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A new method for predicting the maximum vibration amplitude of periodic solution of non-linear system

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Abstract An original method based on the proposed framework for calculating the maximum vibration amplitude of periodic solution of non-linear system is presented. The problem of determining the worst maximum vibration is transformed into a non-linear optimization problem. The harmonic balance method and the Hill method are selected to construct the general non-linear equality and inequality constraints. The resulting constrained maximization problem is then solved by using the MultiStart algorithm. Finally, the effectiveness of the proposed approach is illustrated through two numerical examples. Numerical examples show that the proposed method can, at much lower cost, give results with higher accuracy as compared with numerical results obtained by a parameter continuation method.

Keywords Generalized framework · Periodic solution · The maximum vibration amplitude · Harmonic balance method · Hill's method · The MultiStart algorithm

1 Introduction

There are two categories of solution techniques for determining periodic solutions of non-linear systems: the time domain method such as the shooting method and the frequency method. Within the frequency method, the harmonic balance method (HBM) has formed a versatile tool for various non-linear vibration problems. Many variants on the HBM have emerged, such as the Alternating Frequency/Time domain (AFT) method [1], the high dimensional harmonic balance (HDHB) approach [2, 3] and the constrained harmonic balance method[4]. Recently, Cochelin et al. [5] have proposed another strategy for applying the classical HBM with a large number of harmonics. More recently, an adaptive harmonic balance method was developed in Ref. [6]. By making use of either the harmonic balance method or the shooting method, a set of non-linear equations is formed and a non-linear solver is then used to find zeros of such non-linear equations. However, in order to perform parametric studies the non-linear solver must be used recursively.

Beside the vast research on the method of finding periodic solution, the stability analysis of periodic solution is another important part of non-linear investigation [7–9]. Stability in the time domain is usually determined by determining the monodromy matrix. This procedure is, however, both time consuming and costly to implement. Some other techniques exploit the HBM results directly to determine the local stability of periodic solutions of dynamical systems. For instance, Groll and Ewins [10] combined the harmonic balance method and Hill's approach together to prove the stability of the rotor dynamics rubbing against a casing.

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When a simulation has to be done over some range of parameter variation, continuation methods have to be applied to follow the solutions. Many non-linear vibration problems [11–15] are investigated using the continuation method. Recently, the non-linear normal modes of non-linear dynamic systems have been studied in Refs. [16–18] by using a shooting algorithm with the pseudo-arc length continuation. Within the framework of asymptotic numerical methods (ANM), the approach combining a harmonic balance method developed in [5] and Hill's Method is presented in Ref. [19]. In Refs. [20, 21], the harmonic balance and the arc-length continuation methods have been employed to analysis geometrically non-linear free and forced vibrations of blade disks. More details regarding continuation techniques can be found in Ref. [22].

There is a need for new and efficient methods for finding the peak vibration amplitudes in non-linear dynamical systems. For example, Petrov [23–26] applies the multi-harmonic balance method to compute the worst vibration cases of the mistuned disc assemblies including the friction dampers as well aerodynamic couplings. Also, Chan and Ewins showed the interesting searching strategy in their paper from 2010 [27]. With the help of continuation method, the peak vibration amplitude of non-linear systems can be determined at low computational costs. However, the combined effects of parameter variations on the peak vibration amplitude cannot be investigated via the continuation method when all the parameters studied vary simultaneously. The objective of the present work is to develop a systematic methodology to determine the non-linear resonant peak of non-linear systems.

The remainder of this paper is organized as follows: the general formulation of the proposed framework in determining the maximum vibration amplitude of periodic solution is presented in Sect. 2. An original approach based on the proposed framework is also developed. Validations of the method are then conducted in Sect. 3, which gives some numerical examples. Finally, concluding remarks are presented and discussed in Sect. 4.

2 The proposed framework

The primary purpose of this contribution is to develop a novel framework in calculating the maximum vibration amplitude of periodic solution for non-linear systems. The main idea pursued is to adopt methods and concepts from the non-linear dynamics, including:

1. The time and frequency domain methods formulation leading to the non-linear algebraic equations.
2. The stability concept of periodic solution for non-linear systems.
3. A worst-case performance assessment to avoid the unacceptable computation burden involved with exhaustive search in the presence of parameters uncertainties.

In structural dynamics, a level of forced response is generally dependent on set of values of design parameters for a structure analyzed. In order to facilitate a choice of a set of design parameters, we need to determine not only the resonance peak frequency and response level but also to obtain estimates of the effects of design parameters and these parameters' uncertainty on the maximum vibration amplitude of non-linear structures. Therefore, the optimization formulation for determining the maximum vibration amplitude of periodic solution in non-linear structural dynamics can be expressed as follows:

$$\max \ f(\mathbf{x})$$
$$\text{s.t.} \begin{cases} f_{\mathrm{NAE}}(x) = \mathbf{NonAlgEqu}_i(\mathbf{x}) = 0, \\ \quad i = 1, 2, \ldots, m_e \\ f_{\mathrm{PSS}}(x) = \mathrm{PerSolStability}(\mathbf{x}) \le 0 \\ \mathbf{x}_l \le \mathbf{x} \le \mathbf{x}_u \end{cases} \tag{1}$$

where $\mathbf{x}$ is an optimization variable vector which has a lower bound $\mathbf{x}_l$ and an upper bound $\mathbf{x}_u$, $f : \mathbb{R}^n \to \mathbb{R}$ is the objective function, which returns a scalar value. The objective for this analysis is the maximization of the non-linear system vibration amplitude. The vector function $f_{\mathrm{NAE}}(\mathbf{x}) = \mathbf{NonAlgEqu}(\mathbf{x})$ returns a vector of length m_e containing the values of the equality constraints evaluated at $\mathbf{x}$. m_e is the number of equality constraints. The non-linear algebraic equations $\mathbf{NonAlgEqu}_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m_e$ which denote the motion equation of the non-linear system can be obtained by making use of either the time domain methods or the frequency domain methods (such as the generalized shooting method and the harmonic balance method). The function $f_{\mathrm{PSS}}(\mathbf{x}) = \mathrm{PerSolStability}(x)$ stands for the stability analysis of periodic solution related to $\mathbf{x}$.

The non-linear algebraic equations are considered as the general non-linear equality constraints while the stability analysis of periodic solution is treated as the non-linear inequality constraint $f_{\mathrm{PSS}}(\mathbf{x}) = \mathrm{PerSolStability}(\mathbf{x}) \leq 0$. The proposed framework is the first to employ a global search optimization framework with non-linear equality and inequality constraints.

Based on the generalized framework described above, the non-linear constrained optimization problem is organized into three steps:

1. Modeling and casting the non-linear dynamics motion equation into the non-linear algebraic equations that are considered as general non-linear equality constraints.
2. Setting up the non-linear inequality constraint of the optimization problem in Eq. (1) based on stability analysis of period solution.
3. Implementing a GlobalSearch algorithm for solution of the optimization problem.

Following these steps, a detailed method (named GlobalSearch-HBM-HILL method) which combines the Harmonic balance method and Hill's method along with a GlobalSearch algorithm is proposed in this section. A set of non-linear algebraic equations derived from the harmonic balance method forms the non-linear equality constraints of the optimization problem in Eq. (1). The non-linear inequality constraint to determine the stability of periodic solution is obtained by using Hill's method. With these non-linear equality and inequality constraints, the MultiStart algorithm is selected to find the resonant peak amplitude of non-linear systems. It should be pointed out that the organization of this section is not only based on the three steps of the framework, but also on the logical flow of the concepts. The HBM formulation is illustrated in Sect. 2.1 while the procedures associated with stability analysis are discussed in Sect. 2.2, the complete formulation of the optimization problem is finally described in Sect. 2.3 after its initial development in Sects. 2.1 and 2.2.

2.1 Harmonic balance approach

To obtain the periodic solutions of non-linear system, the harmonic balance method is adopted. The main idea of the method is to impose a harmonic solution with unknown coefficients and period equal to the period of the excitation. To present the method, the following equation of motion is considered:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{D}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \mathbf{f}_{\mathrm{nl}}(\mathbf{x}, \dot{\mathbf{x}}, \omega, t) = 0 \tag{2}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are, respectively, the generalized mass, damping, and stiffness matrices, $\mathbf{x}$ and $\mathbf{f}_{\mathrm{nl}}(\mathbf{x}, \dot{\mathbf{x}}, \omega, t)$ are, respectively, the displacement and force vectors and ω is the excitation frequency.

In the HBM, $\mathbf{x}(t)$ can be written as a Fourier series up to the N_{H}th term:

$$\mathbf{x}(t) = \mathbf{U}_0 + \sum_{k=1}^{N_{\mathrm{H}}} \left[U_{c,k} \cos(k\omega t) + U_{s,k} \sin(k\omega t) \right] \tag{3}$$

Substituting Eq. (3) into Eq. (2) and applying the Galerkin procedure yield the following non-linear function:

$$\mathbf{f}_{\mathrm{NAE}}(\mathbf{U}, \omega) = \mathbf{A}(\omega)\mathbf{U} - \mathbf{b}(\mathbf{U}, \omega) = \mathbf{0} \tag{4}$$

where

$$\mathbf{b} = \begin{bmatrix} C_0^T & \mathbf{C}_1^T & \mathbf{S}_1^T & \cdots & \mathbf{C}_k^T & \mathbf{S}_k^T & \cdots & \mathbf{C}_{N_{\mathrm{H}}}^T & \mathbf{S}_{N_{\mathrm{H}}}^T \end{bmatrix}^T$$

corresponds to the Fourier coefficients of the non-linear forcing term; $\mathbf{A}(\omega)$ and $\mathbf{U}$ are, respectively, defined by

$$\begin{aligned} \mathbf{A} = \mathbf{diag}\Bigg(\mathbf{K}, &\begin{bmatrix} \mathbf{K} - \omega^2\mathbf{M} & \omega\mathbf{D} \\ -\omega\mathbf{D} & K - \omega^2\mathbf{M} \end{bmatrix}, \ldots, \\ &\begin{bmatrix} \mathbf{K} - (k\omega)^2\mathbf{M} & k\omega\mathbf{D} \\ -k\omega\mathbf{D} & \mathbf{K} - (k\omega)^2\mathbf{M} \end{bmatrix}, \ldots, \\ &\begin{bmatrix} \mathbf{K} - (N_{\mathrm{H}}\omega)^2\mathbf{M} & N_{\mathrm{H}}\omega\mathbf{D} \\ -N_{\mathrm{H}}\omega\mathbf{D} & \mathbf{K} - (N_{\mathrm{H}}\omega)^2\mathbf{M} \end{bmatrix} \Bigg) \\ \mathbf{U} = \big[\mathbf{U}_0^T \quad &\mathbf{U}_{c,1}^T \quad \mathbf{U}_{s,1}^T \quad \cdots \quad \mathbf{U}_{c,k}^T \quad \mathbf{U}_{s,k}^T \quad \cdots \quad \mathbf{U}_{c,N_{\mathrm{H}}}^T \quad \mathbf{U}_{s,N_{\mathrm{H}}}^T \big]^T \end{aligned} \tag{5}$$

The difficulty with solving Eq. (4) is in finding a relationship between $\mathbf{b}(\mathbf{U}, \omega)$ and $\mathbf{U}$ since the Fourier coefficients of the non-linear forcing term are implicit functions of the Fourier coefficients of the displacement. To overcome this difficulty, the alternating frequency time technique [1] shown in Eq. (6) is employed.

$$\mathbf{U} \overset{\mathrm{IFFT}}{\Rightarrow} \mathbf{x(t)} \Rightarrow \mathbf{f}_{\mathrm{nl}}(\mathbf{x}, \dot{\mathbf{x}}, \omega, t) \overset{\mathrm{FFT}}{\Rightarrow} \mathbf{b}_{\mathrm{nl}}(\mathbf{U}, \omega) \tag{6}$$

When the conventional HBM is used, Eq. (4) is the set of non-linear equations being directly solved by a Newton–Raphson-type method. However, unlike the traditional implementation of the harmonic balance method, the non-linear equations are used to construct the non-linear equality constraints of optimization problem in Eq. (1).

2.2 Stability analysis with Hill's method

After the periodic solution is obtained, the stability analysis of such periodic solution should be conducted. In the following, the stability analysis of periodic solution is analyzed by the Hill's method, and the developments in Ref. [19] are reviewed to some extent to provide adequate theoretical background to address problem in Eq. (1).

For analyzing the stability of the system, Eq. (2) is rewritten in the state space form as follows:

$$\dot{\mathbf{x}}(t) = f\big(\mathbf{x}(t), t\big) \tag{7}$$

where $\mathbf{x}$ is a N-dimensional state vector and f which may explicitly depend on t (the system is non-autonomous) or not (the system is autonomous) is a non-linear N-dimensional vector field.

Assuming that $\mathbf{x}_0(t)$ is the periodic solution obtained by the HBM, its stability is usually investigated by superposing a small perturbation $\mathbf{y}(t)$ on $\mathbf{x}_0(t)$. Substituting $\mathbf{x}(t) = \mathbf{x}_0(t) + \mathbf{y}(t)$ into Eq. (7) and expanding the result in a Taylor series about $\mathbf{x}_0(t)$ yield the first approximation of Eq. (7):

$$\dot{\mathbf{y}}(t) = \frac{\partial f}{\partial \mathbf{x}}\big(\mathbf{x}_0(t), t\big)\mathbf{y}(t) \tag{8}$$

where $\frac{\partial f}{\partial \mathbf{x}}(\mathbf{x}_0(t), t)$ is the $N \times N$ Jacobian matrix of f evaluated at $\mathbf{x}_0(t)$. Thus the study of the stability of periodic solution of the non-linear system is transformed into the stability analysis of null solution of Eq. (8).

Because system Eq. (8) is a linear system with periodic coefficient, the stability characteristic of null solution of Eq. (8) can be determined by using Floquet's theory. Applying the harmonic balance method to Eq. (8) leads to the following infinite-dimensional eigenvalue problem:

$$(\mathbf{H} - s\mathbf{I})\mathbf{q} = \mathbf{0} \tag{9}$$

where $\mathbf{H}$ is the infinite-dimensional Hill matrix, s is a complex number, $\mathbf{q}$ is an infinite-dimensional vector and $\mathbf{I}$ is the identity matrix of appropriate size.

Following the approach in Ref. [19], matrix $\mathbf{H}$ is truncated to an $N(2\mathrm{N_H} + 1) \times N(2\mathrm{N_H} + 1)$ dimension. Its $N(2\mathrm{N_H} + 1)$ eigensolutions are then computed. Finally, the N Floquet exponents are obtained: α_n, $n = 1, 2, \ldots, N$. Either the Floquet multipliers ρ_n or the Floquet exponents α_n can be used to determine the stability of periodic solutions. With the application of Floquet theory, the necessary and sufficient conditions for stability require $\Re(\alpha_n) < 0$ (or $|\rho_n| < 1$) for all n. Therefore, the following stability criteria can be inferred:

$$\begin{aligned} &f_{\mathrm{PSS}}(\mathbf{U}, \omega) = \max\big(\Re(\alpha)\big) < 0 \quad \text{or} \\ &f_{\mathrm{PSS}}(\mathbf{U}, \omega) = \max\big(|\rho|\big) - 1 < 0 \end{aligned} \tag{10}$$

where $\mathbf{U}$ is the harmonic coefficients corresponding to $\mathbf{x}_0(t)$, the vectors α and ρ are defined as $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]$, $\rho = [\rho_1, \rho_2, \ldots, \rho_N]$, respectively.

The stability of the periodic solutions is investigated by studying the evolution of perturbations to the solutions. Using Hill's method, the stability of periodic solution is examined on a basis of eigenvalues resulting from the eigenvalue problem (9). Finally, the stability study results in the non-linear inequality constraint of the generalized framework which takes the form of Eq. (10).

2.3 Optimization with GlobalSearch

As our objective of optimization is to find the optimal solution aiming to maximize the vibration amplitude of the non-linear system of Eq. (2), thus the non-linear equality and inequality constrains of the optimization problem formulation in Eq. (1) must be organized. Referring to the non-linear equality constraints, the non-linear algebraic equations in Eq. (4) constitute the general non-linear equality constraints. With respect to the non-linear inequality constraint, the stability condition in Eq. (10) forms the general non-linear inequality constraint.

In order to calculate the maximum non-linear resonant peak of non-linear system, Eqs. (4) and (10) have to be combined and solved simultaneously according to the proposed framework described in Eq. (1). Therefore, the following non-linear optimization problem can be formulated:

$$\max \|\mathbf{x}\|_\infty = \max f(\omega, \mathbf{U}, \mathbf{v}_m)$$
$$\text{s.t.} \begin{cases} \mathbf{f}_{\text{NAE}}(\mathbf{U}, \omega, \mathbf{v}_m) \\ \quad = \mathbf{A}(\omega, \mathbf{v}_m)\mathbf{U} - \mathbf{b}(\mathbf{U}, \omega, \mathbf{v}_m) = \mathbf{0} \\ f_{\text{PSS}}(\mathbf{U}, \omega, \mathbf{v}_m) = \max|\rho| - 1 < 0 \end{cases} \tag{11}$$

where $\mathbf{v}_m$ is a set of design parameters and/or uncertainty parameters and ω is the unknown frequency to be determined.

The solution of this non-linear optimization problem with respect to a vector of unknowns $\mathbf{x} = \{\mathbf{U}, \omega, \mathbf{v}_m\}^T$ gives a resonance frequency, ω^{opt} and a vector of harmonic coefficients $\mathbf{U}^{\text{opt}}$ at a set of parameter values $\mathbf{v}_m^{\text{opt}}$. Their accurate and effective calculation is a very important problem, which is discussed in the following section.

The choice of optimization algorithm is very important, because the final results are usually dependent on the specific algorithm in terms of accuracy and local minima sensitivity. Evolutionary algorithms are less sensitive to local minima; however, they are time-consuming, and constraints have to be included as a penalty term to the objective function. On the other hand, gradient-based algorithms can lack in global optimality but allow multiple constraints and are more robust, especially for problems in which a large number of constraints are prescribed. In this investigation, the advanced OptQuest Non-linear Programs (OQNLP) MultiStart gradient-based algorithm from Ref. [28] is implemented and the gradients are approximated by finite differences. For the purpose of completeness sake, short descriptions of optimization algorithms used in this study are briefly outlined below.

2.3.1 The OQNLP MultiStart algorithm for global optimization [28]

The authors of Ref. [28] described the OQNLP MultiStart algorithm for global optimization which enable them to find feasible solutions to a system of non-linear constraints more efficiently. The OQNLP MultiStart algorithm uses the OptQuest Scatter search algorithm to generate candidate starting points for a local NLP solver. This paper adapts the OQNLP MultiStart algorithms to seek a feasible solution to a system of non-linear constraints.

The optimization problem of Eq. (11) can be rewritten as the general non-linear optimization formulation as follows:

$$\min \ f(\mathbf{x})$$
$$\text{s.t.} \begin{cases} g_i(\mathbf{x}) = 0, & i = 1, 2, \ldots, m_e \\ g_i(\mathbf{x}) \leq 0, & i = m_e + 1, \ldots, m \\ \mathbf{x}_l \leq \mathbf{x} \leq \mathbf{x}_u \end{cases} \tag{12}$$

where $\mathbf{x}$ is the vector of length n optimization variables, $f(\mathbf{x})$ is the objective function, which returns a scalar value, and the vector function $g(\mathbf{x})$ returns a vector of length m containing the values of the equality and inequality constraints evaluated at $\mathbf{x}$.

For constrained optimization problem, most methods were based on penalty function methods that transform $f(\mathbf{x})$ into an unconstrained function $F(\mathbf{x})$, consisting of a sum of the objective and the constraints weighted by penalties. By using penalty function methods, the objective is inclined to guide the search toward the feasible solution. The $L1$ exact penalty function is used as a merit function for evaluating candidate starting points. For the problem of Eq. (12) this function is

$$P(\mathbf{x}, w) = f(\mathbf{x}) + \sum_{i=1}^{m} w_i \cdot \text{viol}\big(g_i(\mathbf{x})\big) \tag{13}$$

where w_i are positive penalty weights, the function $\text{viol}(g_i(\mathbf{x}))$ equals the absolute violation of the ith constraint of problem Eq. (12) at the point $\mathbf{x}$.

There are two stages of the algorithm. The algorithm performs n1, n2 iterations for the stage 1 and stage 2, respectively. At each iteration, the starting point generator $SP(xt)$ produces the candidate starting point xt and this point is also used to calculate the $L1$ exact penalty value $P(xt, w)$. After finishing n1 iterations of stage 1, the local solver L is called at the best point that has the smallest value of P in stage 1. In stage 2, the MultiStart algorithm runs the local solver L starting at the points that pass the distance and merit filters. A detailed analysis of the algorithm can be found in Ref. [28].

2.3.2 Sequential quadratic programming

The Sequential Quadratic Programming (SQP) methods are known to be powerful when solving problems with significant non-linearities, so the SQP method will be used in the MultiStart procedure. In the following, the principle of the SQP method is briefly introduced.

The main idea of the SQP method is to formulate the following Quadratic Programming (QP) sub-problem to be solved:

$$\begin{aligned} &\min \ \frac{1}{2}\mathbf{d}_k^T \mathbf{H}_k \mathbf{d}_k + \nabla f(\mathbf{x}_k)^T \mathbf{d}_k \\ &\text{s.t.} \\ &g_i(\mathbf{x}_k) + \nabla g_i(\mathbf{x}_k)^T \mathbf{d}_k = 0, \quad i = 1, 2, \ldots, m_e \\ &g_i(\mathbf{x}_k) + \nabla g_i(\mathbf{x}_k)^T \mathbf{d}_k \le 0, \quad i = m_e + 1, 2, \ldots, m \end{aligned} \tag{14}$$

where $\mathbf{d}_k$ is defined as the search direction and $\mathbf{H}_k$ denotes the Hessian of the Lagrangian function which is updated using the Broyden–Fletcher–Goldfarb–Shanno (BFGS) updated formula.

The method used to solve the QP sub-problem is the active set strategy. The solution of Eq. (14) is used to form a search direction $\mathbf{d}_k$ for a line search procedure. In other words, the solution is used to form the next iterate:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + a_k \mathbf{d}_k \tag{15}$$

The step length parameter α_k is determined by an appropriate line search procedure so that a sufficient decrease in a merit function is obtained. The algorithm above only outlines the principles of the SQP method. Details of the SQP procedure can be found in the literature [29, 30] and references therein.

This section proposes a method that returns the maximum vibration amplitude of a structure. The result is obtained by a concatenation of three methods: harmonic balance method to turn the dynamical problem into an algebraic one, the Hill strategy to determine the stability of the solution (and keep only the stable ones) and finally a MultiStart Algorithm to find the one point that satisfies HBM equalities plus the stability constraint inequality and which maximizes the vibration amplitude.

The complete procedure and description of the GlobalSearch-HBM-HILL method is given in Fig. 1. It should be emphasized that the use of the optimization Algorithm described in Sect. 2.3 is particular important. According our optimization experiments, other optimization methods such as the generic algorithm are very difficult to use to find the real solution of Eq. (11). In addition, it should be noted that the frequency methods which are used to form the non-linear constraints can be replaced with the methods in the time domain.

3 Application to select non-linear dynamical systems

Numerical simulations are presented in this section to demonstrate the validity and the efficiency of the proposed approach. Two numerical examples are presented.

3.1 Duffing oscillator

A classical non-linear Duffing oscillator which has been used extensively in the literature is used as the first example. The accuracy of the proposed method in predicting the maximum vibration response of the forced Duffing oscillator is examined by comparing the predictions with those obtained from the HBM-ANM-HILL method in Refs. [5, 19].

The equation of motion for the Duffing oscillator is given by

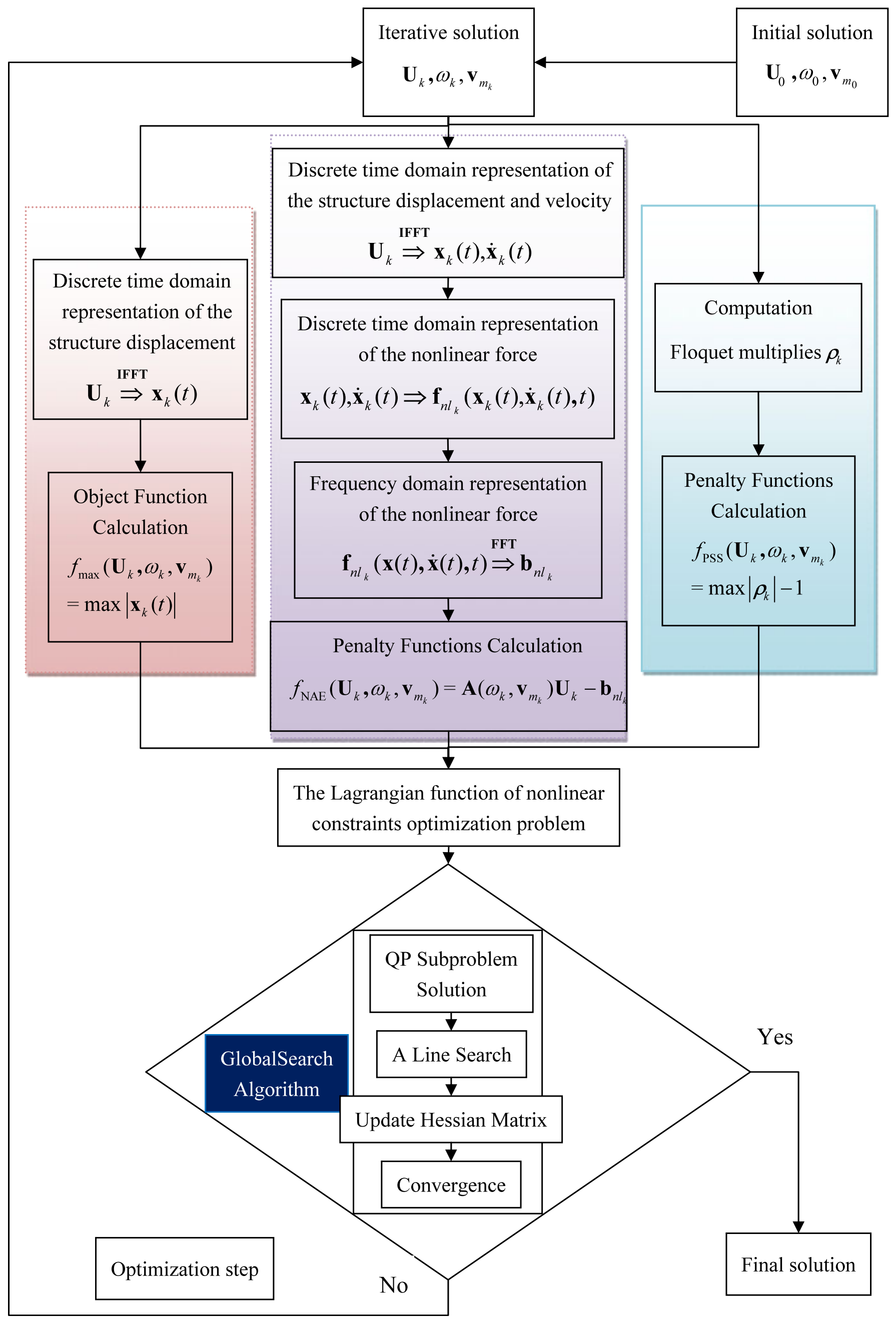
$$\ddot{u} + \mu\dot{u} + u + \beta u^3 = f\cos(\lambda t) \tag{16}$$

where β is the non-linear stiffness coefficient; μ, λ are the damping coefficient and the forcing angular frequency, respectively; f is the force amplitude.

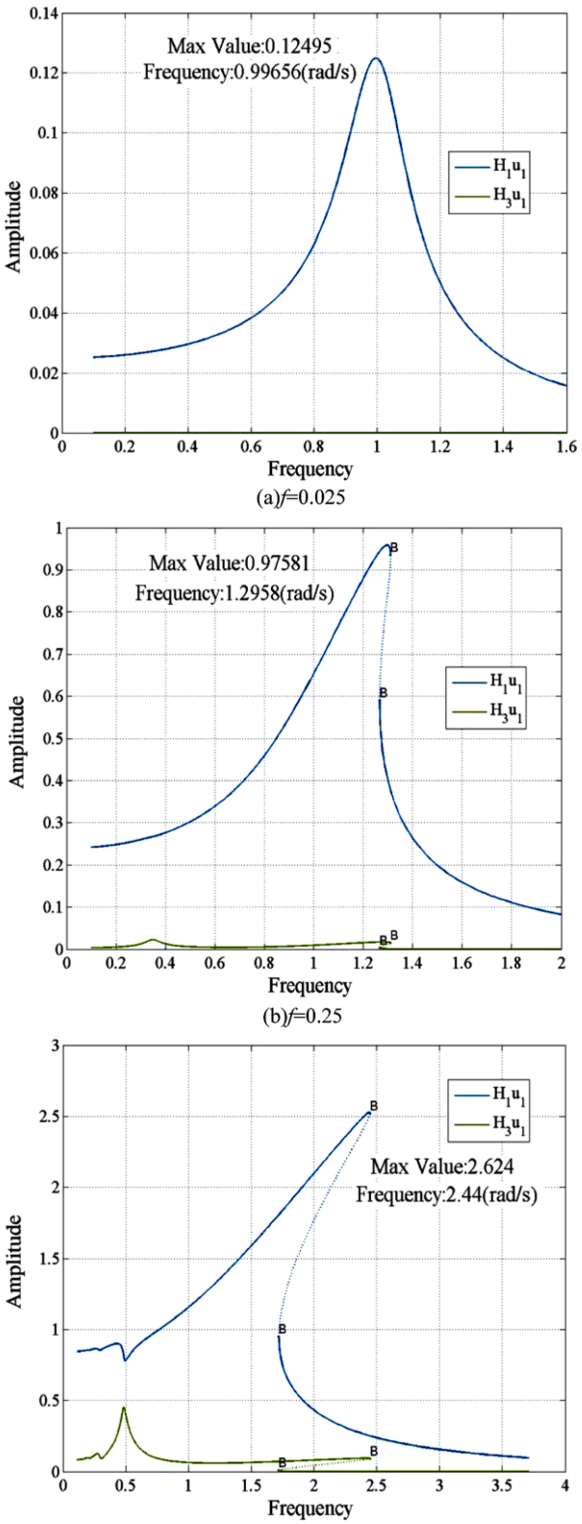
Using numerical continuation of periodic solutions with the excitation frequency as continuation parameter, frequency-response curves shown in Fig. 2 are drawn for specific values of external excitation amplitude $f = 0.025$, 0.25 and 1.25 with $\mu = 0.1$, $\beta = 1$. Notation B in Fig. 2 stands for simple bifurcation in structural dynamics. These curves in Fig. 2 are obtained using the HBM-ANM-HILL method in Refs. [5, 19]. Calculations were made for the frequency range between 0.1 and 3.6 for three levels of the non-linear forces.

Figure 2 shows that the maximum response increases with the increasing in f, whereas the frequency of the maximum response increases. The system reaches the maximum vibration amplitude at three points of which the corresponding frequencies are $\omega = 0.99577, 1.2962, 2.4402$. The frequency-response curve of the Duffing's equation is bended to the right. A classical bent resonance curve can be observed when the non-linear force is strong ($f = 1.25$). Note that only the individual amplitudes of the odd harmonics have been plotted, since the even harmonics are equal to zero. The results shown in Fig. 2 are identical to those reported in Ref. [3].

Fig. 1 Flowchart of the proposed approach



Based on the OQNLP MultiStart algorithm described in Sect. 2.3.1, optimization is then performed to find the maximum vibration amplitude of the forced Duffing oscillator and the parameters of the optimization algorithm are listed as follows: Number of trial points was chosen to be 1000, and the usual value of 200 for the number of Stage1 points has been taken. The maximum number of generations allowed for the SQP algorithm was 600 while the function tolerance and the non-linear constraints tolerance were both set to 10^{-12}. As explained previously, the unknown parameters that have to be determined are the Fourier coefficients U and the frequency ω of the stability periodic solution. A frequency range from 0 to 3 Hz



(a)f=0.025

(b)f=0.25

(c)f=1.25

Fig. 2 Frequency-response curves for various external excitation values

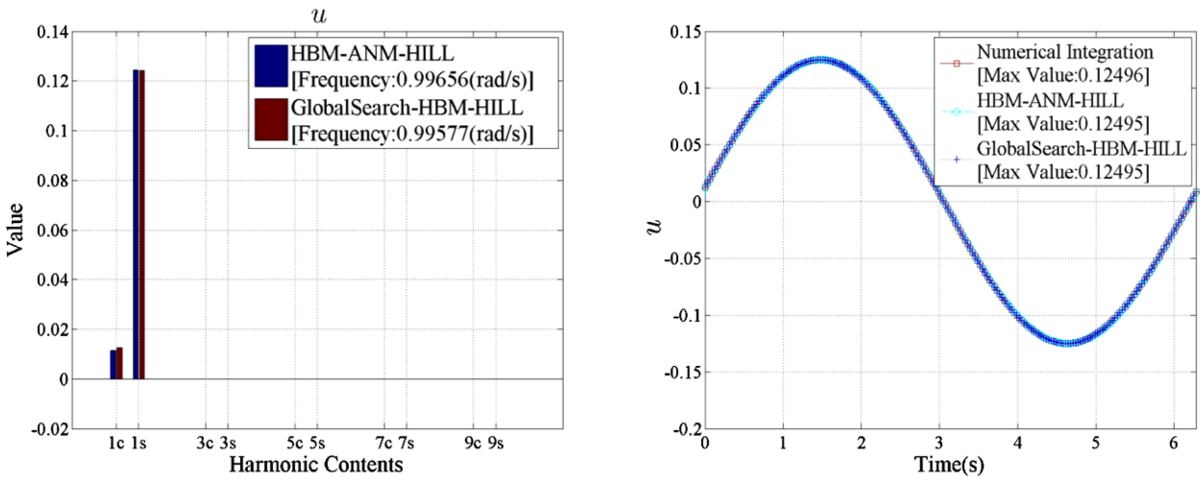
is considered, the vibration displacements covered by this frequency range are to be maximized. The number of harmonics retained N_H is 10.

Numerical results are reported in Fig. 3 where the following notation is used: the blue bars represent the solutions obtained from the HBM-ANM-HILL method while the red bars represent the solutions obtained from the proposed method. It can be observed that the results from the proposed method and the HBM-ANM-HILL method agree very well with each other. Comparison with conventional forced response curves plotted in Fig. 2 shows that the proposed method really finds the resonance peaks. In addition, on the right in Fig. 3 the results from numerical integrations are also compared with the results from the GlobalSearch-HBM-HILL method. From this comparison, it is seen that the numerical integration results are well correlated with the GlobalSearch-HBM-HILL solution.

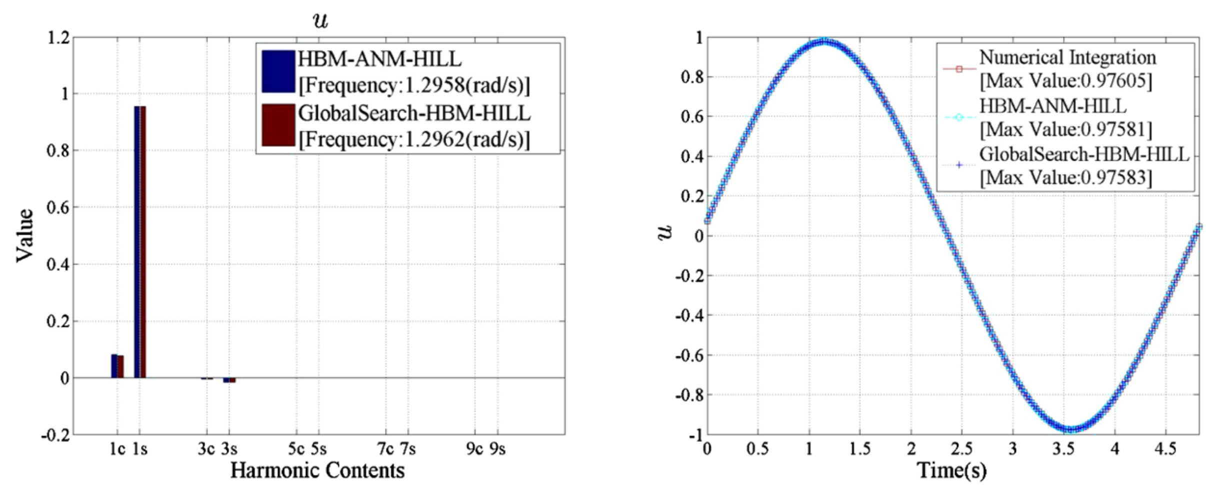
As the Hill matrix is computed during the optimization iterative process, the Floquet multipliers are obtained as a byproduct. Table 1 summarizes the values of the non-linear constraints that were evaluated at these optima solutions. In Table 1, the second column denotes the maximum absolute errors of the non-linear algebraic equations in Eq. (4) and the third column indicates the Floquet multipliers obtained from the eigenproblem described by Eq. (9), where an absolute value less than one means that the periodic solution of the Duffing system is stable. For comparison, the stability results obtained by the HBM-ANM-HILL method are also listed.

As observed from Table 1, the moduli of the Floquet multipliers is less than 1, so the solutions related to the three points obtained by using the two methods are stable. It can be seen that the discrepancies of the results between the GlobalSearch-HBM-HILL method and the HBM-ANM-HILL method are quite small. In addition, it can be seen that the non-linear equality and inequality constraints are satisfied.

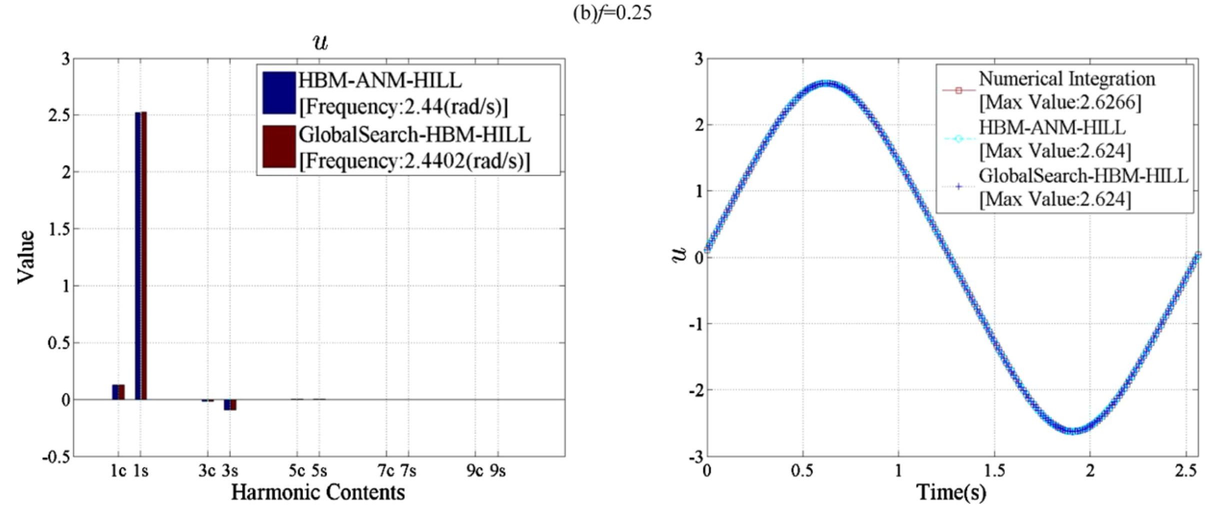
The comparison of the CPU time between the two methods for the three cases is shown in Table 2. As can be seen from Table 2, the HBM-ANM-HILL method took 64.432813, 117.875934, 202.850552 seconds of CPU time for the three cases studied here. It is obvious that the CPU time needed by the proposed method to obtain the worst-case forced response for a given frequency of the interval studied is much less than that of the HBM-ANM-HILL method. The comparison of the



(a)f=0.025



(b)f=0.25



(c) f=1.25

Fig. 3 The optimal solutions of the Duffing oscillator

Table 1 The non-linear equality and inequality constraints of the optimization solutions

f	The maximum absolute error of the non-linear equality constraints	GlobalSearch-HBM- HILL		HBM-ANM-HILL	
		ρ_1, ρ_2	$\|\rho_1\| = \|\rho_2\|$	ρ_1, ρ_2	$\|\rho_1\| = \|\rho_2\|$
0.025	9.3892×10^{-017}	$0.5316 \pm 0.0277\mathrm{i}$	0.5323	$0.5312 \pm 0.0309\mathrm{i}$	0.5321
0.25	5.5511×10^{-016}	$0.6001 \pm 0.1381\mathrm{i}$	0.6158	$0.6015 \pm 0.1323\mathrm{i}$	0.6159
1.25	3.5527×10^{-015}	$0.7612 \pm 0.1346\mathrm{i}$	0.7730	$0.7618 \pm 0.1309\mathrm{i}$	0.7730

Table 2 The comparison of the CPU time between the two methods for the three cases

	The HBM-ANM-HILL method	The proposed method
$f = 0.025$	64.432813	4.4123
$f = 0.25$	117.875934	5.2527
$f = 1.25$	202.850552	5.4224

computational cost shows the outstanding benefits that stem from the use of the present method.

In addition, for the three cases convergence is achieved after 458, 583, 1664 number of function evaluations using the proposed method. Since the final solution of the previous step is provided as the initial solution of the next step in the HBM-ANM-HILL algorithm and no(or very fewer) correction iterations are required in this example, so the number of iteration required by the HBM-ANM-HILL method is not obtained.

3.2 Axially moving beam with lateral excitation

Axially moving systems can be simple models of many engineering devices, such as power transition chains, fiber textiles, band saw blades, magnetic tapes and conveyor belts. A significant amount of research has been carried out on axially moving systems in various pieces of literature [31–33]. The second model represents non-linear vibrations of an axially moving viscoelastic beam. Following Ref. [34], the governing dynamic equations of the problem can be expressed as follows:

$$\begin{aligned} &\ddot{u}_1 + \mu_{11}\dot{u}_1 - \mu_{12}\ddot{u}_2 + k_{11}u_1 + k_{12}u_1u_2^2 + k_{13}u_1^3 \\ &\quad = f_1\cos(\lambda t) \\ &\ddot{u}_2 + \mu_{21}\dot{u}_1 + \mu_{22}\ddot{u}_2 + k_{21}u_2 + k_{22}u_2u_1^2 + k_{23}u_2^3 \\ &\quad = f_2\cos(\lambda t) \end{aligned} \tag{17}$$

where u_1 and u_2 are motion coordinates; μ_{12} and μ_{21} are the gyroscopic coefficients that provide an internal damping effect to the system; μ_{11} and μ_{22} arise from external viscous damping; f_1 and f_2 are forcing coefficients and λ is the exciting frequency.

In the system of this case study, the system parameters under consideration are $\mu_{11} = \mu_{22} = 0.04$, $\mu_{12} = \mu_{21} = 3.2$, $k_{11} = 9.23882$, $k_{12} = k_{22} = 3372\pi^4$, $k_{13} = 421.5\pi^4$, $k_{21} = 72.0226$, $k_{23} = 6744\pi^4$, $f_1 = 0.0055$, $f_2 = 0$. This system has already been studied in Ref. [34] using the incremental harmonic balance method.

For comparison, forced responses are also calculated as a function of excitation frequency using the HBM-ANM-HILL method. Figure 4 shows the frequency-response curves where H_1u_1–H_1u_2 and H_3u_1–H_3u_2 are the amplitudes of the first and the third harmonic terms in u_1 and u_2, respectively. Notation NS in Fig. 4 represents the Neimark–Sacker bifurcation. As shown in Fig. 4, when the force frequency is near the resonance frequency 5.2521(rad/s), the response of the system reaches the top amplitude of the frequency-response curve. The maximum displacement response amplitude is 0.023996.

Using the proposed method, the optimization results which are compared with those counterparts using the HBM-ANM-HILL method are presented in Fig. 5. Observe in Fig. 5 that excellent agreement between the two methods is observed and the GlobalSearch-HBM-HILL method finds very accurately the resonance peak.

The time responses of the non-linear system at the frequency of 5.2527 (rad/s) are shown in Fig. 6 using the GlobalSearch-HBM-HILL method and the

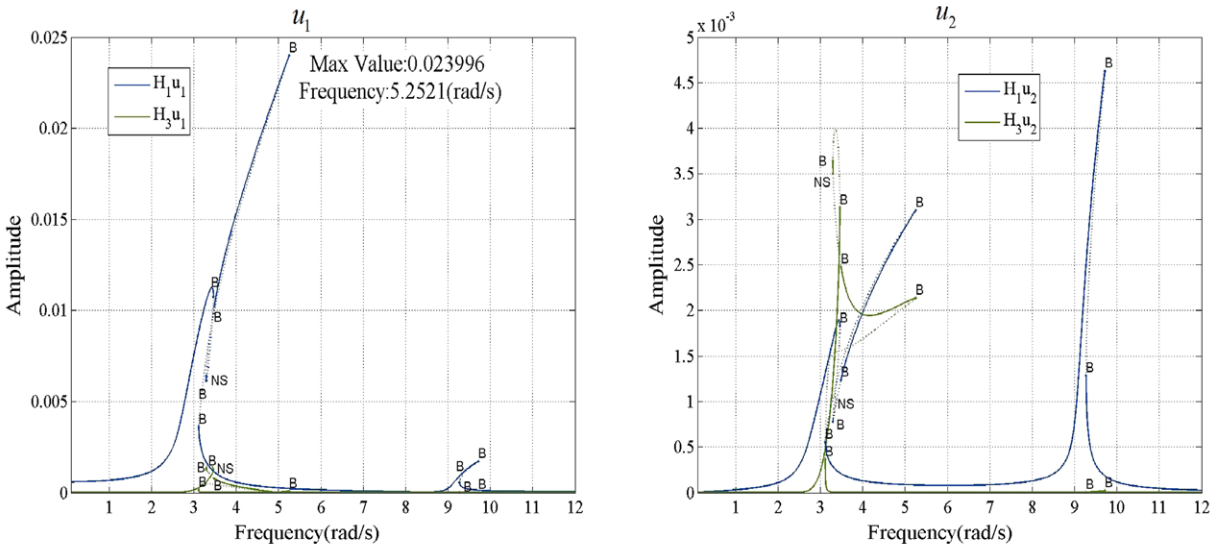


Fig. 4 Non-linear frequency responses of the non-linear system of Eq. (17)

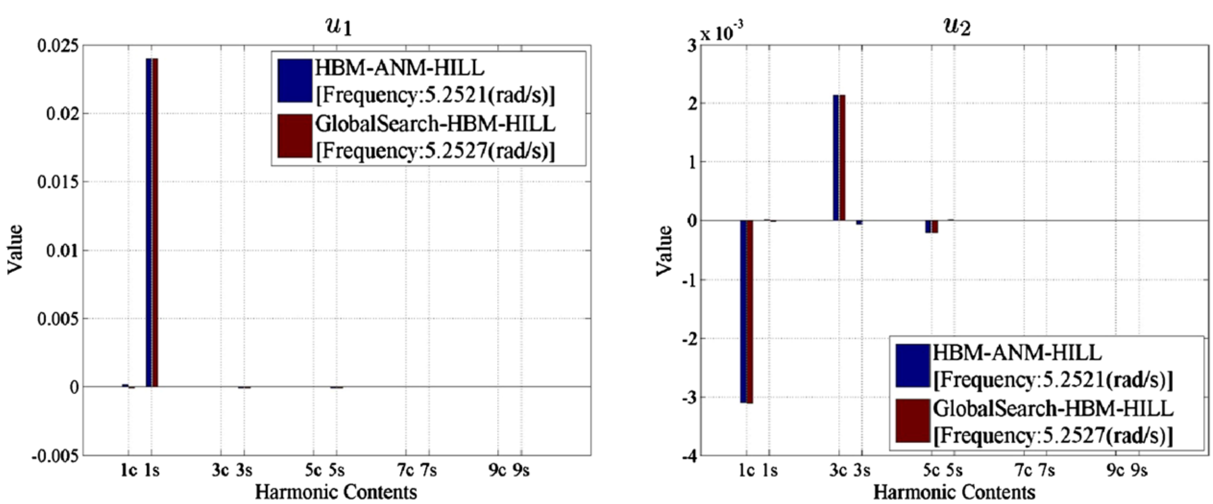
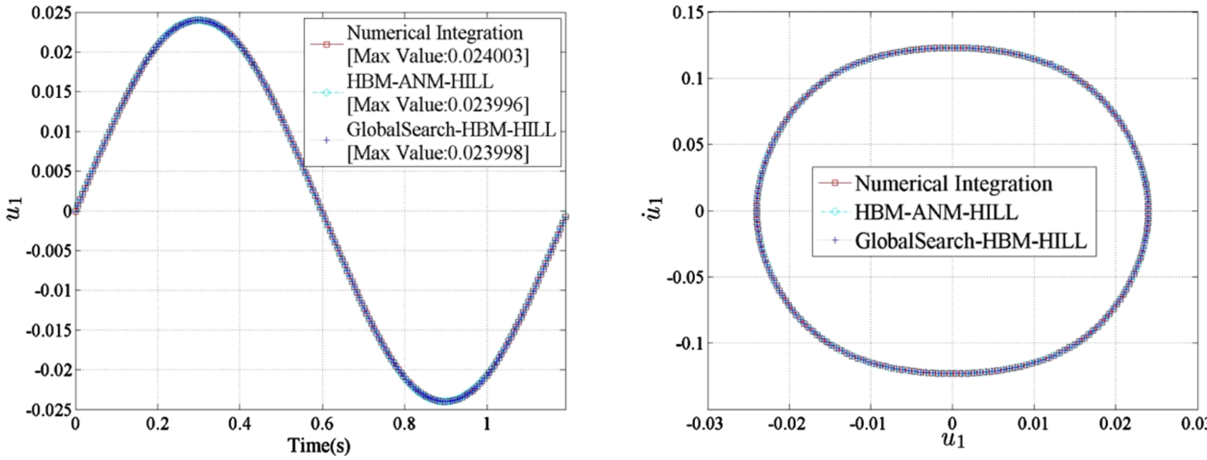


Fig. 5 The optimization results of the proposed method and its counterpart using the continuous method

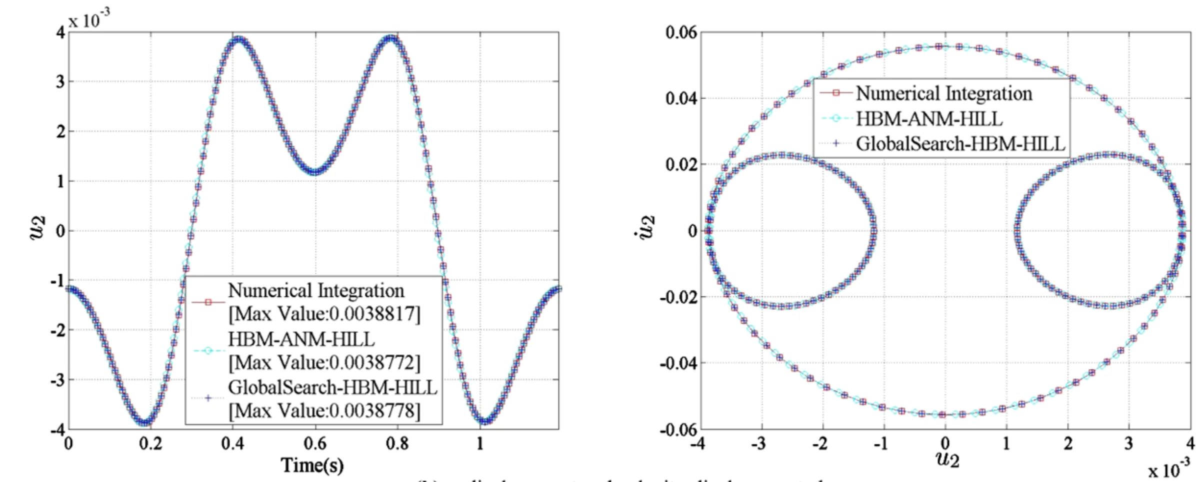
HBM-ANM-HILL method. For comparison, the numerical results obtained by the time domain integration method, which are accurate, are also plotted. The comparison in the time domain between the proposed method results and the time integration shows that there are small errors in amplitude, while the agreement of the proposed method with the HBM-ANM-HILL method is good even though small differences may be noted due to numerical precision.

In Table 3, the Floquet multipliers at these optima solutions obtained by the GlobalSearch-HBM-HILL method are presented along with the counterparts from the HBM-ANM-HILL method. As can be seen from Table 3, these optima solutions is stable because the moduli of the Floquet multipliers are less than 1. In comparison of the Floquet multipliers, the stability results between the HBM-ANM-HILL method and the GlobalSearch-HBM-HILL method are only slight differences. This results reveal that the proposed method can correctly determine the stability of periodic solution of non-linear system.

With respect to the computational cost of this example, the CPU time required by the HBM-ANM-



(a) u_1 displacement and velocity-displacement phase



(b) u_2 displacement and velocity-displacement phase

Fig. 6 Displacement and velocity comparison between different methods

Table 3 The Floquet multipliers obtained by the two methods

	ρ_1, ρ_2	$\lvert\rho_1\rvert = \lvert\rho_2\rvert$	ρ_3, ρ_4	$\lvert\rho_3\rvert = \lvert\rho_4\rvert$
HBM-ANM-HILL	$0.9773 \pm 0.0231i$	0.9775	$-0.9696 \pm 0.1043i$	0.9752
GlobalSearch-HBM-HILL	$0.9770 \pm 0.0333i$	0.9775	$-0.9697 \pm 0.1034i$	0.9752

HILL method is 2797.056463 seconds and the proposed method requires 16.6670 seconds to converge. It is significant that the computational effort in terms of time is smaller for the proposed method than for the HBM-ANM-HILL method. This comparison clearly demonstrates that the exact dynamics of non-linear structures can be captured extremely accurately at a very low computational cost.

4 Conclusions

Based on the proposed framework, an efficient method (abbreviated as GlobalSearch-HBM-HILL) is presented for finding the worst maximum vibration of a time-periodic non-linear dynamical system. The proposed method which combines three methods is formulated as a constrained, non-linear optimization problem. The non-linear algebraic equations derived from the harmonic balance method is applied to form the non-linear equality constraints while the stability constraint inequality is evaluated using the Hill procedure. Then, the MultiStart algorithm is used to optimize the response amplitude within the specified range of physical parameters.

In order to illustrate the efficiency of the proposed method, two non-linear dynamical systems are investigated: the canonical Duffing oscillator and an axially moving beam with lateral excitation. The results obtained by the proposed method compare very well with those obtained via the HBM-ANM-HILL method and the time integration method. It is illustrated that the proposed approach is highly computationally efficient, robust, and accurate. Furthermore, the proposed method is attractive in the dynamic analysis of non-linear systems because it is a general method and can be used to deal with almost any non-linear system. Future investigations will be devoted to extend the presented method to large scale complicated non-linear models in a subsequent paper.

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