2D Image Data Approximation using Savitzky Golay Filter - Smoothing and Differencing

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***Abstract*—Smoothing and differencing is one of the major important and necessary step in the field of signal processing, image processing and also in the field on analytical chemistry. The search for an efficient image smoothing and edge detection method is a challenging task in image processing sector. Savitzky Golay Filters are one among the widely used filters for analytical chemistry. Even though they have exceptional features, they are rarely used in the field of image processing. The designed filter is applied for image smoothing and a mathematical model based on partial derivatives is proposed to extract the edges in images. The smoothing technique of SG filter offers an extremely simple aid in extracting the edge information. An approach using SG filter which can be applied in preserving edge information is one of the major tasks involved in the classification process in the domain of Optical Character Recognition. The paper is focused on designing the Savitzky Golay filter by using the concepts of linear algebra. The main objective of the paper is to portray a clear cut idea about Savitzky Golay filter and to study the design of Savitsky Golay filters based on the concepts of Linear Algebra.**

Keywords:- Savitzky Golay Filter, Least Square method, image processing.

I. INTRODUCTION

An image is a two-dimensional analog or digital signal that gives the visual representation of the co-ordinates along x-axis and y-axis. In the analog image processing, the images are represented over a continuous range of both intensity and its position. In digital image processing the intensities and positions of an image are represented in a discrete set of values. Since most of the images are likely to encounter with noises such as White noise, Salt and pepper, Gaussian noise etc.. processing images becomes a tedious task. Some filters which are used for handling such type of noises include mean filter, median filter, Gaussian smoothing filter etc. Savitzky Golay (SG) filter has been used in the research of analytical chemistry and physics [4][13]. Median filtering is needed to remove impulse noise from an image. Pixel intensity is replaced with the median of pixel intensities within a window centered at that pixel. If a part of the window falls outside of the image, intensities within the portion of the window inside the image is used [9]. Mean filtering is image averaging and the value at a pixel in the output is set to the average of values within a circular window centered at the pixel in the input. The window size determines the neighborhood size where averaging is performed. As the window size increases more noises will be removed, but at the same time the image will be more blurred [9]. A Gaussian is an ideal filter in the sense that it reduces the magnitude of high spatial frequencies in an image proportional to their frequencies. That is, the magnitude of higher frequency components are reduced. This will be at the cost of more computation time when compared to mean filtering [9]. The SG filter is one of the filters that are not so far used in the signal processing community [3].

The main goal of this paper is to describe the design of SG filter with the aid of concepts of Linear Algebra. The filter designed can be used to perform smoothing and differencing operations in images. One of the preprocessing steps involved in OCR is noise removal of the input image, but there is a possibility of losing edge information during this process. In this paper, we discusses an approach using SG filter which can be applied in preserving edge information which is one of the major tasks involved in the classification process in the domain of Optical Character Recognition. Savitzky Golay (SG) filter has been developed and accepted in the field of analytical chemistry [4]. Unlike other filters, Savitzky Golay filter derive the information directly from the time domain itself. SG filter is designed using polynomial basis to give an approximate representation of the signal values. In case of one dimensional signal, let the length of the sampled signal be N and the filter length be 2M+1. Here filter length acts as a window which is convolved with the input signal. In this convolution operation, an approximate representation of the input signal sample is obtained. In other words, input signal sample is fitted through polynomial basis [8]. In each window, the center sample is approximated. In order to find the approximate representation for the all the signal samples, the window is convolved with the signal. Towards the end, this operation gives a smoothed version of the input signal sample.

II. SG FILTER DESIGN USING LINEAR ALGEBRA CONCEPTS

A signal is represented in a vector format and an image is represented in the matrix format. If the data is of size $m \times n$, then it can be represented using the basis from the space $R^{m\times n}$. In general, any transformation of a signal

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can be visualized as the projection of input signal data onto each basis of transformation and the corresponding coefficient is determined. This can be represented using the following equation.

$$y = Ax$$

where y is the observed signal or observed image ,A is the basis matrix of DCT /wavelet /warsh hadamand etc and x is the unknown coefficient vector to be determined. To construct the matrix of polynomial bases, let us consider the filter length to be 2M+1 which ranges between $-M$ to M. The coefficients of the first few bases represent the low pass filter. Similarly, the middle few bases and last few bases represent the band-pass filter and high pass filter respectively.

Consider the length of the sampled signal to be N and the window length to be 2M+1, designed using the values ranging from $-M$ to M, i.e. $-M, -(M-1), ...0, 1, 2, ..., M$. Consider the matrix A of dimension $(2M + 1) \times n$ which spans $R^{(2M+1)\times n}$ space. The columns of the matrix are the basis vector. Here the number of basis (n) is less than the signal length $(n < N)$. So y cannot be exactly represented using A and x then the above equation is represented as,

$$y \approx Ax$$

The signal is approximated using polynomial basis so low pass filtering is used. The length of the polynomial basis varies from $-M$ to M. Fig. 1 represents the polynomial basis of length $2M + 1$. The order of these basis are $0, 1, 2$ and 3. When the order is zero, the degree of t is zero and the corresponding basis vector is $u\,(t) = 1$ where t varies from $-M$ to M. When the order is one the degree of t is one and the corresponding basis vector is $u\,(t) = t$ which is a linear equation. When the order is two the degree of t is two and the corresponding basis vector is $u\,(t) = t^2$ which is a quadratic equation. When the order is three the degree of t is three and the corresponding basis vector is $u\,(t) = t^3$. The representation of the basis in matrix form is given as,

$$y = \begin{pmatrix} y(-m) \\ y(-m-1) \\ \vdots \\ 0 \\ \vdots \\ y(m) \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -m & m^2 & \cdots & m^4 \\ 1 & -(m-1) & (m-1)^2 & \cdots & (m-1)^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & m & m^2 & \cdots & m^4 \end{pmatrix}$$

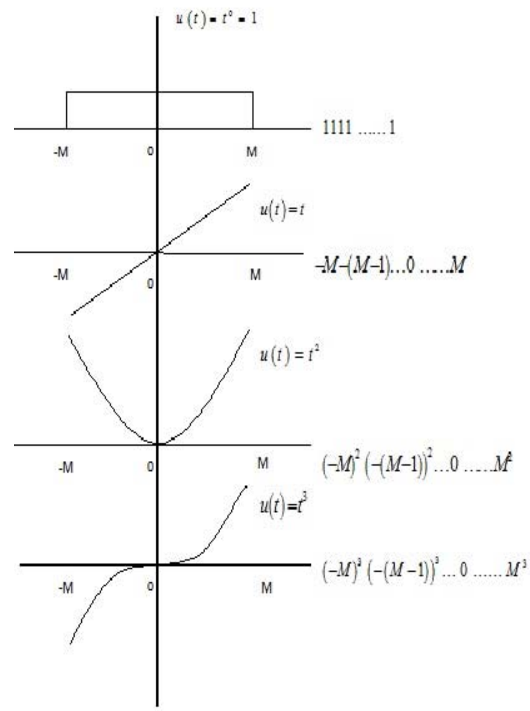

Fig. 1. Polynomial bases

$$x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

In fig. 2 let x_1 and x_2 be the column vectors of matrix A of size $(2M + 1) \times 2$. y is another vector which should span the column space A. From the figure it can be interpreted that y does not lie in the column space of A. In order to overcome this, y is projected on to the basis vectors. Projection of y onto the columns of A we get the error vector e. The error vector shows that y and $\hat{y}$ lie in different subspace [8]. In order to minimize the error vector e, least square method is used. In order to fit the ordered points (x, y) in an image linear least squares methods are used but always with such methods we need to specify the degree of the polynomial that is required to approximate with length of the array. There is no generic way that provides the degree of approximation required to fit to the data.

Higher-order polynomial filters are to be avoided for filtering because at higher frequencies signal gets even more quirky (this cause smoothing issues). The data is expressed as a linear combination of basis of the subspace. The data remains same but expressed using a different set of basis. The vector is projected orthogonal to the subspace. The point is to find the

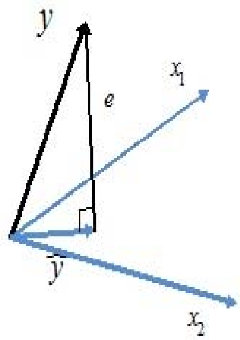

Fig. 2. Projection of y on to column space of A

vector $\hat{y}$ on the subspace that is closest to y which is the projection of y onto the subspace. Least square is a method to approximate data based on an equation that best fits the data variation.

$$y \approx Ax$$

The solution for the least square is given as,

$$\hat{y} = A(A^tA)^{-1}A^ty$$

Since the columns of A are independent, matrix A^tA is invertible. The error term is handled in the above expression. Let $P = A(A^tA)^{-1}A^t$, where P is the projection matrix. Therefore the approximated coefficient values are given by the equation,

$$\hat{y} = Py$$

A. SG filters for 1D data

Let N be the length of the sampled signal and $2M+1$ be the length of Savitzky Golay filter. The filter is moved as a window over the entire signal for approximating the signal values [11][12]. In each step the center value of the original signal within the approximation interval is replaced by the approximated value. From the above equation y is the signal samples; P is the projection matrix and $\hat{y}$ is the smoothened values. Consider the window length to be from $-M$ to M, totally $2M+1$ samples.

The equation of projection matrix along with the dimension is expressed as,

$$P_{m\times m} = A_{m\times n}(A^tA)^{-1}_{n\times n}A^t_{n\times m}$$

If $C = (A^tA)^{-1}A^t$ By evaluating last two equation, the projection matrix is obtained as

$$P_{m\times m} = A_{m\times n}C_{n\times m}$$

If a signal contains $2M+1$ samples, then $2M+1$ basis is required for exact representation of the signal samples. Let the maximum degree of polynomial be 4 and M=2. Therefore -M to M gives $-2, -1, 0, 1, 2$. In this case, the number of basis is less than the length of the signal, N. So with the help of least square method an approximate solution is found which gives approximate representation of the signal.

$$\hat{y}_{5\times 1} = P_{5\times 5} \;\; y_{5\times 1}$$

Where, $P_{m\times m} = A_{m\times n}(A^tA)^t_{n\times n}A^t_{n\times m}$ The corresponding matrix representations are,

$$A = \begin{bmatrix} (-M)^0 & (-M)^1 & (-M)^2 & (-M)^3 & (-M)^4 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ M^0 & M^1 & M^2 & M^3 & M^4 \end{bmatrix}$$

$$A^t = \begin{bmatrix} (-M)^0 & . & . & . & M^0 \\ (-M)^1 & . & . & . & M^1 \\ (-M)^2 & . & . & . & M^2 \\ (-M)^3 & . & . & . & M^3 \\ (-M)^4 & . & . & . & M^4 \end{bmatrix}$$

From the equation,

$$\hat{y}_{5\times 1} = P_{5\times 5} \;\; y_{5\times 1}$$

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \hat{y}_4 \\ \hat{y}_5 \end{bmatrix}_{5\times 1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}_{5\times 5} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}_{5\times 1}$$

$$\hat{y}_3 = (y_1.a_{31}) + (y_2.a_{32}) + (y_3.a_{33}) + (y_4.a_{34}) + (y_5.a_{35})$$

Here $y_1, y_2 \dots y_5$ represents the data samples of length 2M+1. The values $\widehat{y}_1, \widehat{y}_2 \dots \widehat{y}_5$ represent the approximated values. In each approximation only the center value is replaced which can be obtained by projecting the data onto the center row or the matrix i.e. $\widehat{y}_3$ will be replaced in place of y_3. So instead of projecting the data onto the entire bases, it is enough to project the data only to the center row, thus reducing the computation. In other words we take the center row of the projection matrix P as the filter and convolve it with the signal to get the approximated signal representation.

B. SG filter for 2D data

An image consists of two coordinates x and y. Each pixel value is represented using these two coordinates. If x and y coordinates are represented as the rows and columns partition of a matrix, then the pixel value will be the matrix entries.

Fig. 3. Image represented as blocks of equal size

As shown in fig. 3 the image is split and divided into blocks of equal size i.e.. the matrix is split into sub matrices each of size 5×5 i.e.. there will be 25 pixel values in each matrix. Take the first 5×5 matrix and linearize it to a vector of size 25×1. To represent these 25 pixel values 25 bases vectors are required. The basis for Savitzky Golay filters are polynomial values. According to the polynomial order chosen, the number of basis varies. In this case for a 2-D vector the polynomial order of 3 is fixed and then it is represented using 10 bases [5]. They are X_0 varies from $\{-2,-1,0,1,2\}$. Similarly Y_0 varies from $\{-2,-1,0,1,2\}$.

$$\{x_0^0y_0^0, x_0^1y_0^0, x_0^0y_0^1, x_0^1y_0^1, x_0^0y_0^2, x_0^2y_0^0, x_0^3y_0^0, x_0^0y_0^3, x_0^1y_0^2.x_0^2y_0^1\}$$

Let d represent the data samples. So a basis vector of the same length created that is each base vector will be of length 25×1. There are ten bases, so the basis matrix will be of size 25×10. There are 25 data values (pixel values) but only 10 bases are available. So the approximation of the remaining values into the space formed by these bases are obtained using least square method. d is the data samples and $\hat{d}$ is the approximated value.

$$\widehat{d} = Pd$$

Where the projection matrix is given as,

$$P = \begin{pmatrix} x_0^0y_0^0 & x_0^1y_0^0 & x_0^0y_0^1 & x_0^1y_0^1 & x_0^0y_0^2 & \cdots & x_0^0y_0^3 \\ x_1^0y_1^0 & x_1^1y_1^0 & x_1^0y_1^1 & x_1^1y_1^1 & x_1^0y_1^2 & \cdots & x_1^0y_1^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{24}^0y_{24}^0 & x_{24}^1y_{24}^0 & x_{24}^0y_{24}^1 & x_{24}^1y_{24}^1 & x_{24}^0y_{24}^2 & \cdots & x_{24}^0y_{24}^3 \end{pmatrix}$$

We take a block of data multiply it with the projection matrix to get the smoothened coefficient value. Then we take only the coefficient $\widehat{d}_{13}$ and replacing it in the data in place of d_{13} which lies in the middle of the window. $\widehat{d}_{13}$ is the smoothened coefficient and is obtained by projecting the data vector onto the center row of the projection matrix.

$$\begin{bmatrix} \hat{d}(0) \\ \vdots \\ \hat{d}(13) \\ \vdots \\ \hat{d}(24) \end{bmatrix}_{25\times 1} = \begin{bmatrix} 1 & x_0 & . & x_0y_0^2 & y_0^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{13} & . & x_{13}y_{13}^2 & y_{13}^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{24} & . & x_{24}y_{24}^2 & y_{24}^3 \end{bmatrix}_{25\times 25} \begin{bmatrix} d(0) \\ \vdots \\ d(13) \\ \vdots \\ d(24) \end{bmatrix}_{25\times 1}$$

The data is shifted one sample to the right and the samples are taken to be of same length of 2M+1. Then multiply it with the projection matrix to get the second approximated value. The same procedure is repeated for the entire size of an image. Only the center value is required from the approximated vector, which is obtained by projecting the data onto the center row of the projection matrix. So instead of data being shifted and multiplied with the center row of projection matrix (SG filter), here the reverse procedure is done. The center row is converted into a matrix that matches the size of the data block. The row vector of size 1×25 is converted to a matrix of size 5×5. Now this block can be shifted and multiplied with the data blocks. This is nothing but convolution. For each approximation the middle value is replaced with the original image value at the same position. Zero padding is done in order to obtain the convolved output which should be of the same size of the input image.

After the process of image smoothing the second order partial derivatives u_{xx}, u_{yy}, u_{xy} are taken along vertical,horizontal and diagonal directions respectively and u is the smoothened image.

III. **TOTAL VARIATION FILTERING (DENOISING)**

Image restoration becomes harder to accomplish if the image is damaged (blur or noise). Such image could be improved using total variation technique. The two main limitations in image accuracy are categorized as blur and noise. Blur is intrinsic to image acquisition systems and second main image perturbation is noise.

Denoising is the problem of removing noise from an image. Total variation (TV) regularization is a technique that was originally developed for image denoising by Rudin, Osher, and Fatemi [10][1]. The TV regularization technique has since been applied to a multitude of other imaging problems [14]. The model is designed to remove noise and other unwanted fine scale details, while preserving sharp discontinuities (edges). In its original formulation, the ROF model [10] is defined as the constrained optimization problem

$$\min_u \left\{ \int_\Omega |\nabla u| \, d\Omega \right\} \text{s.t} \int_\Omega (u-f)^2 d\Omega = \sigma^2$$

where f is the observed image function which is assumed to be corrupted by Gaussian noise of variance σ^2 and u is the unknown denoised image. The above equation can be transformed into an unconstrained (or Lagrangian) model.

$$\min_u \left\{ \int_\Omega |\nabla u| \, d\Omega + \frac{1}{2\lambda} \int_\Omega (u-f)^2 d\Omega \right\}$$

where λ is the Lagrangian multiplier.
By solving the Euler Lagrangian equation the solution for the above minimization problem is given by [1],

$$(u-f) - \frac{1}{\lambda} \frac{u_{xx}u_y^2 - u_x u_y u_{xy} + u_{yy}u_x^2}{\left(u_x^2 + u_y^2\right)^{3/2}} = 0$$

IV. PROPOSED METHOD

In image processing smoothing and edge detection are the two important sectors. Here SG filter has been applied for image smoothing and using second order partial derivatives edge detection is performed. A test input gray scale image is considered for this purpose. Smoothing is a noise reduction technique in image processing that preserves the edge information of an image. As a result of smoothing the noise spikes in an image are cut down or reduced and finally a blurred image is obtained. Since most of the images have noise (high intensity value) present in it, smoothing acts as a cutting edge method to get the smoothed version of the original signal. In the first stage Savitzky Golay filter is used to smoothen the data by approximating the data values.
The input image is divided into smaller blocks of equal size and SG filter of the same size is constructed. Approximate of the pixel values is done by applying the SG filter over the entire image. This is done by convolving the SG filter with the image. In each stage of convolution the convoluted coefficient is replaced with the image value corresponding to the center position of the SG filter. The filter is shifted and the same process continues till the end of the image. The approximated image is the smoothened SG filter output.

The smoothened output is used for edge detection. The purpose of using smoothened image as the input for edge detection is that, occurrence of noise points in the image can be misjudged, that is noise interference can be wrongly interpreted as real edge and vice versa. Since, as smoothing reduces problems due to noises, there is no misguiding of false edges or missing out the true edges of an image. One of the important feature in an image is its edges. The edges

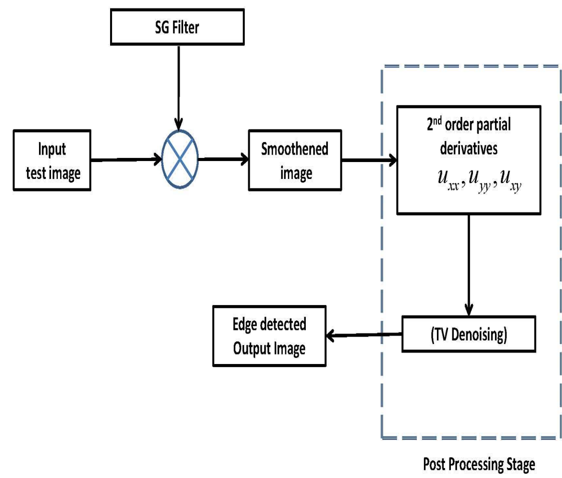

Fig. 4. Block diagram for the Proposed method

of the smoothened image are found by taking second order partial derivatives u_{xx}, u_{yy} and u_{xy}. The block diagram of the proposed work is shown in fig. 4
In the post processing step, fine detection of edges are found by eroding and with total variation filtering. Eroding wears out the grains in the image. Total variation method finely removes almost all the grains present in the image and provides a much better enhanced version of the derived image [15]. Edge detection is a fundamental operation done in image processing [6], pattern recognition and feature extraction. The highlight of edge detection technique is that it reduces the amount of data that is considered as irrelevant and filters them out leaving behind the important structural properties of an image. The edges that are extracted correspond to the boundaries of objects.

V. RESULTS AND DISCUSSIONS

Smoothing is a major task in many image processing applications. There is a possibility of losing edge information during the process of noise removal from the image. The process of smoothing is basically a noise reduction technique in the image processing sector. This piece of work, the image smoothing is carried out using SG filter. By applying SG filter the noise values are smoothened and the image information is retained. The effect of SG filter in image smoothing depends upon the order of polynomials also. Here, in this case we have considered the polynomial order to be 3. The image smoothing output varies with respect to the change in the order of polynomials.

A standard test image of size (256x256) is taken for experimental purpose as shown in fig. 5. SG filter is applied over the test image. The designed SG filter acts as the

Fig. 5. The test image and its corresponding smoothened image

Fig. 7. OCR smoothened image and its corresponding edge detected output

window which convolves with the test image to get the smoothened image which is shown in fig. 5. So when the smoothened image is passed as input for edge detection the edge values can be easily retrieved just by finding the partial derivatives along the coordinate axis. The partial derivatives extract the minute edge information by capturing almost all the variations from the smoothened image. In order to enhance the edge information by keeping the necessary edge values and removing the noises, TV de-noising is applied.

Fig. 6. The edge detected output of the images by taking partial derivatives

By default SG filter is a smoothing filter. But it cannot be used to extract the edge information. In order to extract edge information second order partial derivative of the smoothened image along the x (u_{xx}), y (u_{yy}) and x-y (u_{xy}) direction is taken which captures the edge information. Fig. 6 shows the edge detected output of the image by taking the partial derivatives. The application of TV on the edge detected image retains the continuous edge information and diffuses the other noises with the image background. It can be interpreted that TV denoising removes the noise from the edge detected image and also enhances the edge information. Thus by simple processing of the image edge information can be easily extracted. This method has been applied in the field of OCR and works well in extracting the edge information as shown in fig. 7. Getting the edge information is the major step in OCR applications. Based on this information the efficiency of the system varies. For further retrieval of sharp and quality

Fig. 8. Output obtained after post processing step(total variation)

edge information TV is applied as the post processing step as shown in fig. 8. The regularization operator lambda takes care of edge information in the image. Here the edge information is preserved and noises are removed. In the case of an OCR, no post processing step is required as the edges are sharpened and preserved in the edge detection step itself.

VI. CONCLUSION

The paper portrays a clear cut idea about Savitzky Golay filter which uses least squares that smoothens the image and the derivatives of the smoothened image is taken for preserving the edges of the image. SG Filter provides image smoothing without the loss of image resolution and this smoothing technique of SG filter offers an extremely simple aid in extracting the edge information. Total Variation Denoising method removes the noises present in the image and also enhances the edge information of the image. Results obtained in this approach indicate that, Savitzky Golay filter is also one of the most important filter for smoothing in the field of image processing. The proposed method explains the underlying concepts in a simple mathematical structure (linear algebra method) where even a non-mathematical background person can understand.

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