1 Moments

1.1 Raw Moments

Let X be a random variable with p.d.f $f(x)$,the r^{th} moment of X taken about the origin (or zero) is denoted by $\mu_r^{'}$ defined as $\mu_r^{'} = E(X^r - 0) = E(X^r)$. where $r = 0, 1, 2, 3, ...$

For discrete case:

$$\mu_r^{'} = E(X^r) = \sum_x x^r f(x)$$

and **for continuous case:**

$$\mu_r^{'} = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx$$

The r^{th}moment of X taken about the origin is called the **raw moment** of X.

- When $r = 1$, we have $\mu_1^{'} = E(X^1) = E(X) = \mu$. And this is the mean of X . i.e. the first raw moment of X is the mean of X.
- When $r = 2$, we have $\mu_2^{'} = E(X^2)$ which is the second raw moment.

1.2 Central Moments

Let X be a random variable with p.d.f $f(x)$, then the r^{th} moment about the **mean** of a random variable X, is called the r^{th}central moment of X and it is denoted by μ_r and defined as

$$\mu_r = E(X - \mu)^r$$

When X is discrete we have:

$$E(X - \mu)^r = \sum (X - \mu)^r f(x)$$

$r = 0, 1, 2, 3, - - - - -$

and for continuous case:

$$\mu_r = E(X - \mu)^r = \int_{\infty}^{\infty} (X - \mu)^r f(x)dx$$

When $r = 1$

$$\mu_1 = E(X - \mu)^1 = E(X) - \mu$$

$$= \mu - \mu = 0$$

i.e the first central moment is zero.
When $r = 2$

$$\mu_2 = E(X - \mu)^2$$

is the variance of the distribution of X or simply the variance of X

$$var(X) = E(X^2) - [E(X)]^2$$

$$\mu_2^1 - [\mu_1^1]^2$$

$$\mu_2^1 - \mu^2$$

The variance of a distribution is sometimes called the scale parameter since it determines the spread of the distribution.

The third central moment

The third central moment μ_3 is sometimes called a measure of the asymmetry; skewness ; coefficient of skewness and is given by:

$$= \frac{\mu_3}{\sigma^3} = \frac{E\left(X - \mu\right)^3}{\sigma^3}$$

$$\frac{E(x - \mu)^4}{s^4} - 3$$

The fourth central moment

The fourth central moment μ_4 is used as a measure of kurtosis which is the degree of peakedness near the center of distribution

$$kurtosis = \frac{\mu_4}{\sigma^4} - 3$$

$$= \frac{E\left(X - \mu\right)^4}{\sigma^4} - 3$$

Positive value indicates that a density is more peaked around its center. Negative value indicate that a density is more flat around its center.

Figure 1: Skewness

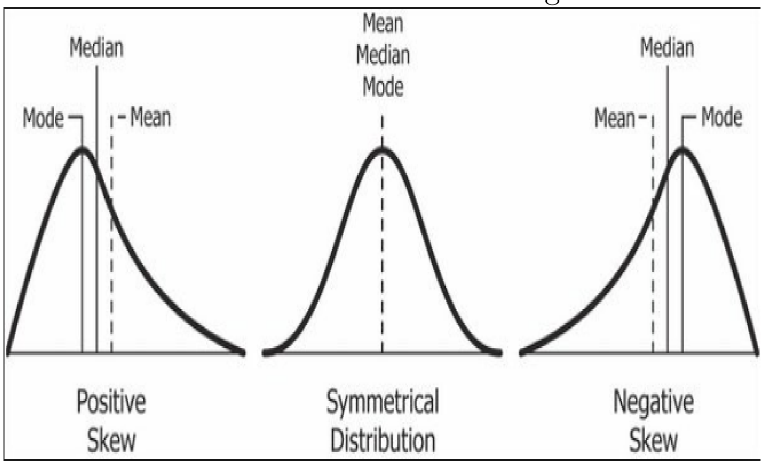
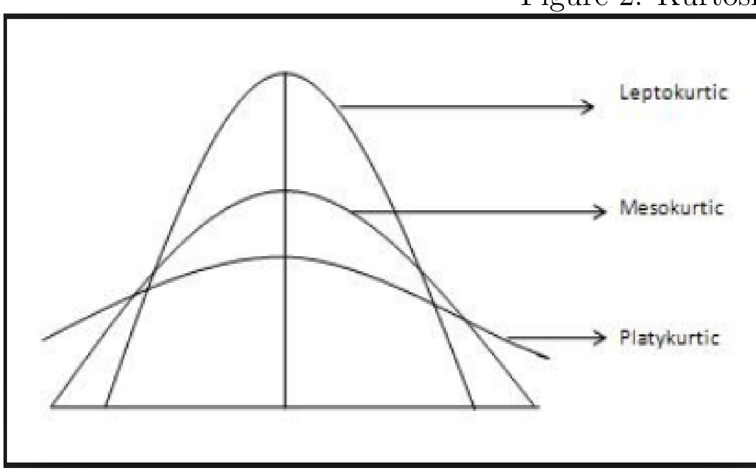


Figure 2: Kurtosis



2 Moment Generating Function

The purpose of moment generating function is in determining moments of distributions. It is a mathematical expression that sometimes (but not always) provides an easy way of finding moments associated with random variables. If a m.g.f exists it is unique and completely determines the distribution of the random variable **Definition**

Let X be a random variable with probability distribution function $f(x)$,then the m.g.f of X is defined as $M_X(t)$ *or* $M(t) = E(e^{tx})$ where t is any real number

If the random variable X is continuous then:

$$M(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

and if the random variable X is discrete then

$$M(t) = E(e^{tx}) = \sum_{x} e^{tx} f(x)$$

After obtaining the m.g.f we can use it to generate moments of a distribution. In particular we look at two ways of obtaining moments from a moment generating fuction namely, by expanding the m.g.f and by differentiating the m.g.f and setting t to zero.

2.1 Expanding the m.g.f

If we are interested in obtaining the r^{th} raw moment, we expand the m.g.f and obtain the r^{th} moment as the coefficient of $\frac{t^r}{r!}$ in the expansion of the m.f.f.

we know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + - - -$$

Then,

$$e^{tx} = 1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + - - -$$

$$M(t) = E(e^{tx})$$

$$= E[1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + - - - - - - -]$$

$$= 1 + E(tx) + E(\frac{(tx)^2}{2!}) + E(\frac{(tx)^3}{3!}) + - - - - - - -$$

$$= 1 + tE(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + - - - - - --$$

$$= 1 + t\mu_1^{'} + \frac{t^2}{2!}\mu_2^{'} + \frac{t^3}{3!}\mu_3^{'} + - - - - - --$$

$$= \sum_{r=0}^{\infty} \mu_r^{'} \frac{t^r}{r!}$$

$$= \sum_{r=0} E(x^r)\frac{t^r}{r!}$$

Thus the r^{th} raw moment $\mu_r^{'}$ of a distribution can be obtained simply as the coefficient of $\frac{t^r}{r!}$ in the expansion of $M(t)$

e.g.

- The first raw moment $E\left(X^1\right) = E\left(X\right) \Rightarrow r = 1$ hence the coefficient of $\frac{t^1}{1!} = t$ in the expansion above is $\mu_1^{'}$
- The second raw moment $E\left(X^2\right) \Rightarrow r = 2$ hence the coefficient of $\frac{t^2}{2!}$ in the expansion above is $\mu_2^{'}$ etc.

2.2 Differentiating the m.g.f.

Let X be a random variable with m.g.f $M\left(t\right)$ then to obtain the r^{th} raw moment, we simply differentiate the m.g.f r times and equate t to zero. i.e.

$$\frac{d^r M(t)}{dt^r}\Big|_{t=0}$$

Proof
Continuous case.
The m.g.f of X is given by:

$$M(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

To obtain the first raw moment we differentiate the m.g.f once with respect to t.
If we differentiate the m.g.f with respect to t we have

$$\frac{d}{dt}M(t) = \int_{-\infty}^{\infty} xe^{tx} f(x)dx = M(t)$$

Table 1: Excercise

X	0	1
$p(x)$	$1-p$	p

Putting $t=0$, in the above equation

$$M(t=0)=\int_{-\infty}^{\infty} xf(x)dx=E(x)=\mu$$

To obtain the second raw moment, we differentiate the m.g.f twice and again equate t to zero.

The second derivative of $M(t)$ is $M^{"}(t)$

$$M"(t)=\int_{-\infty}^{\infty} x^2e^{tx}f(x)dx$$

If we put $t=0$ in the above equation

$$M"(t=0)=\int_{-\infty}^{\infty} x^2f(x)dx=E(x^2)$$

In general if we differentiate the m.g.f r times with respect to t we have

$$\frac{d^{\,r}M(t)}{dt^{\,r}}=\frac{d^{\,r}E(e^{tx})}{dt^{\,r}}=E(x^re^{tx})$$

$$=\int_{-\infty}^{\infty} x^re^{tx}f(x)dx$$

If we let $t=0$ we get

$$\frac{d^{\,r}}{dt^{\,r}}M(t=0)=E(x^r)=\mu_r^1$$

The moments of a distribution may be obtained from the m.g.f by differentiating if it exists

Recall

$$Var(x)=E(x^2)-[E(x)]^2=M"(t=0)-[m'(t=0)]^2$$

Example 1

Let X be a discrete random variable which take only two possible values 0 and 1 with probabilities

Determine the m.g.f and use it to find the mean and variance of X

Solution

Table 2: Solution

X	0	1
$p(x)$	$1-p$	p
e^{tx}	1	e^t
$e^{tx}p(x)$	$1-p$	$e^t p$

$$M(t) = E(e^{tx}) = \sum_{x=0}^{1} e^{tx} f(x)$$

$$= \sum_{x=0}^{1} e^{tx} p(x)$$

$$= 1 \times (1-p) + e^t \times p$$

$$= (1-p) + pe^t$$

if we let $1-p=q$ we have:

$$M(t) = q + pe^t$$

which is the m.g.f

1. **Differentiating the m.g.f**

   Differentiating the m.g.f once

   $$M'(t) = pe^t$$

   Differentiating the m.g.f twice

   $$M"(t) = pe^t$$

   Differentiating m.g.f thrice

   $$M'''(t) = pe^t$$

and so on...

Obtaining mean and variance

Mean:

Using the above differentiations,

The mean is the first raw moment i.e. $E(X)$so we differentiate the m.g.f once and set $t = 0$. So

$$\frac{dM(t)}{dt} = M'(t) = pe^t$$

Setting t to zero we have:

$$\mu = E(x) = M'(t = 0) = pe^0 = p$$

Variance:

$$Var(X) = E\left(X^2\right) - \left[E(X)^2\right]$$

To obtain $E(X^2)$ the second raw moment, so we differentaite the m.g.f twice and set $t = 0$.

$$\frac{d^2M(t)}{dt^2} = M''(t) = pe^t$$

$$M'(t = 0) = E(x^2) = pe^0 = p$$

$$Var(x) = E(x^2) - [E(x)]^2 = M"(t = 0) - [M'(t = 0)]^2 = p - p^2 = p(1 - p) = pq$$

Alternative method: Exapanding the m.g.f.

$$M(t) = 1 - p + pe^t$$

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + - - -$$

hence:

$$= 1 - p + p(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + - - -)$$

$$= 1 - p + p + pt + \frac{pt^2}{2!} + \frac{pt^3}{3!} + - - - - - - -$$

$$= 1 + pt + p\frac{t^2}{2!} + p\frac{t^3}{3!} + - - - - - - -$$

The coefficient of $\frac{t^r}{r!}$ are all equal to p for $r = 1, 2, 3, - - -$hence $E(x^r) = p$ for $r = 1, 2, 3, - - -$

So $E\left(X^2\right) = p$ and $Var\left(X\right) = p - p^2 = p\left(1 - p\right)$
To verify that the moment of the random variable X is p using the definition of expectation

$$E(x) = \sum_{x=0} xp(x)$$

$$= 0 \times (1 - p) + 1 \times p = p$$

$$E(x^2) = \sum_{x=0} x^2 p(x)$$

$$= 0^2 \times (1 - p) + 1^2 \times p = p$$

$$E(x^3) = \sum_{x=0} x^3 p(x)$$

$$= 0^3 \times (1 - p) + 1^3 \times p = p$$

Example 2
Let X be a geometrical random variable with probability distribution given by:

$$f(x) \begin{cases} q^{x-1}p, & x = 1, 2, 3, - - - \\ 0 & otherwise \end{cases}$$

1. Find the m.g.f of X
2. Use the m.g.f to find the mean and variance of X.

Solution
a)

$$M(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} f(x)$$

$$= \sum_{x=1}^{n} e^{tx} q^{x-1} p$$

$$= pq^{-1} \sum_{x=1}^{n} (qe^t)^x$$

$$= pq^{-1}[(qe^t) + (qe^t)^2 + (qe^t)^3 + - - -]$$

$$= pq^{-1}qe^t[1 + (qe^t) + (qe^t)^2 + (qe^t)^3 + - - -]$$

The expansion in the bracket is a geometric progression: $\frac{a}{1-r}$, where $r = qe^t$ so the equation becomes:

$$= pq^{-1}qe^t \frac{1}{1 - qe^t}$$

$q^{-1}$and q cancel out, so we remain with m.g.f being:

$$M\left(t\right) = \frac{pe^t}{1 - qe^t}$$

provided $qe^t < 1$
b) Mean and variance of X

$$M(t) = \frac{pe^t}{1 - qe^t}$$

Mean of X
Differentiating the m.g.f once, using the quotient rule of differentiation we have:
Let pe^t be u and $1 - qe^t$ be v the the quotient rule is:

$$\frac{vdu - udv}{v^2}$$

$du = pe^t$ and $v = -qe^t$
so replacing the above in the quotient rule we have:

$$\frac{\left(1 - qe^t\right)pe^t - pe^t\left(-qe^t\right)}{\left(1 - qe^t\right)^2}$$

$$= \frac{pe^t\left[1 - qe^t + qe^t\right]}{\left(1 - qe^t\right)^2}$$

$$= \frac{pe^t}{\left(1 - qe^t\right)^2}$$

Setting $t = 0$ we have:

$$\frac{pe^0}{\left(1 - qe^0\right)^2} = \frac{p}{\left(1 - q\right)^2}$$

but $1 - q = p$ so we have

$$E\left(x\right) = \frac{p}{p^2} = \frac{1}{p}$$

Variance of X

$$Var\left(x\right) = E\left(X^2\right) - \left(E\left(X\right)\right)^2$$

$$E\left(X^2\right) = \frac{d^2 M(t)}{dt^2}\mid_{t=0}$$

$$\frac{d\frac{pe^t}{\left(1-qe^t\right)^2}}{dt}$$

Using the quotient rule again we have:
let $u = pe^t$ and $v = \left(1 - qe^t\right)^2$, $du = pe^t$ and $dv = 2\left(1 - qe^t\right)\left(-qe^t\right)$
Substituting in the quotient rule we have:

$$\frac{\left(1-qe^t\right)^2 pe^t - -2\left(1-qe^t\right) qe^t pe^t}{\left(1-qe^t\right)^4}$$

$$\frac{pe^t\left[\left(1-qe^t\right)^2 + 2qe^t - 2\left(qe^t\right)^2\right]}{\left(1-qe^t\right)^4}$$

$$= \frac{pe^t\left[1 - 2qe^t + \left(qe^t\right)^2 + 2qe^t - 2\left(qe^t\right)^2\right]}{\left(1-qe^t\right)^2}$$

which becomes:

$$\frac{pe^t\left[1-\left(qe^t\right)^2\right]}{\left(1-qe^t\right)^4}$$

Now $a^2 - b^2 = (a-b)(a+b)$, so $1 - \left(qe^t\right)^2 = 1^2 - \left(qe^t\right)^2 = \left(1 - \left(qe^t\right)\right)\left(1 + \left(qe^t\right)\right)$ so replacing in the above we have:

$$\frac{pe^t\left[1-\left(qe^t\right)^2\right]}{\left(1-qe^t\right)^4} = \frac{pe^t\left[\left(1-\left(qe^t\right)\right)\left(1+\left(qe^t\right)\right)\right]}{\left(1-qe^t\right)^4}$$

$$= \frac{pe^t\left[\left(1+\left(qe^t\right)\right)\right]}{\left(1-qe^t\right)^3}$$

Setting $t = 0$ we have:

$$\frac{pe^t\left[\left(1+\left(qe^t\right)\right)\right]}{\left(1-qe^t\right)^3} = \frac{pe^0\left[\left(1+\left(qe^0\right)\right)\right]}{\left(1-qe^0\right)^3} = \frac{p\left[\left(1+q\right)\right]}{\left(1-q\right)^3}$$

but $1-q=p$ so we have: $\frac{p[(1+q)]}{p^3}=\frac{1+q}{p^2}$

$$Var(x)=E(x^2)-[E(x)]^2$$

$$=\frac{1+q}{p^2}-\frac{1}{p^2}=\frac{1+q-1}{p^2}=\frac{q}{p^2}$$

Example 3
Let X be a random variable with p.d.f given as

$$f(x)=\begin{cases} e^{-x}, & x>0 \\ 0 & ,x\leq 0 \end{cases}$$

- Find the m.g.f of X
- Use the m.g.f to find the mean and variance of X
- Use the m.g.f to find an expression of μ_r^1

Solution
a)

$$M(t)=E(e^{tx})=\int_0^\infty e^{tx}f(x)dx$$

$$=\int_0^\infty e^{tx}e^{-x}dx=\int_0^\infty e^{-x}e^{-x(1-t)}dx$$

$$=\frac{-e^{-x(1-t)}}{1-t}\Big|_0^\infty$$

$$=\frac{1}{1-t}$$

Mean of X

$$E\left(X\right)=\frac{dM(t)}{dt}=(1-t)^{-2}$$

$$E(x)=\frac{dM(t)}{dt}\Big|_{t=0}=1$$

Variance of X, $Var\left(X\right)=E\left(X^2\right)-\left[E\left(x\right)\right]^2$,

$$E\left(X^2\right) = \frac{d^2M(t)}{dt^2} = -2(1-t)^{-3}(-1)$$

$$= 2(1-t)^{-3} = \frac{2}{(1-t)^3}$$

$$E(x^2) = \frac{d^2M(t)}{dt^2} \mid_{t=0} = 2$$

$$var(x) = E(x^2) - [E(x)]^2$$

$$= 2 - 1^2 = 2 - 1 = 1.0$$

c) **By expanding the M.G.F**

$$M(t) = \frac{1}{1-t}$$

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + - - - - - t^n$$

$$= 1 + \frac{1!t}{1!} + \frac{2!t^2}{2!} + \frac{3!t^3}{3!} + - - - - - - - \frac{r!t^r}{r!}$$

and hence

$$\mu_r^1 = r!, \quad where\ r = 1, 2, 3 - --$$

Some important results

i) let $Y = X + a$

$$M_y(t) = E(e^{ty}) = E[e^{t(x+a)}]$$

$$= E(e^{tx})e^{ta}$$

$$= e^{at}E(e^{tx})$$

$$= e^{at}M_x(t)$$

ii) let $y = bx$

$$M_y(t) = E(e^{ty}) = E(e^{tbx})$$

$$= E(e^{(tb)x})$$

$$= M_x(bt)$$

iii) Let

$$y = \frac{x+a}{b}$$

$$M_y(t) = E(e^{ty}) = E[e^{t(\frac{x+a}{b})}]$$

$$= E(e^{\frac{tx}{b}} e^{\frac{ta}{b}})$$

$$= e^{\frac{ta}{b}} E[e^{\left(\frac{t}{b}\right)x}]$$

$$= e^{\frac{at}{b}} M_x(\frac{t}{b})$$

Example

Given the m.g.f of a random variable X

$$M_x(t) = e^{3t+8t^2}$$

Find the m.g.f of the random variable Z where

$$z = \frac{1}{4}(x-3)$$

and use it to find the mean and variance of Z

Solution

$$M_z(t) = E(e^{tz})$$

$$= E(e^{t\frac{1}{4}(x-3)})$$

$$= E(e^{\frac{-3}{4}t} e^{\frac{t}{4}x})$$

$$= e^{\frac{-3}{4}t} E(e^{\left(\frac{t}{4}\right)x})$$

$$= e^{\frac{-3}{4}t} M_x(\frac{t}{4})$$

now $M_x\left(t\right) = e^{3t+8t^2}$therefore

$$M_x\left(\frac{t}{4}\right) = e^{3\frac{t}{4}+8\left(\frac{t}{4}\right)^2}$$

$$\Rightarrow e^{\frac{-3}{4}t}M_x(\frac{t}{4}) = e^{\frac{-3}{4}t}e^{3\frac{t}{4}+8\left(\frac{t}{4}\right)^2}$$

$$= e^{\frac{-3}{4}t}e^{\frac{3}{4}t+\frac{8t^2}{16}}$$

$$= e^{\frac{8t^2}{16}} = e^{\frac{t^2}{2}}$$

Therefore: $M_z\left(t\right) = e^{\frac{t^2}{2}}$

Mean of Z, $E\left(Z\right) = M_z^{'}\left(t\right) = te^{\frac{t^2}{2}}$

and

$$E\left(Z\right) = M_z^{'}\left(t\right)|_{t=0} = 0 \times e^{\frac{0^2}{2}} = 0$$

$$Var(z) = E(z^2) - [E(z)]^2$$

$$M_z^{''}\left(t\right) = 1.e^{\frac{t^2}{2}} + t^2e^{\frac{t^2}{2}}$$

$$E\left(X^2\right) = M_z^{''}\left(t\right)|_{t=0} = 1.e^{\frac{0^2}{2}} + 0 = 1$$

So variance of X

$$= 1 - 0^2 = 1$$

Other moment generating functions include factorial moment generating function, probability generating function etc.