~~##~~ CONSTRAINED OPTIMIZATION

METHOD OF LAGRANGE MULTIPLIERS IS USED. FOR CONSTRAINED OPTIMIZATION PROBLEMS

MOTIVATION

EXAMPLE 1:

$$\min f(x_1, x_2) = 2x_1^2 + 3x_2^2$$

SUBJECT TO : $x_1 + 3x_2 = 10$

SOLVE FOR x_1 : IT IS: $x_1 = 10 - 3x_2$

$\min f(x_1, x_2) = \min\left[2(10-3x_2)^2 + 3x_2^2\right] = \min h(x_2)$

THIS HAS A MINIMUM AND CAN BE READILY OBTAINED. □

EXAMPLE 2

$$\min f(x_1, x_2) = 2x_1^2 + 3x_2^2$$

SUBJECT TO : $x_1 \sin x_1 + 5x_2 \sin x_2 = 10$

IN THIS EXAMPLE, IT IS NOT CONVENIENT TO EXPLICITLY SOLVE FOR x_1 OR x_2 IN TERMS OF x_2 AND x_1 RESPECTIVELY.

□

WE WISH TO: MAXIMIZE $f(x_1, x_2)$

SUBJECT TO $g(x_1, x_2) = b$

STEP 1: LET $x_2 = h(x_1)$

MAX $f[x_1, h(x_1)]$ IMPLIES

$$\frac{d}{dx_1} f(x_1, x_2) = \frac{\partial f(x_1, x_2)}{\partial x_1} + \frac{\partial f(x_1, x_2)}{\partial x_2} \cdot \frac{d x_2}{d x_1}$$

$$= \frac{\partial f(x_1, x_2)}{\partial x_1} + \frac{\partial f(x_1, x_2)}{\partial x_2} \cdot \frac{d h(x_1)}{d x_1} = 0$$

CANNOT BE COMPUTED EXPLICITLY

STEP 2: RECALL THAT $g(x_1, x_2) = b$. THEREFORE

$$\frac{d}{dx_1} g(x_1, x_2) = \frac{\partial g(x_1, x_2)}{\partial x_1} + \frac{\partial g(x_1, x_2)}{\partial x_2} \cdot \frac{d h(x_1)}{d x_1} = 0$$

NOTE

THEREFOR $\frac{d h(x_1)}{d x_1}$ CAN BE COMPUTED FROM THE ABOVE EQUATION.

STEP 3: FROM STEP 2:

$$\frac{d h(x_1)}{d x_1} = - \frac{\partial g(x_1, x_2)}{\partial x_1} \Big/ \frac{\partial g(x_1, x_2)}{\partial x_2}$$

USE STEP 1:

$$\frac{\partial f(x_1, x_2)}{\partial x_1} - \frac{\partial f(x_1, x_2)}{\partial x_2} \left(\frac{\partial g(x_1, x_2)}{\partial x_1} \Big/ \frac{\partial g(x_1, x_2)}{\partial x_2} \right) = 0$$

LET $\lambda = \dfrac{\partial f(x_1, x_2)}{\partial x_2} \Big/ \dfrac{\partial g(x_1, x_2)}{\partial x_2}$

THEN

$$\frac{\partial f(x_1, x_2)}{\partial x_1} - \lambda \frac{\partial g(x_1, x_2)}{\partial x_1} = 0 \qquad \text{— (1)}$$

FROM THE DEFINITION OF λ WE HAVE

$$\frac{\partial f(x_1, x_2)}{\partial x_2} - \lambda \frac{\partial g(x_1, x_2)}{\partial x_2} = 0 \qquad \text{— (2)}$$

THE ORIGINAL CONSTRAINT IS :

$$g(x_1, x_2) = b \qquad \text{— (3)}$$

EQUATIONS 1, 2, & 3 REPRESENT A SET OF NECESSARY CONDITIONS FOR THE EXISTENCE OF A SOLUTION TO THE ORIGINAL PROBLEM.

STEP 4: A SIMPLE WAY AT ARRIVING EQUATIONS 1, 2, & 3 IS AS FOLLOWS. DEFINE THE LAGRANGEAN FUNCTION AS:

$$F(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda\left[b - g(x_1, x_2)\right]$$

DIFFERENTIATE F WITH RESPECT TO x_1, x_2, λ AND EQUATE THE RESULTING EXPRESSIONS TO ZERO. THAT IS

$$\frac{\partial F(x_1, x_2, \lambda)}{\partial x_1} = \frac{\partial f(x_1, x_2)}{\partial x_1} - \lambda \frac{\partial g(x_1, x_2)}{\partial x_1} = 0 \Rightarrow (1)$$

$$\frac{\partial F(x_1, x_2, \lambda)}{\partial x_2} = \frac{\partial f(x_1, x_2)}{\partial x_2} - \lambda \frac{\partial g(x_1, x_2)}{\partial x_2} = 0 \Rightarrow (2)$$

$$\frac{\partial F(x_1, x_2, \lambda)}{\partial \lambda} = b - g(x_1, x_2) = 0 \Rightarrow (3)$$

λ IS CALLED LAGRANGE MULTIPLIER

EXAMPLE MIN $z = 3x_1^2 + 4x_2^2$

SUBJECT TO: $2x_1 - 3x_2 = 10$

SOLUTION THE LAGRANGIAN FUNCTION IS

$$F(x_1, x_2, \lambda) = 3x_1^2 + 4x_2^2 + \lambda(10 - 2x_1 + 3x_2)$$

$$\frac{\partial F(x_1, x_2, \lambda)}{\partial x_1} = 6x_1 - 2\lambda = 0$$

$$\frac{\partial F(x_1, x_2, \lambda)}{\partial x_2} = 8x_2 + 3\lambda = 0$$

$$\frac{\partial F(x_1, x_2, \lambda)}{\partial \lambda} = 10 - 2x_1 + 3x_2 = 0$$

SOLVING THE ABOVE THREE EQUATIONS LEADS TO:

$$x_1 = \frac{80}{43} \; ; \quad x_2 = -\frac{90}{43} \; ; \quad \lambda = \frac{240}{43}$$

WHY DOES TH ABOVE SOLUTION GIVE A MINIMA ?
(AND NOT MAXIMA)

□

CONSTRAINED OPTIMIZATION - EQUALITY CONSTRAINTS

PROBLEM: MINIMIZE (OR MAXIMIZE) $f(x)$; $x \in \mathbb{R}^n$

SUBJECT TO : $g_i(x) = b_i$; $i = 1, 2, \ldots, m$

$f(x)$ = OBJECTIVE FUNCTION

$g_i(x) = b_i$; $1 \leq i \leq m$ = CONSTRAINTS

SOLUTION SET UP THE LAGRANGIAN

$$L(x, \lambda) = f(x) + \lambda \cdot (b - g(x))$$

↑ DOT PRODUCT

WHERE

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$; $b = (b_1, b_2, \ldots, b_m)$;

$g(x) = (g_1(x), g_2(x), \ldots\ldots, g_m(x))$

ASSUME $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ MINIMIZES OR MAXIMIZES SUBJECT TO THE GIVEN CONSTRAINT. THEN EITHER

i) $\nabla g_i(x^*)$; $1 \leq i \leq m$ ARE LINEARLY DEPENDENT, OR

ii) THERE EXISTS A VECTOR $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*)$ SUCH THAT

a) $\nabla L(x^*, \lambda^*) = 0$. THIS IMPLIES

$$\left.\frac{\partial L}{\partial x_1}\right|_{(x^*, \lambda^*)} = \left.\frac{\partial L}{\partial x_2}\right|_{(x^*, \lambda^*)} = \cdots = \left.\frac{\partial L}{\partial x_n}\right|_{(x^*, \lambda^*)} = 0$$

b) AND

$$\left.\frac{\partial L}{\partial \lambda_1}\right|_{(x^*, \lambda^*)} = \left.\frac{\partial L}{\partial \lambda_2}\right|_{(x^*, \lambda^*)} = \cdots = \left.\frac{\partial L}{\partial \lambda_m}\right|_{(x^*, \lambda^*)} = 0$$

THESE ACTUALLY GIVE THE CONSTRAINTS

$g_i(x) = b_i$; $i = 1, 2, \ldots, m$ □

ONCE x^* IS DETERMINED, IT IS NOT ALWAYS CONVENIENT/EASY TO FIGURE WHETHER THEY CORRESPOND TO A MINIMUM, A MAXIMUM OR NEITHER. HOWEVER,

IF $f(x)$ IS CONCAVE ; $g_i(x)$'s ARE LINEAR, THEN ANY FEASIBLE x^* WITH A CORRESPONDING λ^* MAKING $\nabla L(x^*, \lambda^*) = 0$ MAXIMIZES $f(x)$ SUBJECT TO THE CONSTRAINTS.

SIMILARLY, IF $f(x)$ IS CONVEX, $g_i(x)$'s ARE LINEAR, THEN ANY (FEASIBLE) x^* WITH A λ^* MAKING $\nabla L(x^*, \lambda^*) = 0$ MINIMIZES $f(x)$ SUBJECT TO THE CONSTRAINTS.

EXAMPLE 1

MAXIMIZE $f(x) = 5 - (x_1-2)^2 - 2(x_2-1)^2$; $x = (x_1, x_2)$

SUBJECT TO: $x_1 + 4x_2 = 3$

SOLUTION 1

$x_1 = 3 - 4x_2$

$f(x) = 5 - (3-4x_2-2)^2 - 2(x_2-1)^2$

$= 5 - (1-4x_2)^2 - 2(x_2-1)^2$

$\frac{\partial f}{\partial x_2} = -2(1-4x_2)(-4) - 2 \cdot 2(x_2-1) = 12 - 36x_2 = 0 \Rightarrow x_2 = 1/3$

$x_1 = 3 - \frac{4}{3} = \frac{5}{3}$

$\therefore (x_1^*, x_2^*) = (\frac{5}{3}, \frac{1}{3})$

$\frac{\partial^2 f}{\partial x_2^2} = -36 < 0 \Rightarrow$ $f(x)$ IS MAXIMIZED AT $(x_1^*, x_2^*) = (\frac{5}{3}, \frac{1}{3})$

SOLUTION 2

$L(x_1, x_2, \lambda) = 5 - (x_1-2)^2 - 2(x_2-1)^2 + \lambda(3 - x_1 - 4x_2)$

$\frac{\partial L}{\partial x_1} = -2(x_1-2) - \lambda = 0 \Rightarrow x_1 = 2 - \lambda/2$

$\frac{\partial L}{\partial x_2} = -4(x_2-1) - 4\lambda = 0 \Rightarrow x_2 = 1 - \lambda$

$\frac{\partial L}{\partial \lambda} = 3 - x_1 - 4x_2 = 0 \Rightarrow \lambda = \frac{2}{3}, x_1 = \frac{5}{3}, x_2 = \frac{1}{3}$

$\therefore \lambda = 2/3$; $x = (x_1, x_2) = (5/3, 1/3)$

$f(5/3, 1/3) = 5 - (\frac{5}{3}-2)^2 - 2(1/3-1)^2 = 5 - 1/9 - 2 \cdot 4/9 = 4$

$f_{x_1,x_1} = -2$; $f_{x_1 x_2} = 0$; $f_{x_2,x_2} = -4$

$H = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$; $A_1 = [-2]$; $\det A_1 = -2$; $A_2 = H$; $\det A_2 = 8 > 0$

$\Rightarrow$ H IS NEGATIVE DEFINITE

$\therefore (x_1^*, x_2^*) = (5/3, 1/3)$ IS A MAXIMA.

$\square$

EXAMPLE 2

MINIMIZE $\quad f(x_1, x_2, x_3) = x_1 + x_2 + x_3^2$

SUBJECT TO: $\quad x_1 = 1$

$$x_1^2 + x_2^2 = 1$$

SOLUTION 1: $\quad x_1 = 1$

$$1^2 + x_2^2 = 1 \implies x_2 = 0$$

$$f(x_1, x_2, x_3) = 1 + x_3^2$$

$$\frac{\partial f}{\partial x_3} = 2x_3 = 0 \implies x_3 = 0 \; ; \; \frac{\partial^2 f}{\partial x_3^2} = 2 > 0$$

$\implies f$ IS MINIMIZED AT $(x_1, x_2, x_3) = (1, 0, 0)$; $f(1,0,0) = 1$

SOLUTION 2: $\quad L(x_1, x_2, x_3, \lambda_1, \lambda_2) = x_1 + x_2 + x_3^2 + \lambda_1(1 - x_1) + \lambda_2(1 - x_1^2 - x_2^2)$

$$\frac{\partial L}{\partial x_1} = 1 - \lambda_1 - 2\lambda_2 x_1 = 0 \implies 1 - \lambda_1 - 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 1 - 2\lambda_2 x_2 \overset{?}{=} 0 \implies \text{NOT POSSIBLE}$$

$$\frac{\partial L}{\partial x_3} = 2x_3 = 0 \implies x_3 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 1 - x_1 = 0 \implies x_1 = 1$$

$$\frac{\partial L}{\partial \lambda_2} = 1 - x_1^2 - x_2^2 = 0 \implies x_2 = 0$$

$\frac{\partial L}{\partial x_2}$ DOES NOT VANISH AT $(x_1, x_2, x_3) = (1, 0, 0)$

EXPLANATION: LET $g_1(x_1, x_2, x_3) = x_1$

$$g_2(x_1, x_2, x_3) = x_1^2 + x_2^2$$

$\nabla g_1(1,0,0) = (1,0,0)$; $\nabla g_2(1,0,0) = (2,0,0)$ ARE LINEARLY DEPENDENT VECTORS

$\therefore$ CONSTRAINTS GIVE : $x_1 = 1$; $x_2 = 0$

$\implies$ OBJECTIVE FUNCTION $f(x_1, x_2, x_3) = 1 + x_3^2$; WHICH IS MINIMIZED AT $(x_1^*, x_2^*, x_3^*) = (1, 0, 0)$. $\square$

EXAMPLE 3

MINIMIZE $(2x_1^2 + x_2^2) = f(x_1, x_2)$

SUBJECT TO: $x_1 + x_2 = 1$

SOLUTION 1 $\quad x_2 = 1 - x_1$

$$f(x_1, x_2) = 2x_1^2 + (1-x_1)^2$$

$$\frac{df}{dx_1} = 4x_1 - 2(1 - x_1) = 6x_1 - 2 = 0 \Rightarrow x_1 = 1/3 \; ; \; x_2 = 2/3$$

$$\frac{d^2f}{dx_1^2} = 6 > 0 \Rightarrow \text{MINIMUM OF } f(x_1, x_2) \text{ OCCURS AT } (1/3, 2/3)$$

$$f(1/3, 2/3) = 2(1/3)^2 + (2/3)^2 = 2/3$$

SOLUTION 2 $\quad L(x_1, x_2, \lambda) = 2x_1^2 + x_2^2 + \lambda(1 - x_1 - x_2)$

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= 4x_1 - \lambda = 0 \\ \frac{\partial L}{\partial x_2} &= 2x_2 - \lambda = 0 \end{aligned}\right\} \quad \begin{aligned} x_1 &= \lambda/4 \\ x_2 &= \lambda/2 \end{aligned}$$

$$\frac{\partial L}{\partial \lambda} = 1 - x_1 - x_2 = 0 \quad \Rightarrow 1 - \frac{\lambda}{4} - \frac{\lambda}{2} = 0 \Rightarrow \lambda = 4/3$$

$$\Rightarrow x_1 = 1/3 \; ; \; x_2 = 2/3$$

$$f(1/3, 2/3) = 2/3$$

$$f_{x_1 x_1} = 4 \; ; \; f_{x_1 x_2} = 0 \; ; \; f_{x_2 x_2} = 2$$

$$\text{HESSIAN} = H = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\left.\begin{aligned} A_1 &= [4] \; ; \; \det A_1 = 4 > 0 \\ A_2 &= H \; ; \; \det A_2 = 8 > 0 \end{aligned}\right\} \Rightarrow H \text{ IS POSITIVE DEFINITE}$$

$\therefore$ $(1/3, 2/3)$ MINIMIZES $f(x_1, x_2)$ $\square$

CONSTRAINED OPTIMIZATION - EQUALITY & INEQUALITY CONSTRAINTS

PROBLEM: MAXIMIZE $f(x)$; $x \in \mathbb{R}^n$

SUBJECT TO: $g_i(x) = b_i \quad i = 1, 2, \cdots, m$

$$h_j(x) \leq d_j \quad j = 1, 2, \cdots, p$$

$f(x)$ = OBJECTIVE FUNCTION

$g_i(x) = b_i$; $1 \leq i \leq m$; $h_j(x) \leq d_j$, $1 \leq j \leq p$ = CONSTRAINTS

NOTE: i) $\geq$ CONSTRAINTS CAN BE CONVERTED INTO $\leq$ CONSTRAINTS BY MULTIPLYING BOTH SIDES BY -1.

ii) MINIMIZATION PROBLEM CAN BE CONVERTED INTO A MAXIMIZATION PROBLEM BY MULTIPLYING THE OBJECTIVE FUNCTION BY -1.

SOLUTION SET UP THE LAGRANGIAN

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i (b_i - g_i(x)) + \sum_{j=1}^{p} \mu_j (d_j - h_j(x))$$

WHERE $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$; $\mu = (\mu_1, \mu_2, \cdots, \mu_p)$

ASSUME $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ MAXIMIZES $f(x)$ WITH RESPECT TO THE GIVEN CONSTRAINTS. THEN EITHER

i) THE VECTORS $\nabla g_i(x^*)$, $1 \leq i \leq m$ AND

$\nabla h_j(x^*)$; $1 \leq j \leq p$

ARE LINEARLY DEPENDENT

ii) THERE EXISTS VECTORS $\lambda^* = (\lambda_1^*, \lambda_2^*, \cdots, \lambda_m^*)$, AND

$\mu^* = (\mu_1^*, \mu_2^*, \cdots \mu_p^*)$ SUCH THAT

$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) - \sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*) = 0$$

$$\mu_j^* \left(h_j(x^*) - d_j \right) = 0, \qquad j = 1, 2, \ldots, p \qquad \text{(COMPLEMENTARITY)}$$

$$\mu_j^* \geq 0, \quad j = 1, 2, \ldots, p$$

$$g_i(x^*) = b_i \; ; \quad i = 1, 2, \ldots, m$$

$$h_j(x^*) \leq d_j \; ; \quad j = 1, 2, \ldots, p$$

$\square$

THE ABOVE CONDITIONS ARE CALLED KARUSH-KUHN-TUCKER CONDITIONS.

i) IF $\mu_j^* = 0$, THEN THE INEQUALITY IS SAID TO BE NOT TIGHT

ii) IF $\mu_j^* > 0$, " " " " " " TIGHT.

IF IN THE LAST PROBLEM, IF $x \geq 0$, THE KKT CONDITIONS ARE

$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) - \sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*) \leq 0$$

$$\mu_j^* (h_j(x^*) - d_j) = 0 \quad ; j = 1, 2, \ldots, p$$

$$\mu_j^* \geq 0 \quad ; j = 1, 2, \ldots, p$$

$$g_i(x^*) = b_i \quad ; i = 1, 2, \ldots, m$$

$$h_j(x^*) \leq d_j \quad ; j = 1, 2, \ldots, p$$

$$x \geq 0$$

$$\left\{ \frac{\partial f(x^*)}{\partial x_i} - \sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_k} - \sum_{j=1}^{p} \mu_j \frac{\partial h_j(x^*)}{\partial x_k} \right\} x_k = 0 ; \quad 1 \leq k \leq n$$

$\square$

SUFFICIENCY OF CONDITIONS

THE KARUSH-KUHN-TUCKER CONDITIONS GIVE US CANDIDATE OPTIMAL SOLUTIONS x^*.

WHEN ARE THESE CONDITIONS SUFFICIENT FOR OPTIMALITY?

THAT IS, GIVEN x^* WITH λ^* AND μ^* SATISFYING THE KKT CONDITIONS, WHEN CAN WE BE CERTAIN THAT IT IS AN OPTIMAL SOLUTION?

THE MOST GENERAL CONDITION AVAILABLE IS:

1. $f(x)$ IS CONCAVE, AND
2. THE FEASIBLE REGION FORMS A CONVEX REGION.

IT MIGHT BE EASY TO DETERMINE IF THE OBJECTIVE FUNCTION IS CONCAVE BY COMPUTING ITS HESSIAN MATRIX, IT IS NOT EASY TO TELL IF THE FEASIBLE REGION IS CONVEX.

A USEFUL CONDITION IS AS FOLLOWS:

THE FEASIBLE REGION IS CONVEX IF ALL OF $g_i(x)$ ARE LINEAR AND ALL OF $h_j(x)$ ARE CONVEX.

IF THIS CONDITION IS SATISFIED, THEN ANY POINT THAT SATISFIES THE KKT CONDITIONS GIVES A POINT THAT MAXIMIZES $f(x)$ SUBJECT TO THE CONSTRAINTS