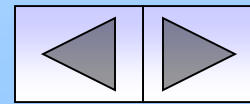


§ 1.3 随机变量的函数

两个问题

1. 随机变量的函数是否仍为同一概率空间上的随机变量?
2. 如何确定其分布?



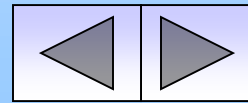
1. 确定随机变量的函数的分布

1) 设已知 (X_1, X_2, \dots, X_n) 的联合分布, 并有 k 个 n 元连续函数:

$$y_1 = g_1(x_1, \dots, x_n), \dots, y_k = g_k(x_1, \dots, x_n)$$

则 $Y_i = g_i(X_1, \dots, X_n)$, ($i=1, 2, \dots, k$) 是随机变量.
 (Y_1, Y_2, \dots, Y_k) 的联合分布函数为:

$$\begin{aligned} F(y_1, y_2, \dots, y_k) &= P\{Y_1 \leq y_1, Y_2 \leq y_2, \dots, Y_k \leq y_k\} \\ &= P\{g_1(X_1, \dots, X_n) \leq y_1, \dots, g_k(X_1, \dots, X_n) \leq y_k\} \end{aligned}$$



当 (X_1, X_2, \dots, X_k) 是连续型的随机变量时

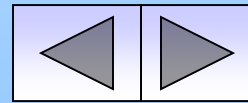
$$F(y_1, y_2, \dots, y_k) = \int \cdots \int_D f(x_1, \dots, x_n) dx_1 \cdots dx_n$$

其中 $D = \{(x_1, \dots, x_n) : g_i(x_1, \dots, x_n) \leq y_i, i = 1, 2, \dots, k\}$

当 (X_1, X_2, \dots, X_k) 是离散型的随机变量时

$$p(y_1, y_2, \dots, y_k) = \sum_{(x_1, \dots, x_n) \in D} P\{X_1 = x_1, \dots, X_n = x_n\}$$

其中 $D = \{(x_1, \dots, x_n) : g_i(x_1, \dots, x_n) = y_i, i = 1, 2, \dots, k\}$

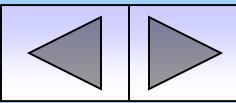


2) 二维随机变量的变换

定理1.3.1 设 (X_1, X_2) 的联合密度为 $f(x_1, x_2)$,若函数

$$\begin{cases} y_1 = g_1(x_1, x_2); \\ y_2 = g_2(x_1, x_2). \end{cases}$$

满足下述条件:



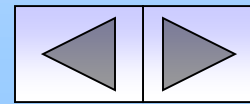
① 存在惟一反函数
$$\begin{cases} x_1 = x_1(y_1, y_2), \\ x_2 = x_2(y_1, y_2). \end{cases}$$

② 有连续的一阶偏导数;

③ Jacobi行列式
$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} \neq 0$$

则 $Y_1 = g_1(X_1, X_2)$, $Y_2 = g_2(X_1, X_2)$ 的联合概率密度为

$$f[x_1(y_1, y_2), x_2(y_1, y_2)] |J|$$



证 Y_1, Y_2 是随机变量, 其联合分布函数为

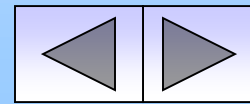
$$F(y_1, y_2) = \iint_{\mathbf{D}} f(x_1, x_2) dx_1 dx_2$$

其中 $\mathbf{D} = \{(x_1, x_2) : g_1(x_1, x_2) \leq y_1, g_2(x_1, x_2) \leq y_2\}$

做积分变换

$$\begin{cases} x_1 = x_1(u_1, u_2); \\ x_2 = x_2(u_1, u_2). \end{cases}$$

$$F(y_1, y_2) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} f[x_1(u_1, u_2), x_2(u_1, u_2)] |J| du_1 du_2$$



$$\begin{aligned}\Rightarrow f(y_1, y_2) &= \frac{\partial^2 F(y_1, y_2)}{\partial y_1 \partial y_2} \\ &= f[x_1(y_1, y_2), x_2(y_1, y_2)] |J|\end{aligned}$$

Ex.2 和的分布 $Y=X_1+X_2$

$$\text{令 } \begin{cases} y_1 = x_1 \\ y_2 = x_1 + x_2 \end{cases} \Rightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_2 - y_1 \end{cases}$$

反函数

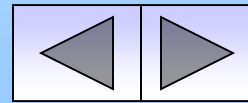
$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1,$$

得 $Y_1=X_1$ 和 $Y_2=X_1+X_2$ 的联合密度为

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(y_1, y_2 - y_1) |J|$$

关于 $Y_2=X_1+X_2$ 的边缘密度为:

$$f_{X_1+X_2}(y_2) = \int_{-\infty}^{+\infty} f_{X_1, X_2}(y_1, y_2 - y_1) dy_1$$



若 X_1, X_2 相互独立, 则

$$f_{X_1+X_2}(y_2) = \int_{-\infty}^{+\infty} f_{X_1}(y_1) f_{X_2}(y_2 - y_1) dy_1$$

称为两个函数的
卷积或褶积

