1 Univariate continuous probability distributions

In this chapter we shall study the probability distribution for continues random variable that occur prominently in statistical theory and application.

Examples of some of the distributions are uniform distribution, normal distribution ,Gamma distribution exponential and beta distribution. We shall discuss the exponential, uniform and the normal distribution.

1.1 The Uniform Distribution

A random variable X is said to be uniformly distributed over an interval (a, b) if it's density is given by

$$f(x) = \frac{1}{b-1}, a < x < b$$

This is written as $x \sim u(a,\, b)$where a and b are known as the parameters of distribution. to show it's p.d.f

$$\int_a^b f(x)dx = 1$$

$$\int_a^b \frac{1}{b-a}dx = [\frac{x}{b-a}]_a^b = \frac{b-a}{b-a} = 1$$

mean and variance

$$\mu = E(x) = \int xf(x)dx$$

$$= \int_a^b \frac{x}{b-a}dx = [\frac{x^2}{2(b-a)}]_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{(a+b)(b-a)}{2(b-a)} = \frac{a+b}{2}$$

$$E(x^2) = \int_a^b x^2 f(x)dx$$

$$= \int_a^b x^2 \frac{1}{b-a}dx$$

$$= \frac{x^3}{3(b-a)}]_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{b^3 - a^3}{3(b-a)} - \frac{(a+b)^2}{4}$$

$$= \frac{4(b^3 - a^3) - 3(b-a)(a+b)^2}{12(b-a)}$$

$$= \frac{(b-a)^3}{12(b-a)} = \frac{(b-a)^2}{12}$$

Cumulative distribution function

$$F(x) = p(X \leq x) = \int_a^x \frac{1}{b-a} du = \frac{u}{b-a}]_a^x = \frac{x-a}{b-a}$$

Therefore:

$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & x \geq b \end{cases}$$

Example

A random variable X follows a uniform distribution with p.d.f

$$f(x) = k, \;\; 3 \leq x \leq 6$$

Find

1. value of k
2. $E(x)$ and $var(x)$
3. $p(x > 5)$

Solution

1. $\int_3^6 f(x) = 1$

$$\int_3^6 k dx = 1$$

$$[kx]_3^6 = 1$$

$$6k - 3k = 1$$

$$3k = 1$$

$$k = \frac{1}{3}$$

2. $E(x) = \frac{a+b}{2} = \frac{6+3}{2} = 4.5$
   $Var(x) = \frac{(b-a)^2}{12} = \frac{(6-3)^2}{12} = \frac{3^2}{12} = \frac{9}{12} = \frac{3}{4}$

3. $p(x > 5) = \int_5^6 \frac{1}{3} dx = [\frac{1}{3}x]_5^6$
   $= \frac{6-5}{3} = \frac{1}{3}$

Example

The daily amount of coffee in liters dispensed by a machine located in an airport lobby is a random variable X having a uniform distribution with $a = 7, b = 10$. Find the probability that on a given day the amount of coffee dispensed by this will be

1. At most 8.8 liters
2. More than 7.4 liters but less than 9.5 liters
3. At least 8.5 liters

Solution

p.d.f

$$f(x) = \begin{cases} \frac{1}{10-7} = \frac{1}{3}, & 7 \leq x \leq 10 \\ 0, & elsewhere \end{cases}$$

1. $p(x \leq 8.8) = \int_7^{8.8} \frac{1}{3} dx$
   $= \frac{x}{3}]_7^{8.8} = \frac{8.8-7}{3} = \frac{1.8}{3} = 0.6$

2. $p(7.4 < x < 9.5) = \int_{7.4}^{9.5} \frac{1}{3} dx = \frac{1}{3}x \mid_{7.4}^{9.5} = \frac{9.5-7.4}{3} = \frac{2.1}{3} = 0.7$

3. $p(x \geq 8.5) = \int_{8.5}^{10}$
   $= \frac{1}{3}x]_{8.5}^{10} = \frac{10-8.5}{3} = \frac{1.5}{3} = 0.5$

1.2 Exponential Distribution