Machine Learning

ASSIGNMENT 5

84 points

Notation:

$$\begin{aligned}\mathbb{R} &= (-\infty, \infty) = \text{ set of real numbers} \\ \mathbb{R}^{+} &= (0, \infty) = \text{ set of positive real numbers}\end{aligned}$$

Probability Theory Notation

1. The expectation of random variable X is $\mathcal{E}(X) \triangleq \mu$. This is also called the mean value or average value of random variable X.
2. The second moment of random variable X is $\mathcal{E}\left(X^2\right)$.
3. The variance of random variable X is $Var(X)$.
$$Var(X) = \mathcal{E}\left(X^2\right) - \mu^2$$
4. The moment generating function (MGF) of random variable X is
$$\mathcal{M}_X(t) = \mathcal{E}\left(e^{tX}\right)$$

1. (30 points) The probability density function of a standard normal random variable X is $\phi(\cdot)$, where
$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad x \in \mathbb{R}$$
Prove from first principles:
   (a) $\int_{-\infty}^{\infty} \phi(x)\, dx = 1$
   (b) The first moment of random variable X is $\mathcal{E}(X) = 0$.
   (c) The second moment of random variable X is $\mathcal{E}\left(X^2\right) = 1$.
   (d) $Var(X) = 1$.
   (e) The moment generating function of random variable X is
$$\mathcal{M}_X(t) = \exp\left(\frac{t^2}{2}\right)$$
   (f) Use the MGF of random variable X to determine $\mathcal{E}(X)$, and $\mathcal{E}\left(X^2\right)$.

   Hint: See any standard textbook on probability theory.

2. (30 points) A random variable X has a normal (or Gaussian) distribution, if the probability density function of X is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \quad x \in \mathbb{R}$$

where $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$ are its parameters. Prove that:

(a) $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

(b) The first moment of random variable X is $\mathcal{E}(X) = \mu$.

(c) The second moment of random variable X is $\mathcal{E}(X^2) = (\sigma^2 + \mu^2)$.

(d) $Var(X) = \sigma^2$.

(e) The moment generating function (MGF) of random variable X is

$$\mathcal{M}_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

(f) Use the MGF of random variable X to determine $\mathcal{E}(X)$, and $\mathcal{E}(X^2)$.

Hint: Use the results from problem 1, and the method of transformation of variables.

3. (24 points) Let the number of data points be $n = 3$. The data points are

$$(0,2),\ (1,3),\ \text{and}\ (2,1)$$

Using the notation in class notes, find $\overline{x}$, $\overline{y}$, S_x^2, S_y^2, S_{xy}, r, a, and b. Summarize your results below

---

$\overline{x} =$ $\qquad\qquad$ $\overline{y} =$

$S_x^2 =$ $\qquad\qquad$ $S_y^2 =$

$S_{xy} =$ $\qquad\qquad$ $r =$

$a =$ $\qquad\qquad$ $b =$
