_Y.$$

其中因 $R_{YX}(\tau)=E[Y(t)X(t+\tau)]$

$$=E[Y(t)]E[X(t+\tau)]=m_X m_Y=R_{XY}(\tau).$$

思考题:

1) X是平稳过程，X和X'是联合平稳过程吗？

2) **为什么需要特别研究平稳过程的自相关和互相关函数?**

3) **联合平稳过程意义?**