Complex Variable

Egoist *some math victim*

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hope this textbook will be helpful *COMPLEX VARIABLES, Second Edition, Stephen D. Fisher.*

1 The Complex Plane

1.1 The Complex Numbers and the Complex Plane

- **Basic Definitions**
  real numbers; complex numbers; real part of z (x = **Re** z), imaginary part of z (y = **Im** z), the modulus (or the absolute value) $|z| = \sqrt{\mathbf{Re}\ z^2 + \mathbf{Im}\ z^2}$; complex conjugate.
  Then we have
  $$|x| \leq |z|, \quad |y| \leq |z|, \quad , |z| \leq |x| + |y| \tag{1}$$
  Basis Computation Rules follow the ordinary rules of arithmetic.

- **Polar Representation**
  1. Definition (polar representation): The polar coordinate system gives
  $$x = r\cos\theta, \quad y = r\sin\theta \tag{2}$$
  where $r = \sqrt{x^2 + y^2}$ and θ is the angle measured from the positive x-axis to the line segment from the origin to $P(x, y)$, then
  $$z = |z|(\cos\theta + I\sin\theta) \tag{3}$$
  2. Theorem (De Moivre's Theorem):
  $$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin\theta \tag{4}$$
  3. Definition (argument): nonzero complex number z, $\forall\theta$
  $$\arg z = \theta, \quad \text{is equivalent to} \quad , z = |z|(\cos\theta + I\sin\theta) \tag{5}$$
  A concrete choice of arg z is made by defining **Arg** z to be that number θ_0 in the interval $[-\pi, \pi)$ such that
  $$z = |z|(\cos\theta_0 + I\sin\theta_0) \tag{6}$$
  Then we may can write
  $$\mathrm{Arg}(zw) = \mathrm{Arg}\ z + \mathrm{Arg}\ w, \quad \mod 2\pi \tag{7}$$
  where the expression (mod 2π) means that the two sides of this last formula differ by some integer multiple of 2π.

- **Complex Numbers as Vectors**
  The angel α between the vector z and w is found by using
  $$\cos\alpha = \frac{\mathbf{Re}(z\bar{w})}{|z||w|} \tag{8}$$
  In summary, the usual xy-plane has a natural interpretation as the location of the complex variable $z = x + iy$ and all the rules for the geometry of the vectors $P(x, y)$ can be recast in terms of z. Henceforth, then, we refer to the xy-plane as the complex plane, or simply, the plane. The x-axis will be called the real axis, and the y-axis will be called the imaginary axis.

- **A Formal View of the Complex Numbers**
  Two complex numbers $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$.
  Basic Computation Rules; (There is a thing we need to remember: the additive identity is $\mathbf{0} = (0, 0)$, and the multiplicative identity is $\mathbf{1} = (1, 0)$.)
  A nonzero $z = (x, y)$ necessarily satisfies the condition $x^2 + y^2 > 0$, and its unique multiplicative inverse is
  $$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right). \tag{9}$$

The complex number $(0, 1)$ has the interesting property that its square is -1. We denote (0,1) by the symbol **i**.

1.2 Some Geometry

- **The Triangle Inequality**

$$|z + w| \leq |z| + |w| \tag{10}$$

- **Straight Line**
The equation of a nonvertical straight line $y = mx + b$, m and b real, can be formulated as

$$0 = Re((m + i)z + b) \tag{11}$$

More generally, if a=A + iB is a nonzero complex number and b is any complex number (not just a real number), then

$$0 = Re(az + b) \tag{12}$$

is exactly the straight line $Ax - By + Re(b) = 0$, this formulation also includes the vertical lines,$x = \mathbf{Re}\ z =$ constant.

- **Roots of Complex Numbers**
A complex number z satisfying the equation $z^n = w$ is called an **nth root of w**.
We define

$$\theta_k = \frac{\psi}{n} + k(\frac{2\pi}{n}), \quad k = 0, 1, ..., n - 1 \tag{13}$$

Then $n\theta_k = \psi + 2\pi k$ and so $\cos n\theta_k = \cos\psi$ and $\sin n\theta_k = \sin\psi$. Complex number $z_0, z_1, ..., z_{n-1}$ are defined by the rule

$$z_k = |w|^{\frac{1}{n}}(\cos\theta_k + i\sin\theta_k), \quad k = 0, 1, ..., n - 1 \tag{14}$$

Then each of $z_0, z_1, ..., z_{n-1}$ is distinct, and each satisfies

$$z_k^n = w, k = 0, 1, ..., n - 1 \tag{15}$$

Moreover, these complex numbers $z_0, ..., z_{n-1}$ are the only possible roots of the equation $z^n = w$. For if

$$\cos n\theta = \cos\psi, \quad \sin n\theta = \sin\psi \tag{16}$$

then we have $n\theta = \psi + 2\pi j$, for some integer j. The values $j = 0, ..., n-1$ yield distinct numbers $\cos\theta_j + i\sin\theta$, whereas other values of j just give a repetition of numbers already obtained.
For a nonzero complex number a, we define $a^z = e^{z\log\, a}$

- **Circles**
In this part, we give a way to use complex numbers to describe circles.
If p and q are distinct complex numbers, then those complex numbers z with

$$|z - p| = |z - q| \tag{17}$$

are equidistant from p and q. If $/rho$ is a positive real number not equal to 1, those z with

$$|z - p| = \rho|z - q| \tag{18}$$

form a circle. The proof is as below.
Suppose that 0¡ρ¡1. Let $z = w + q$ and $c = p - q$; then the equation becomes

$$|w - c| = \rho|w| \tag{19}$$

Upon squaring and transposing terms, this can be written as

$$|w|^2(1 - \rho^2) - 2\mathbf{Re}\ w\bar{c} + |c|^2 = 0 \tag{20}$$

We complete the square of the left side and find that

$$(1 - \rho^2)|w|^2 - 2\mathbf{Re}\ w\hat{c} + \frac{|c|^2}{1 - \rho^2} = \frac{|c|^2\rho^2}{1 - \rho^2} \tag{21}$$

Equivalently

$$|w - \frac{c}{1 - \rho^2}| = c\frac{\rho}{1 - \rho^2} \tag{22}$$

Thus, w lies on the circle of radius $R = \frac{|c|\rho}{(1-\rho^2)}$ centered at the point $\frac{c}{(1-\rho^2)}$, and so z lies on the circle of the same radius R centered at the point

$$z_0 = \frac{p - \rho^2 q}{1 - \rho^2} = \frac{1}{1 - \rho^2}p - \frac{\rho^2}{1 - \rho^2}q \tag{23}$$

Then, we use a similar method to derive a special pattern called Circles of Apollonius.
Let C_1 be the family of circles of the form

$$|z - p| = \rho|z - q|, \quad 0 < \rho < \infty \tag{24}$$

where we include the case $\rho = 1$ for completeness. Let L be the perpendicular bisenctor of the line segment from p to q. Take C_2 to be the family of circles through p and q and centered on the line L. We shall show that each circle in the family C_1 is perpendicular to each circle in the family C_2 at their two points of intersection. The computation is considerably simplified by locating the origin at the point of intersection of the line L and the line L', which passes through p and q. L' can then be taken to be the real axis and L to be the imaginary axis; in this way, we may assume that $0 < p = -q$. A circle from the family C_1 is then centered at a point on the real axis, and because there is no loss in assuming $0 < \rho < 1$, the center of that circle is at the point $s = \frac{p(1+\rho^2)}{(1-\rho^2)}$. The center of the circle from the family C_2 is at the point $t = i\alpha$ (α real), and this circle must pass through p and -p. Let $z = x + iy$ be on both circles Since z is on the

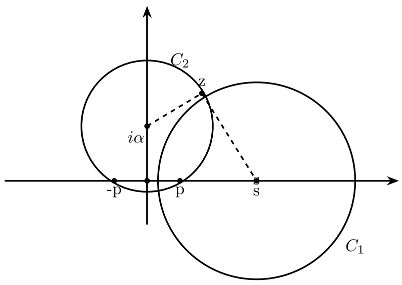

Figure 1: *Two Families of mutually perpendicular circles*

circle C_1,

$$|z - p| = \rho|z + p|, \tag{25}$$

and consequently,

$$x^2(1-\rho^2) - 2\rho x(1+\rho^2) + p^2(1-\rho^2) + y^2(1-\rho^2) = 0 \tag{26}$$

For notational convenience, set $\nu = \frac{(1+\rho^2)}{(1-\rho^2)}$; the above equation then becomes

$$x^2 - 2\rho\nu x + p^2 + y^2 = 0 \tag{27}$$

On the other hand, since $z = x + iy$ is also on the circle C_2,

$$|z - i\alpha| = |p - i\alpha| = |-p - i\alpha| = \sqrt{p^2 + \alpha^2} \tag{28}$$

and so

$$x^2 + y^2 - 2\alpha y = p^2 \tag{29}$$

1.3 Subsets of the Plane

- Preliminary Definitions; (omitted)
- **Some Sums**

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \tag{30}$$

$$1 + x + 2x^2 + \cdots + nx^n = 1 - (n+1)\frac{x^{n+1}}{1-x} + x\frac{1 - x^{n+1}}{(1-x)^2} \tag{31}$$

1.4 The Exponential, Logarithm, and Trigonometric Functions

- **Definition (the exponential function):**

$$e^z = e^x(\cos y + i \sin y), \quad z = x + iy \tag{32}$$

- **Properties (the exponential function):**
  1. z and w are two complex numbers then

$$e^{z+w} = e^z e^w \tag{33}$$

  2. $|e^z| = e^{\textbf{Re}\ z}$, especially $|e^{it}| = 1$, t real. Additionally, $e^{2\pi Im} = 1$ for $m = 0, \pm 1, \cdots$ and $e^{\pi i} = -1$.
  3. The function $f(z) = e^z$ never has the value zero. And, if w is any nonzero complex number, $e^z = w$ has infinitely many solutions.

- **The Logarithm Function:**
  z is any nonzero complex number,

$$\text{Log} z = \ln|z| + i\text{Arg}\ z, \quad z \neq 0 \tag{34}$$

- **Trigonometric Functions:**
  1.**Definition:**

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \tag{35}$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \tag{36}$$

  2. If $\cos z + \alpha = \cos z, \quad for z$, then

$$e^{iz}(e^{i\alpha} - 1) = e^{-i\alpha)}e^{-iz}(e^{i\alpha} - 1) \tag{37}$$

  If $e^{i\alpha} - 1$ is nonzero, it can be canceled

$$e^{iz} = e^{-i\alpha)}e^{-iz} \tag{38}$$

- **Inverse Trigonometric Functions**

$$\text{Arcsin}\ z = -i\text{Log}(iz + \sqrt{1 - z^2}), \quad z \in D \tag{39}$$

$$\text{Arccos}\ z = -i\text{Log}(z + \sqrt{z^2 - 1}), \quad z \in D \tag{40}$$

$$\text{Arctan}\ z = \frac{i}{2}\text{Log}(\frac{1 - iz}{1 + iz}), \quad z \neq \pm i \tag{41}$$

1.5 Line Integrals and Green Theorem

- **Definition (Curves):** A curve γ is a continuous complex-valued function $\gamma(t)$ defined for t in some interval $[a, b]$ in the real axis. The curve is simple if $\gamma(t_1) \neq \gamma(t_2)$ whenever $a \leq t_1 < t_2 < b$ and it's closed if $\gamma(a) = \gamma(b)$.

- **Smooth Curve:** Suppose γ is a curve,; seperate the complex number $\gamma(t)$ into its real and imaginary parts and write $\gamma(t) = x(t) + iy(t), a \leq t \leq b$. The function $x(t)$ and $y(t)$ are real functions.
  Assume $x(t)$ and $y(t)$ are both differentiable at t_0, then we say that γ is differentiable at t_0.
  A curve γ is smooth if $\gamma^{'}(t)$ not only exists, but is also continuous on [a,b] as well. Moreover, if the curve is piecewise smooth, it is composed of a finite number of smooth curves, the end of one coinciding with the beginning of the next.
  Each curve γ is oriented by increasing t. Similarly, we can define the reverse orientation. A simple closed curve γ is positively oriented if, for each point p on the inside of y, the argument of $\gamma - -p$ increases by 2π as t increases from a to b.

- **Integral:**
  1. **Definition(Integral):** Suppose $g(t) = \sigma(t) + i\tau(t)$ is a continuous complex-valued function on the interval [a,b]. Define the integral of g over [a,b] by

  $$\int_a^b g(t)dt = \int_a^b \sigma(t)dt + i\int_a^b \tau(t)dt \tag{42}$$

  2.**Definition (line integral of u)**:

  $$\int_\gamma u(z)dz = \int_a^b u(\gamma(t))\gamma'(t)dt \tag{43}$$

  For a piecewise smooth curve, the line integral of u along γ becomes

  $$\int_\gamma u(z)dz = \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} u(\gamma(s))\gamma'(s)ds \tag{44}$$

  The properties of the Integration is quite similar with the real-valued functions' integration.
  3. The length of the curve that is the range of $\gamma(t)$ is given by

  $$\text{length}(\gamma) = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt \tag{45}$$
  $$= \int_a^b |\gamma'(t)|dt \tag{46}$$

  Thus,

  $$|\int_\gamma u(z)dz| \leq (max_{z\in\gamma}|u(z)|)length(\gamma) \tag{47}$$

- **Greem Theorem and Greem Formula**
  1. **Green Theorem**: Green's Theorem is formulated for a domain Ω whose boundary Γ consists of a finite number of disjoint, piecewise smooth simple closed curves $\gamma_1, \gamma_2, \cdots, \gamma_n$. We orient the boundary Γ of Ω positively by requiring that Ω remain on the left as we walk along Γ. Thus, the outer piece of the boundary of Ω is oriented counterclockwise, and each inner piece of Γ is oriented clockwise.
  Assume that there is some open set D that contains both Ω and Γ and, on D, f has continuous partial derivatives with respect to both x and y. That is, if $f = p + iq$, then

  $$\frac{\partial f}{\partial x} = \frac{\partial p}{\partial x} + i\frac{\partial q}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial p}{\partial y} + i\frac{\partial q}{\partial y} \tag{48}$$

  where all of $\frac{\partial p}{\partial x}, \frac{\partial q}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial q}{\partial y}$ are continuous on D.
  Theorem: $\int_\Gamma f(z)dz = i\int\int_\Omega \{\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\}dxdy$, which can also be stated in another way:

  $$\int_\Gamma \{udx + vdy\} = \int\int_\Omega (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})dxdy \tag{49}$$

  2.**Green Formula:**

  $$\int_\Gamma (g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n})ds = \int\int_\Omega (g \triangle f - f \triangle g)dxdy \tag{50}$$

- **Harmonic**
  1. **Definition**: A function $u(z) = u(x, y)$ with continuous first and second partial derivatives with respect to both x and y is harmonic on an open set D if

  $$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on D.} \tag{51}$$

  2. **Theorem**: Suppose that u is real-valued harmonic function on an open set D and D contains a domain Ω and the boundary Γ of Γ. Assume that Γ consists of a finite number of disjoint, piecewise smooth simple closed curves. If $u = 0$ on Γ, then $u = 0$ in Ω as well.