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An optimized decomposition method for nonlinear ordinary and partial differential equations

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ABSTRACT

In this paper, firstly, an innovative decomposition method, called the optimized decomposition method, is suggested to analytically solve nonlinear ODEs. The proposed technique designs a new optimal construction of the series solutions based on a linear approximation of the nonlinear equation. Then, an efficient adaptation of the optimized decomposition method that will expand the application of the method to nonlinear PDEs is developed. Actual comparison between the suggested method and the Adomian decomposition method is carried out through numerical simulation of some test problems. The study demonstrates that the proposed method works successfully in dealing with nonlinear differential equations and gives better accuracy and convergence compared to Adomian decomposition method. The new proposed method reported in this work is believed to be implemented more widely to handle nonlinear models in applied sciences.

1. Introduction

The increasing interest in applications which deal with dynamical systems that exhibit nonlinearities has motivated the development and the investigation of analytical as well as numerical techniques to solve nonlinear ODEs and PDEs. The Adomian decomposition method (ADM), formulated first by Adomian [1,2], is one of the most robust computational techniques that are used to get analytically precise approximate solutions for large categories of nonlinear differential equations including ODEs, PDEs, integral equations, integro-differential equations, etc. A remarkable attractiveness of the Adomian's method is that it has demonstrated to be efficient and reliable in providing rapid convergent series solutions with straightforwardly computable terms. Particular attention is paid in [3] to discuss the convergence analysis of the method. According to its methodology, in order to solve nonlinear operator equation, the ADM requires the representation of any nonlinear expression in the form of a special infinite series known as Adomian polynomials. Consequently, some efforts have been made to calculate these polynomials [4,5]. Some adjustments, modifications and approaches have been developed significantly to extend the application, improve the performance, or facilitate the calculations of the method [6–12]. In recent years, essential highlights of the ADM and its capability to treat numerous applications in nonlinear sciences, including ODEs [13,14], PDEs [15–24], integral equations [25–30], fractional differential equations [31–36] and others [37–42], have been presented. However, the ADM has some drawbacks. Although the series solution can be rapidly convergent in a very small region, it may has very slow convergence rate in wider regions and, in this case, the truncated series solution is an inaccurate solution in those regions [43]. Also, in the application of the ADM to boundary value problems, it is found that the ADM approximations could not always satisfy all the boundary conditions of the nonlinear problem [44].

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The principle of the ADM, which expresses the solution as an infinite series of functions, consists of partitioning the equation under consideration into linear and nonlinear parts. The selection of the linear part is generally designed to realize an easily invertible linear operator with resulting simple operations. In the present study, we suggest a novel method, called the optimized decomposition method, to produce analytical approximate solutions for nonlinear differential equations. Two approaches of the method will be introduced to deal with ODEs and PDEs. We implement a linearization, Taylor series approximation, of the nonlinear equation to construct the linear operator. Then we design an efficient decomposition technique that will deliver faster convergence and better accuracy.

2. Optimized decomposition method for ODEs

In this section, we describe in detail the proposed technique, the optimized decomposition method (ODM), for obtaining analytical solutions to nonlinear ODEs. Now, we discuss the main ideas of the suggested method to handle the following nonlinear ODE

$$L[u(t)] = N[u(t)] + g(t), \tag{1}$$

where L represents the linear differential operator $L = \frac{d^n}{dt^n}$, $n = 1$ or 2, $N[u(t)]$ indicates the nonlinear terms and g is a given function. Firstly, we will establish the ADM series solution to Eq. (1). Performing the integral operator L^{-1}, the inverse of the differential operator L, to Eq. (1), where $L^{-1}(\cdot) = \int_0^t (\cdot)\, dt$ if $n = 1$ and $L^{-1}(\cdot) = \int_0^t \int_0^t (\cdot)\, dt\, dt$ if $n = 2$, and using the considered initial condition(s), we get

$$u(t) = f(t) + L^{-1}(N[u(t)]), \tag{2}$$

where

$$f(t) = \begin{cases} L^{-1}[g(t)] + u(0), & \textit{if} \quad L = \frac{d}{dt}, \\ L^{-1}[g(t)] + u(0) + u'(0)\, t, & \textit{if} \quad L = \frac{d^2}{dt^2}. \end{cases} \tag{3}$$

The ADM [1,2] recommends the solution $u(t)$ be decomposed by the series

$$u(t) = \sum_{k=0}^{\infty} v_k(t), \tag{4}$$

and the nonlinear part $N[u(t)]$ is represented by

$$N[u(t)] = \sum_{k=0}^{\infty} P_k(t), \tag{5}$$

where $P_k(t)$, which are called the Adomian polynomials [4,5], can be determined from the relation

$$P_k(t) = \frac{1}{k!} \frac{d^k}{d\theta^k} \left[N\left(\sum_{i=0}^{k} \theta^i v_i(t) \right) \right] \Big|_{\theta=0}. \tag{6}$$

Substituting the infinite series (4) and (5) into both sides of (2) gives the ADM solution $u(t) = \sum_{k=0}^{\infty} v_k(t)$ for Eq. (1), where the component functions $\{v_k(t)\}_{k=0}^{\infty}$ can be evaluated by using the formula

$$\begin{cases} v_0(t) &= f(t), \\ v_{k+1}(t) &= L^{-1}[P_k(t)], \qquad k \geq 0. \end{cases} \tag{7}$$

Further details, properties, modifications and algorithms for determining Adomian polynomials of the ADM can be found in the literature [1–12]. Now, we will describe the main steps of the ODM to deal with Eq. (1). Initially, we will derive an optimized linear operator depending on a linear approximation, Taylor series approximation, of the nonlinear Eq. (1). For this purpose, based upon the assumption that the two variables function $F(L[u], u) = L[u] - N[u]$ can be linearized by a first-order Taylor series expansion at $t = 0$, solving the algebraic equation $F(L[u](0), u(0)) = 0$ for $L[u](0)$, say $u(0) = u_0$ and $L[u](0) = u_0^*$. Then the linear approximation of the function $F(L[u], u)$ near the point (u_0^*, u_0) is definitely given by [45–47]

$$F(L[u], u) \approx \frac{\partial F}{\partial L[u]}(u_0^*, u_0)\, L[u] + \frac{\partial F}{\partial u}(u_0^*, u_0)\, u. \tag{8}$$

Hence, using the approximation given in (8), Eq. (1) can be written as

$$R[u(t)] = N[u(t)] + C_0\, u(t) + g(t), \tag{9}$$

where

$$R[u] = L[u] + C_0\, u, \tag{10}$$

and

$$C_0 = \frac{\frac{\partial F}{\partial u}(u_0^*, u_0)}{\frac{\partial F}{\partial L[u]}(u_0^*, u_0)}. \tag{11}$$

The suggested decomposition method (ODM) will be established based on the implementation of the constant C_0, the coefficient of u in the designed linear operator R, within our decomposition. Now, it is obvious that the linear differential operator R cannot be simply inverted. So, to design the iterative formula of the suggested decomposition method, the component functions of the decomposition series will be assigned in a way that they can be comfortably determined.

The ODM assumes that the solution $u(t)$ of Eq. (1) be decomposed by the series $u(t) = \sum_{k=0}^{\infty} w_k(t)$, where the component functions $\{w_k(t)\}_{k=0}^{\infty}$ are determined recursively by

$$\begin{cases} w_0(t) &= f(t), \\ w_1(t) &= L^{-1}[Q_0(t)], \\ w_2(t) &= L^{-1}\big[Q_1(t) + C_0 w_1(t)\big], \\ w_{k+1}(t) &= L^{-1}\big[Q_k(t) + C_0\big(w_k(t) - w_{k-1}(t)\big)\big], \qquad k \geq 2, \end{cases} \tag{12}$$

and

$$Q_k(t) = \frac{1}{k!}\frac{d^k}{d\theta^k}\Big[N\Big(\sum_{i=0}^{k}\theta^i w_i(t)\Big)\Big]\Big|_{\theta=0}, \tag{13}$$

such that $N\big[\sum_{k=0}^{\infty} w_k(t)\big] = \sum_{k=0}^{\infty} Q_k(t)$. Clearly, if the decomposition series $\sum_{k=0}^{\infty} w_k(t)$ converges, that is $\lim_{k\longrightarrow\infty} w_k = 0$, then, using the deformation Eq. (12),

$$\sum_{k=0}^{\infty} w_k(t) = f(t) + \sum_{k=0}^{\infty} L^{-1}[Q_k(t)]. \tag{14}$$

This is due to the cancellation of adjacent component functions. Hence,

$$u(t) = \sum_{k=0}^{\infty} w_k(t) = f(t) + L^{-1}\sum_{k=0}^{\infty} Q_k(t) = f(t) + L^{-1}N\big[\sum_{k=0}^{\infty} w_k(t)\big] = f(t) + L^{-1}N[u(t)], \tag{15}$$

and so $L[u(t)] = g(t) + N[u(t)]$. Therefore, $u(t) = \sum_{k=0}^{\infty} w_k(t)$ is completely a solution of the nonlinear differential Eq. (1). The presented decomposition method is an optimized method in the direction that the designed approximation $R[u] = L[u] + C_0 u$ is the best linear interpolation to the nonlinear function $F(L[u], u)$ near $t = 0$.

Simply, we can notice that if we take $C_0 = 0$ then the ODM reduces to the ADM. We will see that the ODM, which is particularly relevant in case of nonlinear problems, is highly capable of greatly improving the accuracy of the series solutions compared to ADM. For practical applications, an approximation to the solution $u(t)$ of Eq. (1) is made by truncating the series $\sum_{k=0}^{\infty} w_k(t)$ to a certain degree.

Remark 1. In order to facilitate the calculations of the ADM and ODM, based on the assumption that the function $f(t)$ can be expressed in Taylor series as $f(t) = \sum_{k=0}^{\infty} f_k(t)$, the component functions $\{\upsilon_k(t)\}_{k=0}^{\infty}$ and $\{w_k(t)\}_{k=0}^{\infty}$ can be determined recursively by [8,48]

$$\begin{cases} \upsilon_0(t) &= f_0(t), \\ \upsilon_{k+1}(t) &= f_{k+1}(t) + L^{-1}[P_k(t)], \qquad k \geq 0, \end{cases} \tag{16}$$

and

$$\begin{cases} w_0(t) &= f_0(t), \\ w_1(t) &= f_1(t) + L^{-1}[Q_0(t)], \\ w_2(t) &= f_2(t) + L^{-1}\big[Q_1(t) + C_0 w_1(t)\big], \\ w_{k+1}(t) &= f_{k+1}(t) + L^{-1}\big[Q_k(t) + C_0\big(w_k(t) - w_{k-1}(t)\big)\big], \qquad k \geq 2. \end{cases} \tag{17}$$

3. Optimized decomposition method for PDEs

In this section, we describe an extension of the optimized decomposition method (ODM), presented in the previous section, to solve analytically nonlinear PDEs of parabolic and hyperbolic types of the form

$$\frac{\partial^n}{\partial t^n}u(x,t) = \gamma\frac{\partial^2}{\partial x^2}u(x,t) + M[u(x,t)], \qquad t > 0, \tag{18}$$

where $n = 1$ or 2, $\gamma \in R$, M is a nonlinear given function of u, associated with the following initial and boundary conditions

$$\begin{cases} u(x,0) = f(x), \\ u(x,t) \longrightarrow 0, \ \ as \ |x| \longrightarrow \infty, \ \ t > 0, \end{cases} \quad if \ \ n = 1, \tag{19}$$

and

$$\begin{cases} u(x,0) = f(x), \quad \frac{\partial}{\partial t}u(x,0) = g(x), \\ u(x,t) \longrightarrow 0, \ \ as \ |x| \longrightarrow \infty, \ \ t > 0, \end{cases} \quad if \ \ n = 2. \tag{20}$$

The new extension modifies the series solution decomposition suggested in Section 2, which is used for the analytic treatment of nonlinear ODEs, in a favourable way that can be used to deal with nonlinear PDEs. Initially, performing the inverse operator L^{-1} of $L = \frac{\partial^n}{\partial t^n}$ to both sides of Eq. (18) and employing the associated initial and boundary conditions, the PDE given in Eq. (18) becomes

$$u(x,t) = h(x,t) + L^{-1}\Big\{\gamma \frac{\partial^2}{\partial x^2}u(x,t) + M[u(x,t)]\Big\}, \tag{21}$$

where

$$h(x,t) = \begin{cases} f(x), & if \ \ n = 1, \\ f(x) + g(x)t, & if \ \ n = 2. \end{cases} \tag{22}$$

First, we recall the main steps of the ADM to deal with the nonlinear PDE given in Eq. (18). The ADM [1,2] recommends the solution $u(x,t)$ be exhibited by the infinite series of component functions

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), \tag{23}$$

and the nonlinear term $M[u(x,t)]$ is decomposed as follows

$$M[u(x,t)] = \sum_{k=0}^{\infty} P_k(x,t), \tag{24}$$

where the so-called Adomian polynomials $P_k(x,t)$, $k \geq 0$, are depending on $u_0, u_1, \cdots, u_k$ and can be derived using the formula

$$P_k(x,t) = \frac{1}{k!}\frac{d^k}{d\theta^k}\Big[M\Big(\sum_{i=0}^{k}\theta^i u_i(x,t)\Big)\Big]\Big|_{\theta=0}. \tag{25}$$

Now, substituting the decomposition series (23) and (24) into both sides of (21) produces the ADM solution $u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$ for Eq. (18), where the component functions $u_k(x,t)$, $k \geq 0$, can be evaluated using the iteration formula

$$\begin{cases} u_0(x,t) &= h(x,t), \\ u_{k+1}(t) &= L^{-1}\Big\{\gamma \frac{\partial^2}{\partial x^2}u_k(x,t) + P_k(x,t)\Big\}, \qquad k \geq 0. \end{cases} \tag{26}$$

In general, as discussed earlier, the ADM uses an easily invertible linear operator, such as $L = \frac{\partial^n}{\partial t^n}$ ($n = 1$ or 2), whose inverse is a simple integral operator. However, the suggested extension of the ODM implements a linear approximation of the nonlinear problem as the linear operator. Now, to illustrate the basic ideas of the ODM as a tool for solving the nonlinear PDE given in Eq. (18), let us use the assumption that the nonlinear function $F(\frac{\partial^n}{\partial t^n}u, u_{xx}, u) = \frac{\partial^n}{\partial t^n}u - \gamma\frac{\partial^2}{\partial x^2}u - M[u]$, where $n = 1$ or 2, can be linearized by a first-order Taylor series expansion at $t = 0$. Under this assumption, the linear approximation to the function $F(\frac{\partial^n}{\partial t^n}u, u_{xx}, u)$ when $t = 0$ can be readily obtained as

$$F\Big(\frac{\partial^n}{\partial t^n}u, u_{xx}, u\Big) \approx \frac{\partial^n}{\partial t^n}u - \gamma\frac{\partial^2}{\partial x^2} - C(x)u, \tag{27}$$

where

$$C(x) = \frac{\partial M}{\partial u}\Big|_{t=0}. \tag{28}$$

Thus, based on the approximation given in Eq. (27), the nonlinear PDE given in Eq. (18) can be reformulated as

$$R[u(x,t)] = M[u(x,t)] - C(x)u(x,t), \qquad t > 0, \tag{29}$$

where $R[u] = \frac{\partial^n}{\partial t^n}u - \gamma u_{xx} - C(x)u$ and $C(x) = \frac{\partial M}{\partial u}\Big|_{t=0}$. The proposed adaption of the ODM will be developed based on employing the function $C(x)$, the coefficient of u in the designed linear operator R, within our decomposition. Now, instead of using the inverse operator of the linear operator R, which cannot be easily invertible, we design the iterative formula of the proposed adaption of ODM in a way that the component functions of the decomposition series can be straightforwardly determined.

Our adaption of the ODM proposes the solution $u(x,t)$ be represented by the infinite series $u(x,t)=\sum_{k=0}^{\infty}v_k(x,t)$, where the component functions $v_k(x,t)$, $k\geq 0$, are determined recursively by

$$\begin{cases} v_0(x,t) &= h(x,t), \\ v_1(x,t) &= L^{-1}\big\{Q_0(x,t)+\gamma\frac{\partial^2}{\partial x^2}v_0(x,t)\big\}, \\ v_2(x,t) &= L^{-1}\big\{\big(Q_1(x,t)+\gamma\frac{\partial^2}{\partial x^2}v_1(x,t)\big)-\big(\gamma\frac{\partial^2}{\partial x^2}+C(x)\big)v_1(x,t)\big\}, \\ v_{k+1}(x,t) &= L^{-1}\big\{\big(Q_k(x,t)+\gamma\frac{\partial^2}{\partial x^2}v_k(x,t)\big)-\big(\gamma\frac{\partial^2}{\partial x^2}+C(x)\big)\big(v_k(x,t)-v_{k-1}(x,t)\big)\big\}, \qquad k\geq 2, \end{cases} \tag{30}$$

such that

$$Q_k(x,t)=\frac{1}{k!}\frac{d^k}{d\theta^k}\Big[M\Big(\sum_{i=0}^{k}\theta^i v_i(x,t)\Big)\Big]\Big|_{\theta=0}, \tag{31}$$

and $M\big[\sum_{k=0}^{\infty}v_k(x,t)\big]=\sum_{k=0}^{\infty}Q_k(x,t)$. Evidently, if the decomposition series $\sum_{k=0}^{\infty}v_k(x,t)$ converges then $\lim_{k\longrightarrow\infty}v_k=0$. Thus, because of cancellation of adjacent terms in the iteration formula (30), we get

$$\sum_{k=0}^{\infty}v_k(x,t)=h(x,t)+\sum_{k=0}^{\infty}L^{-1}\big\{Q_k(x,t)+\gamma\frac{\partial^2}{\partial x^2}v_k(x,t)\big\}. \tag{32}$$

Hence

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty}v_k(x,t), \\ &= h(x,t)+L^{-1}\sum_{k=0}^{\infty}\big\{Q_k(x,t)+\gamma\frac{\partial^2}{\partial x^2}v_k(x,t)\big\}, \\ &= h(x,t)+L^{-1}\big\{M\big[\sum_{k=0}^{\infty}v_k(x,t)\big]+\gamma\frac{\partial^2}{\partial x^2}\sum_{k=0}^{\infty}v_k(x,t)\big\}, \end{aligned} \tag{33}$$

and so $L[u(x,t)] = M[u(x,t)]+\gamma\frac{\partial^2}{\partial x^2}u(x,t)$. Therefore, $u(x,t)=\sum_{k=0}^{\infty}v_k(x,t)$ is a solution of the nonlinear PDE given in Eq. (18). The ODM, as discussed in Section 2, is an optimized method in the sense that the approximation $R[u]=\frac{\partial^n}{\partial t^n}u-\gamma u_{xx}-C(x)u$ is the best linear approximation to the function $F(\frac{\partial^n}{\partial t^n}u,u_{xx},u)$ near $t=0$. As a special case, if we replace $C(x)$ by 0 in (30) then the ODM reduces to the ADM. Analytical approximate solutions to Eq. (18), using ODM, can be provided by truncating the series $\sum_{k=0}^{\infty}v_k(x,t)$ to a certain degree.

4. Test problems

This section is performed to explore the performance and the accuracy of the suggested method, ODM, for solving nonlinear ODEs and PDEs. Test problems including nonlinear ODEs, nonlinear and linear PDEs have been solved by means of ODM and ADM. Then numerical comparisons between the ODM solutions and the ADM solutions are made. The numerical simulation results are carried out by using the Mathematica software.

4.1. Nonlinear ODEs

Example 1. First, we start by considering the nonlinear Riccati differential equation

$$\frac{d}{dt}u(t)=1-u^2(t), \qquad t>0, \tag{34}$$

with the initial condition

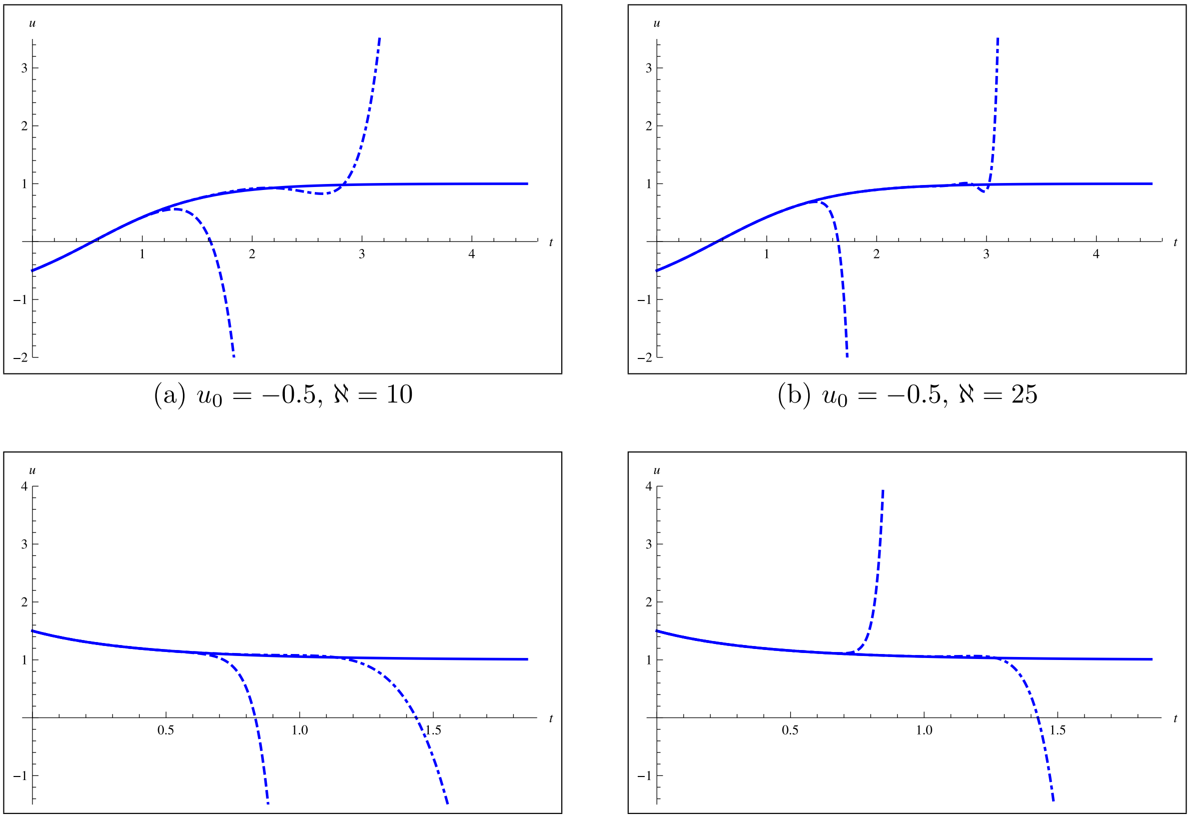
$$u(0)=u_0, \tag{35}$$

where $u_0\in R$. The exact solution of the Riccati Eq. (34) is given by

$$u(t)=\frac{\tanh(t)+u_0}{u_0\tanh(t)+1}. \tag{36}$$

Linearizing the function $F(L[u],u)=\frac{d}{dt}u+u^2-1$ near the point (u_0^*,u_0), we obtain the linear approximation

$$F(L[u],u)\approx\frac{d}{dt}u+C_0\,u. \tag{37}$$



(a) $u_0 = -0.5$, $\aleph = 10$ (b) $u_0 = -0.5$, $\aleph = 25$

(c) $u_0 = 1.5$, $\aleph = 15$ (d) $u_0 = 1.5$, $\aleph = 30$

Fig. 1. Plots of the approximate solutions and exact solution for Eq. (34): (——) exact solution; (– – –) approximate solution $\sum_{k=0}^{\aleph} v_k(t)$ using ADM; (– · – ·) approximate solution $\sum_{k=0}^{\aleph} w_k(t)$ using ODM.

where $C_0 = 2u_0$. The ADM exhibits the solution $u(t) = \sum_{k=0}^{\infty} v_k(t)$, where the components $\{v_k(t)\}_{k=0}^{\infty}$ can be, according to Remark 1, evaluated by using the relation

$$\begin{cases} v_0(t) &= u_0, \\ v_1(t) &= t + \int_0^t P_0(t)\,dt, \\ v_{k+1}(t) &= \int_0^t P_k(t)\,dt, \qquad k \geq 1, \end{cases} \tag{38}$$

and the polynomial $P_k(t)$ is defined by the relation

$$P_k(t) = \frac{-1}{k!}\frac{d^k}{d\theta^k}\left[\left(v_0(t) + \theta v_1(t) + \theta^2 v_2(t) + \theta^3 v_3(t) + \cdots\right)^2\right]\Big|_{\theta=0}. \tag{39}$$

According to our decomposition method and Remark 1, the ODM suggests the solution $u(t) = \sum_{k=0}^{\infty} w_k(t)$, where the components $\{w_k(t)\}_{k=0}^{\infty}$ are evaluated recursively by using the formula

$$\begin{cases} w_0(t) &= u_0, \\ w_1(t) &= t + \int_0^t Q_0(t)\,dt, \\ w_2(t) &= \int_0^t \left[Q_1(t) + 2u_0 w_1(t)\right] dt, \\ w_{k+1}(t) &= \int_0^t \left[Q_k(t) + 2u_0\left(w_k(t) - w_{k-1}(t)\right)\right] dt, \qquad k \geq 2, \end{cases} \tag{40}$$

and $Q_k(t)$ is defined by

$$Q_k(t) = \frac{-1}{k!}\frac{d^k}{d\theta^k}\left[\left(w_0(t) + \theta w_1(t) + \theta^2 w_2(t) + \theta^3 w_3(t) + \cdots\right)^2\right]\Big|_{\theta=0}. \tag{41}$$

Fig. 1 displays the ADM approximate solutions $\sum_{k=0}^{\aleph} v_k(t)$, the ODM approximate solutions $\sum_{k=0}^{\aleph} w_k(t)$ and the exact solutions of $u(t)$ to Eq. (34) for some specified values of u_0 and $\aleph$, where $\aleph \in N$. We can observe, from the numerical approximation results drawn in Fig. 1, that the approximate solutions exhibited using ODM are better than the ones found by ADM. Furthermore, it is clear that the ODM solutions converge faster than the ADM solutions.

Example 2. Now, we consider the following nonlinear Duffing equation,

$$\frac{d^2}{dt^2}u(t) = -au(t) - bu^3(t), \qquad t > 0, \tag{42}$$

where $a, b \in R$, with the initial conditions

$$u(0) = u_0, \qquad \frac{du}{dt}(0) = 0. \tag{43}$$

Evidently, the linear approximation to the function $F(L[u], u) = \frac{d^2}{dt^2}u + au + bu^3$ near the point (u_0^*, u_0) is given by

$$F(L[u], u) \approx \frac{d^2}{dt^2}u + C_0\, u. \tag{44}$$

where $C_0 = a + 3bu_0^2$. The ADM gives the solution $u(t) = \sum_{k=0}^{\infty} v_k(t)$, where

$$\begin{cases} v_0(t) &=& u_0, \\ v_{k+1}(t) &=& \int_0^t \int_0^t P_k(t)\,dt\,dt, \qquad k \geq 0, \end{cases} \tag{45}$$

while the ODM suggests the solution $u(t) = \sum_{k=0}^{\infty} w_k(t)$, where

$$\begin{cases} w_0(t) &=& u_0, \\ w_1(t) &=& \int_0^t \int_0^t Q_0(t)\,dt\,dt, \\ w_2(t) &=& \int_0^t \int_0^t \left[Q_1(t) + (a + 3bu_0^2)w_1(t)\right] dt\,dt, \\ w_{k+1}(t) &=& \int_0^t \int_0^t \left[Q_k(t) + (a + 3bu_0^2)\big(w_k(t) - w_{k-1}(t)\big)\right] dt\,dt, \qquad k \geq 2. \end{cases} \tag{46}$$

Here the polynomials $P_k(t)$ and $Q_k(t)$ are defined as

$$P_k(t) = \frac{-1}{k!}\frac{d^k}{d\theta^k}\left[a\big(v_0(t) + \theta v_1(t) + \theta^2 v_2(t) + \theta^3 v_3(t) + \cdots\big) + b\big(v_0(t) + \theta v_1(t) + \theta^2 v_2(t) + \theta^3 v_3(t) + \cdots\big)^3\right]\Big|_{\theta=0}, \tag{47}$$

and

$$Q_k(t) = \frac{-1}{k!}\frac{d^k}{d\theta^k}\left[a\big(w_0(t) + \theta w_1(t) + \theta^2 w_2(t) + \theta^3 w_3(t) + \cdots\big) + b\big(w_0(t) + \theta w_1(t) + \theta^2 w_2(t) + \theta^3 w_3(t) + \cdots\big)^3\right]\Big|_{\theta=0}. \tag{48}$$

Fig. 2 displays the ADM approximate solutions $\sum_{k=0}^{\aleph} v_k(t)$, the ODM approximate solutions $\sum_{k=0}^{\aleph} w_k(t)$ and the RK4 method approximate solutions of $u(t)$ to Eq. (42) for some specified values of a, b and u_0 when $\aleph = 30$. As observed in the previous example and from the numerical approximation results shown in Fig. 2, it is clear that the approximate solutions exhibited using ODM are more accurate than those obtained by ADM.

4.2. Nonlinear PDEs

Example 3. Next, we consider the nonlinear FitzHugh–Nagumo equation

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) - u(x,t)(1 - u(x,t))(a - u(x,t)), \tag{49}$$

where $0 < a < 1/2$, together with the initial condition

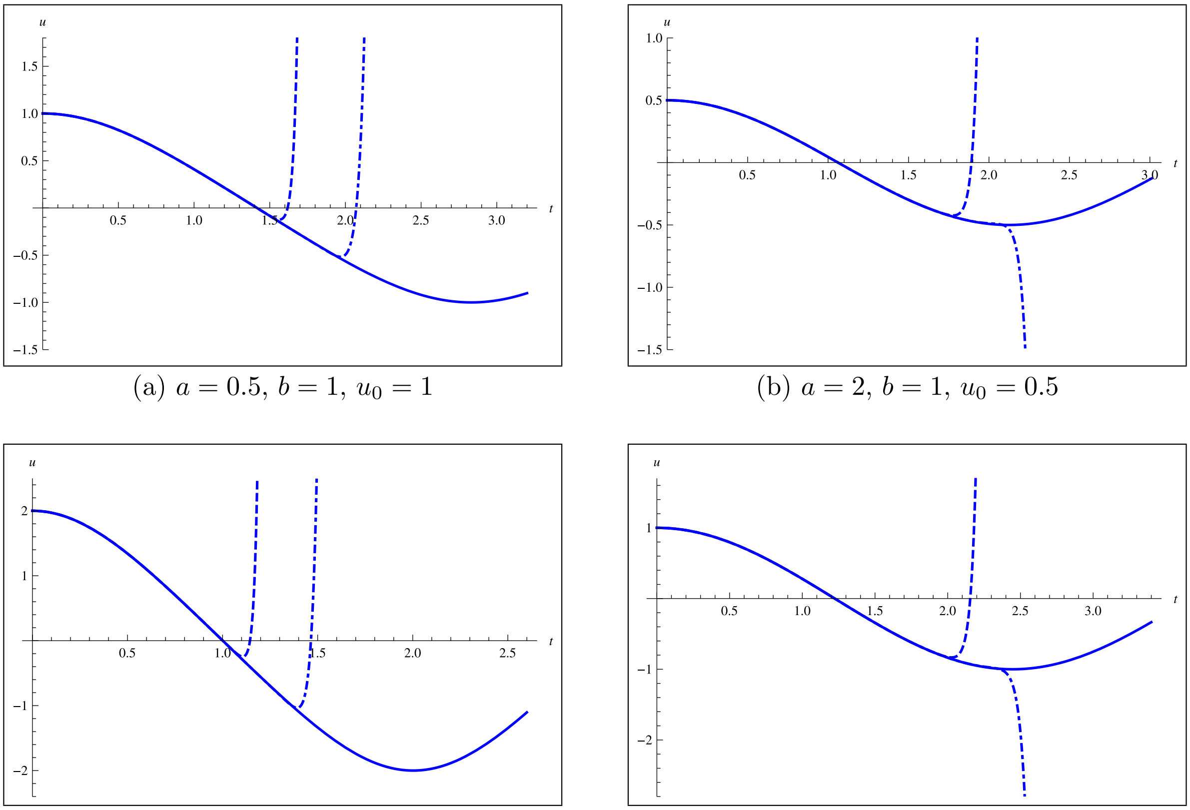
$$u(x,0) = \frac{a\,Exp[ax/\sqrt{2}]}{Exp[ax/\sqrt{2}] + 1}. \tag{50}$$

The FitzHugh–Nagumo equation is a nonlinear reaction–diffusion model which arises in genetics, biology, and heat and mass transfer [49,50]. On the basis of our approach, the linear approximation to the function $F(\frac{\partial}{\partial t}u, u_{xx}, u) = \frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u + u(1-u)(a-u)$, when $t = 0$, is given by

$$F(\frac{\partial}{\partial t}u, u_{xx}, u) \approx \frac{\partial}{\partial t}u - \frac{\partial^2}{\partial x^2}u + \big(a - 2(a+1)u(x,0) + 3u^2(x,0)\big)u. \tag{51}$$

In this case, comparing with (28), we obtain $C(x) = -\big(a - 2(a+1)u(x,0) + 3u^2(x,0)\big)$. The ADM provides the series solution $u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)$, using the linear operator $L = \frac{\partial}{\partial t}$, where the component functions $\{u_k(x,t)\}_{k=0}^{\infty}$ are calculated frequently using the relation

$$\begin{cases} u_0(x,t) &=& \frac{a\,Exp[ax/\sqrt{2}]}{Exp[ax/\sqrt{2}]+1}, \\ u_{k+1}(t) &=& \int_0^t \left\{\frac{\partial^2}{\partial x^2}u_k(x,t) - P_k(x,t)\right\} dt, \qquad k \geq 0, \end{cases} \tag{52}$$



(a) $a = 0.5,\ b = 1,\ u_0 = 1$ (b) $a = 2,\ b = 1,\ u_0 = 0.5$

(c) $a = 1,\ b = 0.5,\ u_0 = 2$ (d) $a = 1.5,\ b = 0.2,\ u_0 = 1$

Fig. 2. Plots of the approximate solutions for Eq. (42): (——) RK4 solution; (− − −) approximate solution $\sum_{k=0}^{30} v_k(t)$ using ADM; (−·− ·) approximate solution $\sum_{k=0}^{30} w_k(t)$ using ODM.

and the polynomial $P_k(x,t)$ is defined as

$$P_k(x,t) = \frac{1}{k!}\frac{d^k}{d\theta^k}\Big[\Big(\sum_{i=0}^{k}\theta^i u_i(x,t)\Big)\Big(1-\sum_{i=0}^{k}\theta^i u_i(x,t)\Big)\Big(a-\sum_{i=0}^{k}\theta^i u_i(x,t)\Big)\Big]\Big|_{\theta=0}. \tag{53}$$

Consequently, the first component functions of the ADM solution can be obtained as

$$\begin{cases} u_1(x,t) &= \frac{(a-2)a^2}{2}\frac{Exp[ax/\sqrt{2}]}{\left(1+Exp[ax/\sqrt{2}]\right)^2}\,t, \\ u_2(x,t) &= \int_0^t \left\{\frac{\partial^2}{\partial^2}u_1(x,t) - au_1(x,t) + 2(a+1)u_0(x,t)u_1(x,t) - 3u_0^2(x,t)u_1(x,t)\right\}dt, \\ \quad\vdots \end{cases} \tag{54}$$

According to our decomposition method, implementing the linear approximation given in (51), the ODM suggests the series solution $u(x,t) = \sum_{k=0}^{\infty} v_k(x,t)$, where the component functions $\{v_k(x,t)\}_{k=0}^{\infty}$ are to be determined using the formula

$$\begin{cases} v_0(x,t) &= \frac{a\,Exp[ax/\sqrt{2}]}{Exp[ax/\sqrt{2}]+1}, \\ v_1(x,t) &= \int_0^t \left\{Q_0(x,t) + \frac{\partial^2}{\partial x^2}v_0(x,t)\right\}dt, \\ v_2(x,t) &= \int_0^t \left\{\left(Q_1(x,t) + \frac{\partial^2}{\partial x^2}v_1(x,t)\right) - \left(\frac{\partial^2}{\partial x^2} + C(x)\right)v_1(x,t)\right\}dt, \\ v_{k+1}(x,t) &= \int_0^t \left\{\left(Q_k(x,t) + \frac{\partial^2}{\partial x^2}v_k(x,t)\right) - \left(\frac{\partial^2}{\partial x^2} + C(x)\right)\left(v_k(x,t) - v_{k-1}(x,t)\right)\right\}dt, \qquad k \geq 2, \end{cases} \tag{55}$$

where $C(x)=-\big(a-2(a+1)u(x,0)+3u^2(x,0)\big)$, $u(x,0)=a\,Exp[ax/\sqrt{2}]/(Exp[ax/\sqrt{2}]+1)$ and the polynomial $Q_k(x,t)$ can be derived using the formula

$$Q_k(x,t)=\frac{-1}{k!}\frac{d^k}{d\theta^k}\Big[\Big(\sum_{i=0}^{k}\theta^i\upsilon_i(x,t)\Big)\Big(1-\sum_{i=0}^{k}\theta^i\upsilon_i(x,t)\Big)\Big(a-\sum_{i=0}^{k}\theta^i\upsilon_i(x,t)\Big)\Big]\Big|_{\theta=0}. \tag{56}$$

For $k\geq 1$, simplifying the polynomial $Q_k(x,t)$, the relation (55) can be reduced to

$$\begin{aligned}\upsilon_k(x,t)=\textstyle\int_0^t\Big\{&\tfrac{\partial^2}{\partial x^2}\upsilon_{k-1}(x,t)-a\upsilon_{k-1}(x,t)+(a+1)\sum_{j=0}^{k-1}\upsilon_j(x,t)\upsilon_{k-j-1}(x,t)\\ &-\sum_{j=0}^{k-1}\sum_{i=0}^{j}\upsilon_i(x,t)\upsilon_{j-i}(x,t)\upsilon_{k-j-1}(x,t)-\Big(\frac{\partial^2}{\partial x^2}+C(x)\Big)\big[\chi_k\upsilon_{k-1}-\chi_{k-1}\upsilon_{k-2}\big]\Big\}\,dt,\end{aligned} \tag{57}$$

where

$$\chi_k=\begin{cases}1, & k>1,\\ 0, & k\leq 1.\end{cases} \tag{58}$$

As a result, using the formula (57), the first component functions of the ODM solution can be extracted as

$$\begin{cases}\upsilon_1(x,t) &= \frac{(a-2)a^2}{2}\frac{Exp[ax/\sqrt{2}]}{\big(1+Exp[ax/\sqrt{2}]\big)^2}\,t,\\ \upsilon_2(x,t) &= 0,\\ \quad\vdots\end{cases} \tag{59}$$

Now, to test the accuracy of the suggested extension of the ODM and to make a clear and explicit comparison between the ODM and the ADM, we have calculated approximate solutions for Eq. (49) using both methods when the space variable x is fixed. The exact solution of the FitzHugh–Nagumo Eq. (49), where $0<a<1/2$, under the initial condition given in Eq. (50) is given by

$$u(x,t)=\frac{a\,Exp\big[ax/\sqrt{2}+a(a/2-1)t\big]}{Exp\big[ax/\sqrt{2}+a(a/2-1)t\big]+1}. \tag{60}$$

Fig. 3 displays the ODM approximate solutions $\sum_{k=0}^{\aleph}\upsilon_k(x,t)$, the ADM approximate solutions $\sum_{k=0}^{\aleph}u_k(x,t)$ and the exact solutions of the FitzHugh–Nagumo Eq. (49) for some certain values of a and $\aleph$ when $x=1$. It is easily observed, form the numerical results pictured in Fig. 3, that the ODM approximate solutions are more accurate than ADM approximate solutions and that the ODM often gives a better convergence of the results than the ADM.

Example 4. Next, we consider the nonlinear Klein–Gordon equation

$$\frac{\partial^2}{\partial t^2}u(x,t)=\frac{\partial^2}{\partial x^2}u(x,t)+au(x,t)+bu^2(x,t), \tag{61}$$

where $a>0$ and $b<0$, together with the initial conditions

$$u(x,0)=\frac{-3a}{2b\,cosh^2[\sqrt{a}x/2]}, \tag{62}$$

and

$$\frac{\partial u}{\partial t}(x,0)=\frac{3a}{b}\sqrt{\frac{a}{2}}\frac{sinh[\sqrt{a}x/2]}{cosh^3[\sqrt{a}x/2]}. \tag{63}$$

The nonlinear Klein–Gordon equation models many nonlinear phenomena in physical science applications such as nonlinear optics, quantum field theory and solid state physics [51,52]. In view of our approach, the linear approximation to the function $F(\frac{\partial^2}{\partial t^2}u,u_{xx},u)=\frac{\partial^2}{\partial t^2}u-\frac{\partial^2}{\partial x^2}u-au-bu^2$, when $t=0$, is given by

$$F(\frac{\partial^2}{\partial t^2}u,u_{xx},u)\approx\frac{\partial^2}{\partial t^2}u-\frac{\partial^2}{\partial x^2}u-\big(a+2bu(x,0)\big)u. \tag{64}$$

In this case, comparing with (28), we obtain $C(x)=a+2bu(x,0)$. The ADM gives the series solution $u(x,t)=\sum_{k=0}^{\infty}u_k(x,t)$, using the linear operator $L=\frac{\partial^2}{\partial t^2}$, where the component functions $\{u_k(x,t)\}_{k=0}^{\infty}$ are calculated frequently using the relation

$$\begin{cases}u_0(x,t) &= \frac{-3a}{2b\,cosh^2[\sqrt{a}x/2]}+\frac{3a}{b}\sqrt{\frac{a}{2}}\frac{sinh[\sqrt{a}x/2]}{cosh^3[\sqrt{a}x/2]}\,t,\\ u_{k+1}(t) &= \int_0^t\int_0^t\Big\{\frac{\partial^2}{\partial x^2}u_k(x,t)+P_k(x,t)\Big\}\,dt\,dt, \qquad k\geq 0,\end{cases} \tag{65}$$

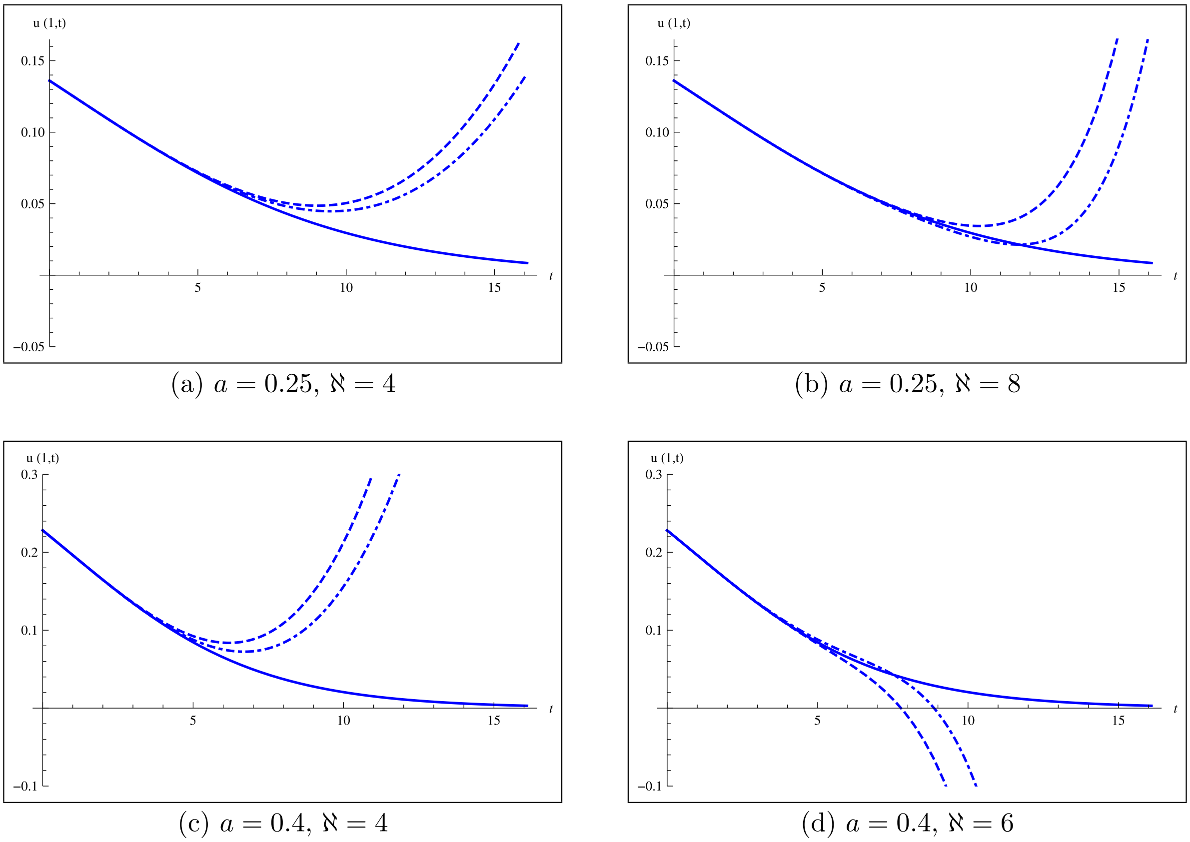


Fig. 3. Plots of the approximate solutions and exact solution for FitzHugh–Nagumo Eq. (49) when $x = 1$: (——) exact solution $u(1,t)$; (— — —) approximate solution $\sum_{k=0}^{\aleph} u_k(1,t)$ using ADM; (— · — ·) approximate solution $\sum_{k=0}^{\aleph} \upsilon_k(1,t)$ using ODM.

and the polynomial $P_k(x,t)$ is defined as

$$P_k(x,t) = \frac{1}{k!}\frac{d^k}{d\theta^k}\Big[a\sum_{i=0}^{k}\theta^i u_i(x,t) + b\Big(\sum_{i=0}^{k}\theta^i u_i(x,t)\Big)^2\Big]\Big|_{\theta=0}. \tag{66}$$

According to our decomposition method, implementing the linear approximation given in (64), the ODM suggests the series solution $u(x,t) = \sum_{k=0}^{\infty} \upsilon_k(x,t)$, where the component functions $\{\upsilon_k(x,t)\}_{k=0}^{\infty}$ are to be determined using the formula

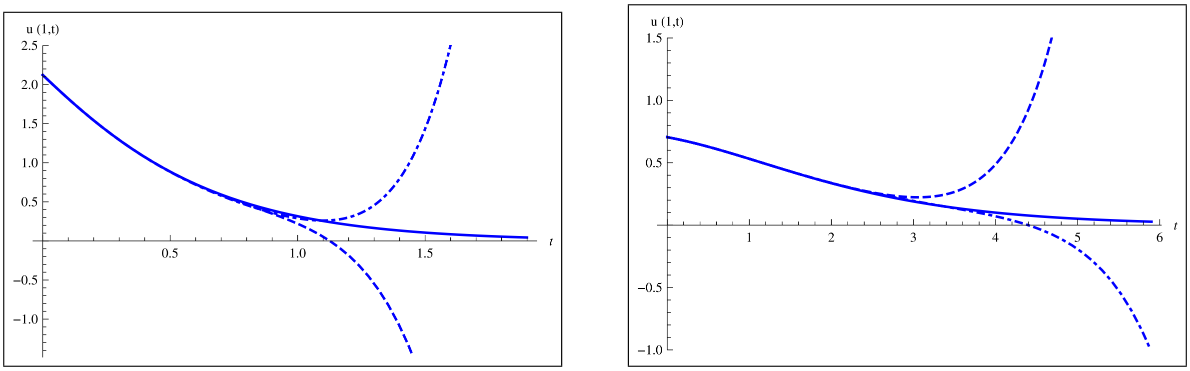
$$\begin{cases} \upsilon_0(x,t) &= \frac{-3a}{2b\cosh^2[\sqrt{a}x/2]} + \frac{3a}{b}\sqrt{\frac{a}{2}}\,\frac{\sinh[\sqrt{a}x/2]}{\cosh^3[\sqrt{a}x/2]}\,t, \\ \upsilon_1(x,t) &= \int_0^t\int_0^t\big\{Q_0(x,t) + \frac{\partial^2}{\partial x^2}\upsilon_0(x,t)\big\}\,dt\,dt, \\ \upsilon_2(x,t) &= \int_0^t\int_0^t\big\{\big(Q_1(x,t) + \frac{\partial^2}{\partial x^2}\upsilon_1(x,t)\big) - \big(\frac{\partial^2}{\partial x^2} + C(x)\big)\upsilon_1(x,t)\big\}\,dt\,dt, \\ \upsilon_{k+1}(x,t) &= \int_0^t\int_0^t\big\{\big(Q_k(x,t) + \frac{\partial^2}{\partial x^2}\upsilon_k(x,t)\big) - \big(\frac{\partial^2}{\partial x^2} + C(x)\big)\big(\upsilon_k(x,t) - \upsilon_{k-1}(x,t)\big)\big\}\,dt\,dt, \qquad k \geq 2, \end{cases} \tag{67}$$

where $C(x) = a + 2bu(x,0)$ and the polynomial $Q_k(x,t)$ can be derived using the formula

$$Q_k(x,t) = \frac{1}{k!}\frac{d^k}{d\theta^k}\Big[a\sum_{i=0}^{k}\theta^i \upsilon_i(x,t) + b\Big(\sum_{i=0}^{k}\theta^i \upsilon_i(x,t)\Big)^2\Big]\Big|_{\theta=0}. \tag{68}$$

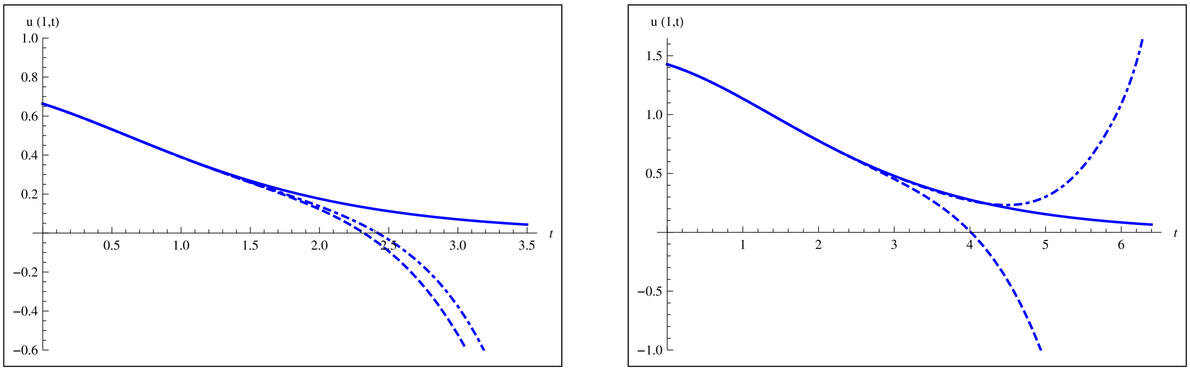
For $k \geq 1$, simplifying the polynomial $Q_k(x,t)$, the relation (67) can be reduced to

$$\begin{aligned} \upsilon_k(x,t) = \int_0^t\int_0^t \Big\{ &\frac{\partial^2}{\partial x^2}\upsilon_{k-1}(x,t) + a\upsilon_{k-1}(x,t) + b\sum_{j=0}^{k-1}\upsilon_j(x,t)\upsilon_{k-j-1}(x,t) \\ &-\big(\frac{\partial^2}{\partial x^2} + C(x)\big)\big[\chi_k\upsilon_{k-1} - \chi_{k-1}\upsilon_{k-2}\big]\Big\}\,dt\,dt. \end{aligned} \tag{69}$$



(a) $a = 2.5,\ b = -1,\ \aleph = 3$ (b) $a = 0.25,\ b = -0.5,\ \aleph = 4$

Fig. 4. Plots of the approximate solutions and exact solution for Klein–Gordon Eq. (61) when $x = 1$: (——) exact solution $u(1,t)$; (– – –) approximate solution $\sum_{k=0}^{\aleph} u_k(1,t)$ using ADM; (– · – ·) approximate solution $\sum_{k=0}^{\aleph} v_k(1,t)$ using ODM.



(a) $a = 0.5,\ b = -1$ (b) $a = 0.2,\ b = -0.2$

Fig. 5. Plots of the approximate solutions for Klein–Gordon Eq. (61), when $x = 1$, using ODM: (——) exact solution $u(1,t)$; (– – –) ODM approximate solution $\sum_{k=0}^{\aleph} v_k(1,t)$ when $\aleph = 3$; (– · – ·) ODM approximate solution $\sum_{k=0}^{\aleph} v_k(1,t)$ when $\aleph = 5$.

The exact solution of the Klein–Gordon Eq. (61) under the initial conditions given in Eqs. (62) and (63) is given by

$$u(x,t) = \frac{-3a}{2b\, cosh^2[\sqrt{a}/2(x+\sqrt{2}t)]}. \tag{70}$$

Fig. 4 displays the ODM approximate solutions $\sum_{k=0}^{\aleph} v_k(x,t)$, the ADM approximate solutions $\sum_{k=0}^{\aleph} u_k(x,t)$ and the exact solutions of the Klein–Gordon Eq. (61) for some certain values of a and b when $x = 1$. From the numerical results drawn in Fig. 4, it is clear that the ODM numerical solutions are more accurate than those obtained by the ADM. In Fig. 5, we display the ODM approximate solutions $\sum_{k=0}^{\aleph} v_k(x,t)$ of the Klein–Gordon Eq. (61), where $x = 1$, for some certain values of a and b when $\aleph = 3$ and $\aleph = 5$. Obviously, by adding further terms in the truncated series approximations we can improve the accuracy of the method.

4.3. Linear PDEs

Example 5. Finally, we consider the linear PDE

$$\frac{\partial^2}{\partial t^2}u(x,t) + \frac{\partial^2}{\partial x^2}u(x,t) + u(x,t) = 0, \tag{71}$$

together with the initial conditions

$$u(x,0) = 1 + \sin x, \qquad \frac{\partial u}{\partial t}(x,0) = 0. \tag{72}$$

In view of our approach and Remark 1, using Taylor series expansion of $1+\sin x$, the ODM suggests the series solution $u(x,t)=\sum_{k=0}^{\infty} v_k(x,t)$, where the component functions $\{v_k(x,t)\}_{k=0}^{\infty}$ are to be determined using the formula

$$\begin{cases} v_0(x,t) &= 1, \\ v_1(x,t) &= x-\int_0^t\int_0^t\left\{v_0(x,t)+\frac{\partial^2}{\partial x^2}v_0(x,t)\right\}dt\,dt, \\ v_2(x,t) &= -\frac{1}{3!}x^3-\int_0^t\int_0^t\left\{\left(v_1(x,t)+\frac{\partial^2}{\partial x^2}v_1(x,t)\right)-\left(\frac{\partial^2}{\partial x^2}-C(x)\right)v_1(x,t)\right\}dt\,dt, \\ v_{k+1}(x,t) &= \frac{(-1)^k}{(2k+1)!}x^{2k+1}-\int_0^t\int_0^t\left\{\left(v_k(x,t)+\frac{\partial^2}{\partial x^2}v_k(x,t)\right)\right. \\ & \left.\quad -\left(\frac{\partial^2}{\partial x^2}-C(x)\right)\left(v_k(x,t)-v_{k-1}(x,t)\right)\right\}dt\,dt, \qquad k\geq 2, \end{cases} \tag{73}$$

where $C(x)=-1$. Consequently, we obtain

$$\begin{cases} v_0(x,t) &= \mathbf{1}, \\ v_1(x,t) &= \mathbf{x}-\frac{\mathbf{t^2}}{\mathbf{2!}}, \\ v_2(x,t) &= -\frac{\mathbf{x^3}}{\mathbf{3!}}, \\ v_3(x,t) &= \frac{\mathbf{x^5}}{\mathbf{5!}}+\frac{\mathbf{t^4}}{\mathbf{4!}}-x\frac{t^2}{2!}, \\ v_4(x,t) &= -\frac{\mathbf{x^7}}{\mathbf{7!}}+x\frac{t^2}{2!}+\frac{x^3}{3!}\frac{t^2}{2!}, \\ v_5(x,t) &= \frac{\mathbf{x^9}}{\mathbf{9!}}-\frac{\mathbf{t^6}}{\mathbf{6!}}-\frac{x^5}{5!}\frac{t^2}{2!}-\frac{x^3}{3!}\frac{t^2}{2!}+x\frac{t^4}{4!}, \\ v_6(x,t) &= -\frac{\mathbf{x^{11}}}{\mathbf{11!}}+\frac{x^7}{7!}\frac{t^2}{2!}+\frac{x^5}{5!}\frac{t^2}{2!}-\frac{x^3}{3!}\frac{t^4}{4!}-2x\frac{t^4}{4!}, \\ v_7(x,t) &= \frac{\mathbf{x^{13}}}{\mathbf{13!}}+\frac{\mathbf{t^8}}{\mathbf{8!}}-\frac{x^9}{9!}\frac{t^2}{2!}-\frac{x^7}{7!}\frac{t^2}{2!}+x\frac{t^4}{4!}+2\frac{x^3}{3!}\frac{t^4}{4!}+\frac{x^5}{5!}\frac{t^4}{4!}-x\frac{t^6}{6!}. \\ & \vdots \end{cases} \tag{74}$$

Therefore, cancelling the noise terms, the solution in a series form is given by

$$u(x,t)=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\right)+\left(1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\cdots\right), \tag{75}$$

and so the solution in a closed form is given by $u(x,t)=\sin x+\cos t$.

5. Concluding remarks

In this paper, an effective decomposition method, called the optimized decomposition method (ODM), to deal with nonlinear ODEs has been proposed. Then, an extension of the ODM to handle nonlinear PDEs has been developed. The formulation of the suggested approaches of the ODM which employ Taylor series approximation of the considered nonlinear equation have been derived. The proposed decomposition method provides an efficient technique for obtaining accurate analytic approximate solutions to nonlinear differential equations. The performed numerical simulations confirm that the ODM offers better accuracy and convergence compared to the ADM.

The suggested method could become a powerful and promising technique for solving accurately nonlinear differential systems. Finally, we suggest the following future extensions to our work: studying the convergence of the ODM and extending the application of the proposed method to many other nonlinear problems in applied sciences.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] G. Adomian, A review of the decomposition method in applied mathematics, J. Math. Anal. Appl. 135 (1988) 501–544.
[2] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers, Boston, 1994.
[3] Y. Cherruault, G. Saccomandi, B. Some, New results for convergence of Adomian's method applied to integral equations, Math. Comput. Model. 16 (2) (1992) 85–93.
[4] A.M. Wazwaz, A new algorithm for calculating Adomian polynomials for nonlinear operators, Appl. Math. Comput. 111 (1) (2000) 33–51.
[5] R. Rach, A new definition of the Adomian polynomials, Kybernetes 37 (2008) 910–955.
[6] K. Abbaoui, Y. Cherruault, V. Seng, Practical formulae for the calculus of multivariable Adomian polynomials, Math. Comput. Model. 22 (1995) 89–93.
[7] A.M. Wazwaz, A reliable modification of Adomian decomposition method, Appl. Math. Comput. 102 (1) (1999) 77–86.
[8] A.M. Wazwaz, S.M. El-Sayed, A new modification of the adomian decomposition method for linear and nonlinear operators, Appl. Math. Comput. 122 (3) (2001) 393–405.
[9] T.A. Abassy, Improved Adomian decomposition method, Comput. Math. Appl. 59 (1) (2010) 42–54.
[10] A.M. Wazwaz, The combined Laplace transform-Adomian decomposition method for handling nonlinear Volterra integro-differential equations, Appl. Math. Comput. 216 (4) (2010) 1304–1309.

[11] J.S. Duan, Convenient analytic recurrence algorithms for the Adomian polynomials, Appl. Math. Comput. 217 (2011) 6337–6348.
[12] J.S. Duan, R. Rach, A.M. Wazwaz, T. Chaolu, Z. Wang, A new modified Adomian decomposition method and its multistage form for solving nonlinear boundary value problems with Robin boundary conditions, Appl. Math. Model. 37 (20–21) (2013) 8687–8708.
[13] A.M. Wazwaz, Adomian decomposition method for a reliable treatment of the Emden-Fowler equation, Appl. Math. Comput. 161 (2) (2005) 543–560.
[14] L. Blanco-Cocom, A.G. Estrella, E. Avila-Vales, Solving delay differential systems with history functions by the Adomian decomposition method, Appl. Math. Comput. 218 (10) (2012) 5994–6011.
[15] A.M. Wazwaz, Construction of solitary wave solutions and rational solutions for the KdV equation by Adomian decomposition method, Chaos Solitons Fractals 12 (12) (2001) 2283–2293.
[16] A.M. Wazwaz, The modified decomposition method for analytic treatment of differential equations, Appl. Math. Comput. 173 (1) (2006) 165–176.
[17] S.A. El-Wakil, M.A. Abdou, New applications of Adomian decomposition method, Chaos Solitons Fractals 33 (2) (2007) 513–522.
[18] A.M. Wazwaz, A comparison between the variational iteration method and Adomian decomposition method, J. Comput. Appl. Math. 207 (1) (2007) 129–136.
[19] Q. Mao, S. Pietrzko, Free vibration analysis of stepped beams by using Adomian decomposition method, Appl. Math. Comput. 217 (7) (2010) 3429–3441.
[20] H. Zhu, H. Shu, M. Ding, Numerical solutions of two-dimensional Burgers' equations by discrete Adomian decomposition method, Comput. Math. Appl. 60 (3) (2010) 840–848.
[21] L. Bougoffa, R.C. Rach, Solving nonlocal initial–boundary value problems for linear and nonlinear parabolic and hyperbolic partial differential equations by the Adomian decomposition method, Appl. Math. Comput. 225 (2013) 50–61.
[22] Z. Hai-Xing, A. Hong-Li, C. Yong, A Laplace decomposition method for nonlinear partial differential equations with nonlinear term of any order, Commun. Theor. Phys. 61 (1) (2016) 23–31.
[23] H.O. Bakodah, N.A. Al-Zaid, M. Mirzazadeh, Q. Zhou, Decomposition method for solving Burgers' equation with Dirichlet and Neumann boundary conditions, Optik 130 (2017) 1339–1346.
[24] M. Asma, W.A. Othman, B.R. Wong, A. Biswas, Optical soliton perturbation with quadratic-cubic nonlinearity by Adomian decomposition method, Optik 164 (2018) 632–641.
[25] R. Novin, M.A. Araghi, Y. Mahmoudi, A novel fast modification of the Adomian decomposition method to solve integral equations of the first kind with hypersingular kernels, J. Comput. Appl. Math. 343 (1) (2018) 619–634.
[26] A.M. Wazwaz, Necessary conditions for the appearance of noise terms in decomposition solution series, Appl. Math. Comput. 81 (1997) 265–274.
[27] S. Abbasbandy, Numerical solutions of the integral equations: Homotopy perturbation method and Adomian's decomposition method, Appl. Math. Comput. 173 (1) (2006) 493–500.
[28] L. Bougoffa, A. Mennouni, R.C. Rach, Solving Cauchy integral equations of the first kind by the Adomian decomposition method, Appl. Math. Comput. 219 (9) (2013) 4423–4433.
[29] Y. Mahmoudi, A new modified Adomian decomposition method for solving a class of hypersingular integral equations of second kind, J. Comput. Appl. Math. 255 (2014) 737–742.
[30] A.M. Wazwaz, R. Rach, Two reliable methods for solving the Volterra integral equation with a weakly singular kernel, J. Comput. Appl. Math. 302 (2016) 71–80.
[31] S. Momani, Z. Odibat, Analytical solution of a time-fractional Navier–Stokes equation by Adomian decomposition method, Appl. Math. Comput. 177 (2) (2006) 488–494.
[32] Z. Odibat, S. Momani, Approximate solutions for boundary value problems of time-fractional wave equation, Appl. Math. Comput. 181 (1) (2006) 767–774.
[33] S. Momani, Z. Odibat, Numerical comparison of methods for solving linear differential equations of fractional order, Chaos Solitons Fractals 31 (5) (2007) 1248–1255.
[34] Y. Hu, Y. Luo, Z. Lu, Analytical solution of the linear fractional differential equation by Adomian decomposition method, J. Comput. Appl. Math. 215 (1) (2008) 220–229.
[35] L. Song, W. Wang, A new improved Adomian decomposition method and its application to fractional differential equations, Appl. Math. Model. 37 (3) (2013) 1590–1598.
[36] J.S. Duan, T. Chaolu, R. Rach, L. Lu, The Adomian decomposition method with convergence acceleration techniques for nonlinear fractional differential equations, Comput. Math. Appl. 66 (5) (2013) 728–736.
[37] A.M. Wazwaz, Analytical approximations and Padé's approximants for Volterra's population model, Appl. Math. Comput. 100 (1999) 13–25.
[38] L. Wu, L. Xie, J. Zhang, Adomian decomposition method for nonlinear differential-difference equations, Commun. Nonlinear Sci. Numer. Simul. 14 (1) (2009) 12–18.
[39] S. Tapaswini, S. Chakraverty, Dynamic response of imprecisely defined beam subject to various loads using Adomian decomposition method, Appl. Soft Comput. 24 (2014) 249–263.
[40] A. Hasseine, Z. Barhoum, M. Attarakih, H.-J. Bart, Analytical solutions of the particle breakage equation by the Adomian decomposition and the variational iteration methods, Adv. Powder Technol. 62 (1) (2015) 105–112.
[41] D. Adair, M. Jaeger, Simulation of tapered rotating beams with centrifugal stiffening using the Adomian decomposition method, Appl. Math. Model. 40 (4) (2016) 3230–3241.
[42] P. Moradweysi, R. Ansari, K. Hosseini, F. Sadeghi, Application of modified Adomian decomposition method to pull-in instability of nano-switches using nonlocal Timoshenko beam theory, Appl. Math. Model. 54 (2018) 594–604.
[43] Y.C. Jiao, Y. Yamamoto, C. Dang, Y. Hao, An aftertreatment technique for improving the accuracy of Adomian's decomposition method, Comput. Math. Appl. 43 (6–7) (2002) 783–798.
[44] J. He, A new approach to nonlinear partial differential equations, Commun. Nonlinear Sci. Numer. Simul. 2 (4) (1997) 230–235.
[45] Z. Odibat, On the optimal selection of the linear operator and the initial approximation in the application of the homotopy analysis method to nonlinear fractional differential equations, Appl. Numer. Math. 137 (2019) 203–212.
[46] Z. Odibat, S. Kumar, A robust computational algorithm of homotopy asymptotic method for solving systems of fractional differential equations, J. Comput. Nonlinear Dynam. 14 (8) (2019) 081004.
[47] Z. Odibat, D. Baleanu, A linearization-based approach of homotopy analysis method for non-linear time-fractional parabolic PDEs, Math. Methods Appl. Sci. (2019) http://dx.doi.org/10.1002/mma.5829.
[48] Z. Odibat, A new modification of the homotopy perturbation method for linear and nonlinear operators, Appl. Math. Comput. 189 (1) (2007) 746–753.
[49] R. Fitzhugh, Impulse and physiological states in models of nerve membrane, Biophys. J. 1 (1961) 445–466.
[50] J.S. Nagumo, S. Arimoto, S. Yoshizawa, An active pulse transmission line simulating nerve axon, Proc. IRE 50 (1962) 2061–2070.
[51] P.J. Caudrey, J.C. Eilbeck, J.D. Gibbon, The sine-Gordon equation as a model classical field theory, Nuovo Cimento B 25 (2) (1975) 497–512.
[52] R.K. Dodd, I.C. Eilbeck, J.D. Gibbon, H.C. Morris, Solitons and Nonlinear Wave Equations, Academic Press, London, 1982.