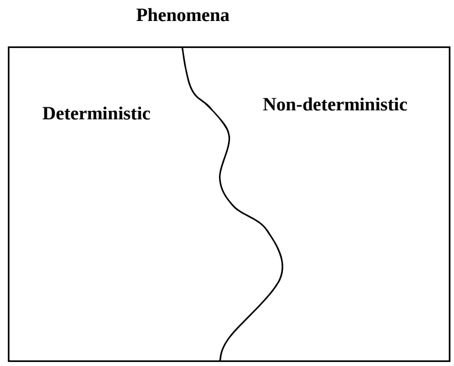
Probability Theory
by
W. H. Laverty
(modified)

Probability – Models for random phenomena



Deterministic Phenomena

- There exists a mathematical model that allows "***perfect***" prediction the phenomena's outcome.
- Many examples exist in Physics, Chemistry (the exact sciences).

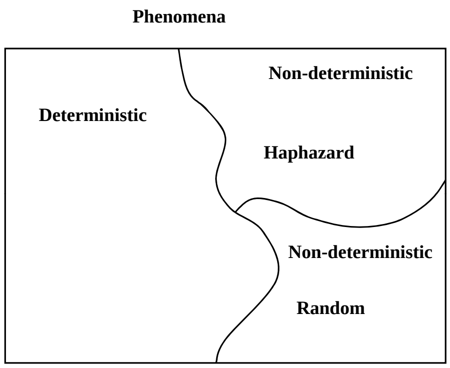
Non-deterministic Phenomena

- **No** mathematical model exists that allows "***perfect***" prediction the phenomena's outcome.

Non-deterministic Phenomena

- may be divided into two groups.

1. **Random phenomena**
   – Unable to predict the outcomes, but in the long-run, the outcomes exhibit statistical regularity.
2. **Haphazard phenomena**
   – unpredictable outcomes, but no long-run, exhibition of statistical regularity in the outcomes.



Haphazard phenomena

– unpredictable outcomes, but no long-run, exhibition of statistical regularity in the outcomes.
– Do such phenomena exist?
– Will any non-deterministic phenomena exhibit long-run statistical regularity eventually?

Random phenomena

– Unable to predict the outcomes, but in the long-run, the outcomes exhibit statistical regularity.

Examples

1. Tossing a coin – outcomes $S = \{\textbf{Head, Tail}\}$

Unable to predict on each toss whether is Head or Tail.

In the long run can predict that 50% of the time heads will occur and 50% of the time tails will occur

2. Rolling a die – outcomes

$S = \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}$

Unable to predict outcome but in the long run can one can determine that each outcome will occur 1/6 of the time.

Use symmetry. Each side is the same. One side should not occur more frequently than another side in the long run. If the die is not balanced this may not be true.

Definitions

The sample Space, S

The **sample space**, S, for a random phenomena is the set of all possible outcomes.

Examples

1. Tossing a coin – outcomes $S = \{\textbf{Head, Tail}\}$

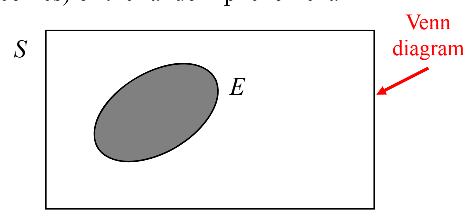
2. Rolling a die – outcomes

$S = \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}$

$= \{1, 2, 3, 4, 5, 6\}$

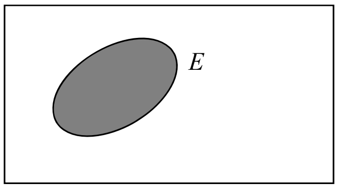
An Event , E

The **event**, E, is any subset of the **sample space**, S. i.e. any set of outcomes (not necessarily all outcomes) of the random phenomena



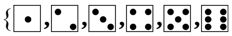
The **event**, E, is said to **have occurred** if after the outcome has been observed the outcome lies in E.

S



Examples

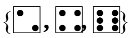
1. Rolling a die – outcomes

S ={}

={1, 2, 3, 4, 5, 6}

E = the event that an even number is rolled

= {2, 4, 6}

={}

Special Events

The Null Event, The empty event - ϕ

ϕ = { } = the event that contains no outcomes

The Entire Event, The Sample Space - S

S = the event that contains all outcomes

The empty event, ϕ , never occurs.

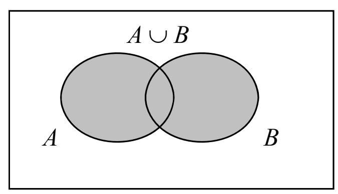
The entire event, S, always occurs.

Set operations on Events

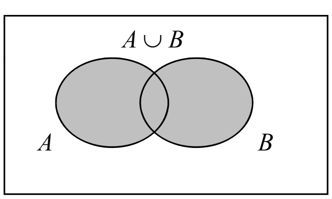
Union

Let A and B be two events, then the **union** of A and B is the event (denoted by $A \cup B$) defined by:

$A \cup B$ = {e| e belongs to A **or** e belongs to B}



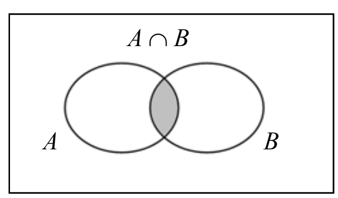
The event $A \cup B$ **occurs** if the event A **occurs or** the event and B **occurs** .



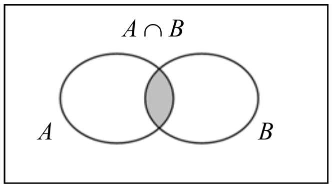
Intersection

Let A and B be two events, then the **intersection** of A and B is the event (denoted by $A \cap B$) defined by:

$A \cap B$ = {e| e belongs to A **and** e belongs to B}



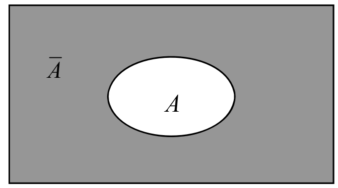
The event $A \cap B$ **occurs** if the event A **occurs and** the event and B **occurs** .



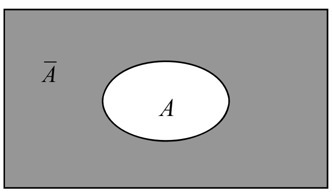
Complement

Let A be any event, then the **complement** of A (denoted by $\bar{A}$) defined by:

$\bar{A}$ = {e| e **does not** belongs to A}



The event $\bar{A}$ **occurs** if the event A **does not occur**



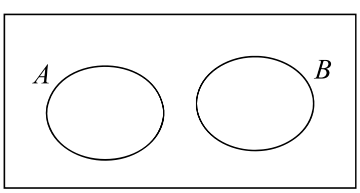
In problems you will recognize that you are working with:

1. **Union** if you see the word **or**,
2. **Intersection** if you see the word **and**,
3. **Complement** if you see the word **not**.

Definition: mutually exclusive

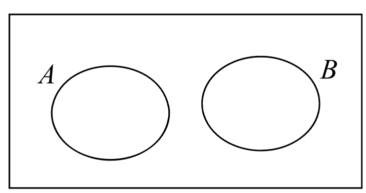
Two events A and B are called **mutually exclusive** if:

$$A \cap B = \phi$$



If two events A and B are are **mutually exclusive** then:

1. They have no outcomes in common.
   They can't occur at the same time. The outcome of the random experiment can not belong to both A and B.



Probability

Definition: probability of an Event E.

Suppose that the sample space $S = \{o_1, o_2, o_3, \ldots o_N\}$ has a finite number, N, of oucomes.

Also each of the outcomes is equally likely (because of symmetry).

Then for any event E

$$P[E] = \frac{n(E)}{n(S)} = \frac{n(E)}{N} = \frac{\text{no. of outcomes in } E}{\text{total no. of outcomes}}$$

Note: the symbol $n(A) =$ no. of elements of A

Thus this definition of $P[E]$, i.e.

$$P[E] = \frac{n(E)}{n(S)} = \frac{n(E)}{N} = \frac{\text{no. of outcomes in } E}{\text{total no. of outcomes}}$$

Applies only to the special case when

1. The sample space has a finite no.of outcomes, and
2. Each outcome is equi-probable

   If this is not true a more general definition of probability is required.

Rules of Probability

Rule The additive rule (Mutually exclusive events)

$$\boxed{\begin{array}{l} P[A \cup B] = P[A] + P[B] \\ \text{i.e.} \\ P[A \text{ or } B] = P[A] + P[B] \end{array}}$$

if $A \cap B = \phi$

(A and B mutually exclusive)

If two events A and B are are **mutually exclusive** then:

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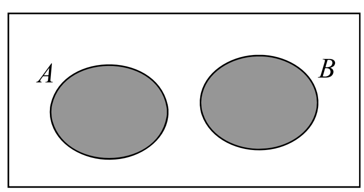

$P[A \cup B] = P[A] + P[B]$
i.e.
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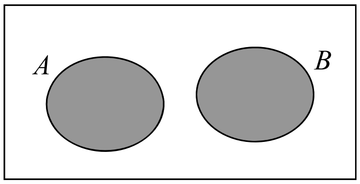


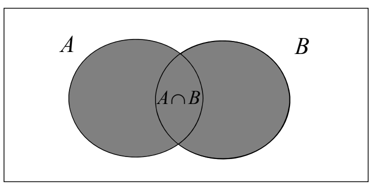
Rule The additive rule
(In general)

$P[A \cup B] = P[A] + P[B] - P[A \cap B]$

or

$P[A \text{ or } B] = P[A] + P[B] - P[A \text{ and } B]$

Logic $A \cup B$



When $P[A]$ is added to $P[B]$ the outcome in $A \cap B$ are counted twice
hence
$P[A \cup B] = P[A] + P[B] - P[A \cap B]$

$P[A \cup B] = P[A] + P[B] - P[A \cap B]$

Example:
Saskatoon and Moncton are two of the cities competing for the World university games. (There are also many others). The organizers are narrowing the competition to the **final 5 cities.**
There is a 20% chance that Saskatoon will be amongst the **final 5**. There is a 35% chance that Moncton will be amongst the **final 5** and an 8% chance that both Saskatoon and Moncton will be amongst the **final 5**. What is the probability that Saskatoon or Moncton will be amongst the **final 5.**

Solution:
Let A = the event that Saskatoon is amongst the **final 5**.
Let B = the event that Moncton is amongst the **final 5**.
Given $P[A] = 0.20$, $P[B] = 0.35$, and $P[A \cap B] = 0.08$
What is $P[A \cup B]$?
Note: "and" $\equiv \cap$**,** "or" $\equiv \cup$.

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$
$$= 0.20 + 0.35 - 0.08 = 0.47$$

Rule for complements

2. $P\left[\bar{A}\right] = 1 - P\left[A\right]$

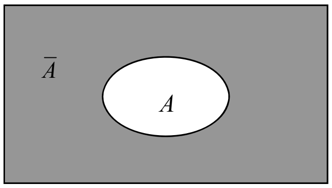
or

$P\left[\text{not } A\right] = 1 - P\left[A\right]$

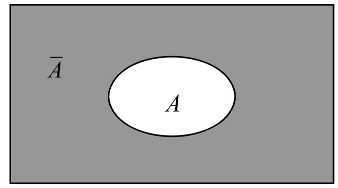
Complement

Let A be any event, then the **complement** of A (denoted by $\bar{A}$) defined by:

$\bar{A}$ = {e| e **does not** belongs to A}



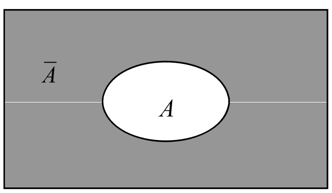
The event $\bar{A}$ **occurs** if the event A **does not occur**



Logic:

$\bar{A}$ and A are **mutually exclusive**.

and $S = A \cup \bar{A}$



thus $1 = P[S] = P[A] + P[\bar{A}]$

and $P[\bar{A}] = 1 - P[A]$

Conditional Probability

Conditional Probability

- Frequently before observing the outcome of a random experiment you are given information regarding the outcome
- How should this information be used in prediction of the outcome.
- Namely, how should probabilities be adjusted to take into account this information
- Usually the information is given in the following form: You are told that the outcome belongs to a given event. (i.e. you are told that a certain event has occurred)

Definition

Suppose that we are interested in computing the probability of event A and we have been told event B has occurred.

Then the conditional probability of A given B is defined to be:

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \quad \text{if} \quad P[B] \neq 0$$

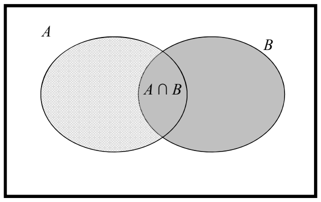
Rationale:

If we're told that event B has occurred then the sample space is restricted to B.

The probability within B has to be normalized, This is achieved by dividing by $P[B]$

The event A can now only occur if the outcome is in of $A \cap B$. Hence the new probability of A is:

$$P\left[A|B\right] = \frac{P\left[A \cap B\right]}{P\left[B\right]}$$



An Example

The academy awards is soon to be shown.

For a specific married couple the probability that the husband watches the show is 80%, the probability that his wife watches the show is 65%, while the probability that they both watch the show is 60%.

If the husband is watching the show, what is the probability that his wife is also watching the show

Solution:

The academy awards is soon to be shown.

Let B = the event that the husband watches the show

$P[B]$= 0.80

Let A = the event that his wife watches the show

$P[A]$= 0.65 and $P[A \cap B]$= 0.60

$$P\left[A|B\right] = \frac{P\left[A \cap B\right]}{P\left[B\right]} = \frac{0.60}{0.80} = 0.75$$

Independence

Definition

Two events A and B are called **independent** if

$$P\left[A \cap B\right] = P\left[A\right]P\left[B\right]$$

Note if $P\left[B\right] \neq 0$ and $P\left[A\right] \neq 0$ then

$$P\left[A|B\right] = \frac{P\left[A \cap B\right]}{P\left[B\right]} = \frac{P\left[A\right]P\left[B\right]}{P\left[B\right]} = P\left[A\right]$$

and $$P\left[B|A\right] = \frac{P\left[A \cap B\right]}{P\left[A\right]} = \frac{P\left[A\right]P\left[B\right]}{P\left[A\right]} = P\left[B\right]$$

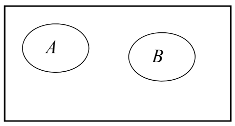
Thus in the case of independence the conditional probability of an event is not affected by the knowledge of the other event

Difference between **independence** and **mutually exclusive**

mutually exclusive

Two mutually exclusive events are independent only in the special case where

$P[A]=0$ and $P[B]=0$. (also $P[A\cap B]=0$

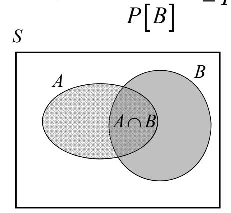


Mutually exclusive events are highly dependent otherwise. A and B **cannot** occur simultaneously. If one event occurs the other event does not occur.

Independent events

$$P[A\cap B]=P[A]P[B]$$

or $$\frac{P[A\cap B]}{P[B]}=P[A]=\frac{P[A]}{P[S]}$$



The ratio of the probability of the set A within B is the same as the ratio of the probability of the set A within the entire sample S.

The multiplicative rule of probability

$$P[A\cap B]=\begin{cases} P[A]P[B|A] & \text{if } P[A]\neq 0 \\ P[B]P[A|B] & \text{if } P[B]\neq 0 \end{cases}$$

and

$$P[A\cap B]=P[A]P[B]$$

if A and B are **independent**.

Probability

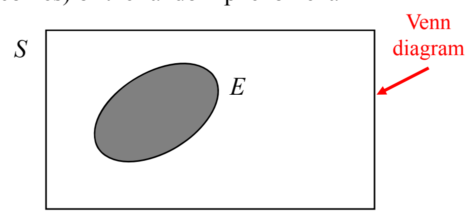
Models for **random** phenomena

The sample Space, S

The **sample space**, S, for a random phenomena is the set of all possible outcomes.

An Event , E

The **event**, E, is any subset of the **sample space**, S. i.e. any set of outcomes (not necessarily all outcomes) of the random phenomena



Definition: probability of an Event E.

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Also each of the outcomes is equally likely (because of symmetry).

Then for any event E

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Note: the symbol $n(A) =$ no. of elements of A

Thus this definition of $P[E]$, i.e.

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Applies only to the special case when

1. The sample space has a finite no.of outcomes, and
2. Each outcome is equi-probable

   If this is not true a more general definition of probability is required.

Summary of the Rules of Probability

The additive rule

$$\boxed{P[A \cup B] = P[A] + P[B] - P[A \cap B]}$$

and

$$P[A \cup B] = P[A] + P[B] \text{ if } A \cap B = \phi$$

The Rule for complements

for any event E

$$\boxed{P[\bar{E}] = 1 - P[E]}$$

Conditional probability

$$\boxed{P[A|B] = \frac{P[A \cap B]}{P[B]}}$$

The multiplicative rule of probability

$$P[A\cap B]=\begin{cases}P[A]P[B|A] & \text{if } P[A]\neq 0\\ P[B]P[A|B] & \text{if } P[B]\neq 0\end{cases}$$

and

$$P[A\cap B]=P[A]P[B]$$

if A and B are **independent**.

This is the definition of independent

Counting techniques

Finite uniform probability space

Many examples fall into this category

1. Finite number of outcomes
2. All outcomes are equally likely
3. $P[E]=\dfrac{n(E)}{n(S)}=\dfrac{n(E)}{N}=\dfrac{\text{no. of outcomes in } E}{\text{total no. of outcomes}}$

Note: $n(A)=$ no. of elements of A

To handle problems in case we have to be able to count. Count $n(E)$ and $n(S)$.