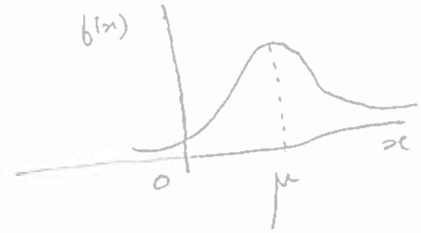
NORMAL DISTRIBUTION

$X \sim N(\mu, \sigma^2) \iff X$ HAS NORMAL (GAUSSIAN) DISTRIBUTION

$\mu \in \mathbb{R} \; ; \; \sigma > 0$



i) PROBABILITY DENSITY FUNCTION OF RV X

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \; ; \quad x \in \mathbb{R}$$

ii) $E(X) = \mu \; ; \quad VAR(X) = \sigma^2$

iii) $\mu = 0 \; ; \; \sigma = 1 \; ;$ THEN X IS STANDARD NORMAL RV

iv) $MGF = M_X(t) = E(e^{tX}) = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]$

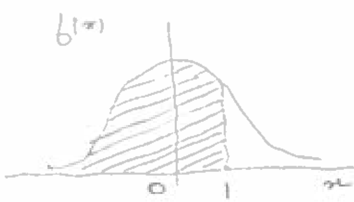
v) $Z = \frac{X-\mu}{\sigma} \; ; \quad Z \sim N(0,1)$

vi) $Y = aX + b \; ; \quad a \in \mathbb{R} \; ; \; b \in \mathbb{R}$

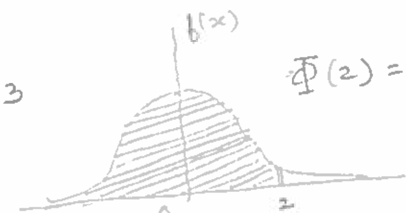
Y HAS NORMAL DISTRIBUTION WITH MEAN $= \mu_Y = a\mu + b$

VARIANCE $= \sigma_Y^2 = a^2 \sigma^2$ ▫

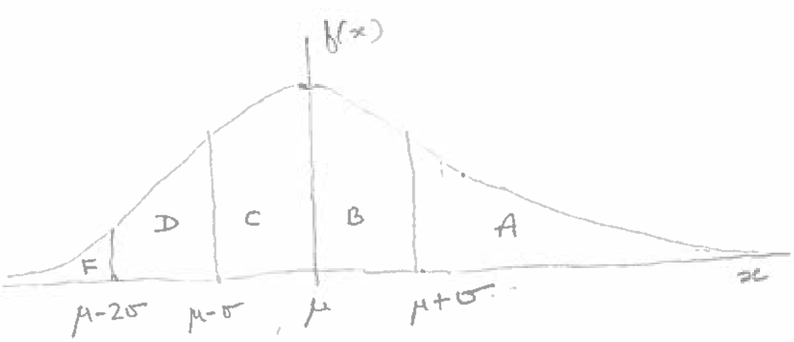
GRADING ON THE CURVE



$\Phi(1) = 0.8413$



$\Phi(2) = 0.9772$



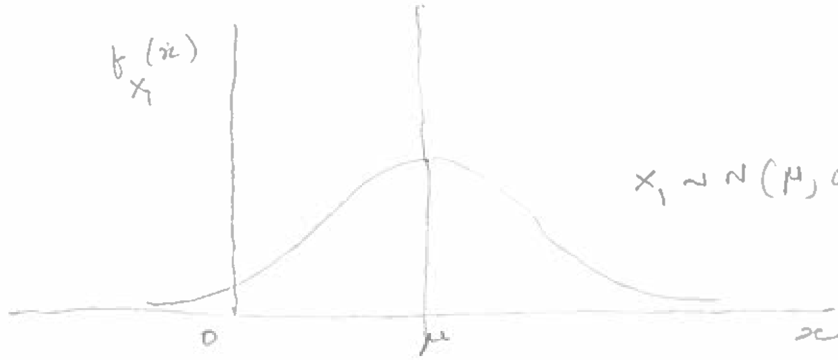
A: $1 - 0.8413 = 0.1587$

B: $0.8413 - 0.5 = 0.3413$

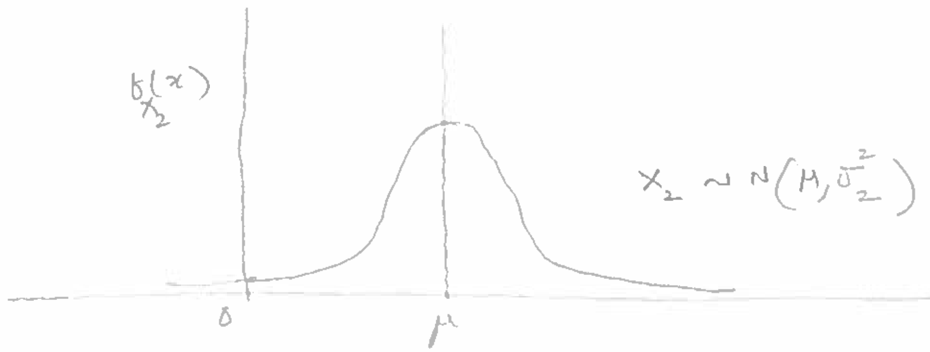
C: 0.3413

D: $0.9773 - 0.8413 = 0.1359$

F: $1 - 0.9772 = 0.0228$



$X_1 \sim N(\mu, \sigma_1^2)$



$X_2 \sim N(\mu, \sigma_2^2)$ $\quad \sigma_2^2 < \sigma_1^2$

$X_3 \sim N(\mu, \sigma_3^2)$ $\quad \sigma_1^2 < \sigma_3^2$

CHECK RESULTS

⓪ $f(x) \geq 0$

① $\int_{-\infty}^{\infty} f(x)\,dx \overset{?}{=} 1$

PROOF: $\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$

$\frac{x-\mu}{\sigma} = y$

$dx = \sigma\,dy$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy = 1$$

② $M_X(t) = E(e^{tX}) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{tx} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$

$$= \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]$$

PROOF: USE METHOD OF COMPLETING THE SQUARE.

$$-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 + tx = -\frac{1}{2\sigma^2}\left[x^2 - 2x\mu - 2xt\sigma^2 + \mu^2\right]$$

$$= -\frac{1}{2\sigma^2}\left[x^2 - 2x(\mu + t\sigma^2) + \mu^2\right]$$

$$= -\frac{1}{2\sigma^2}\left[\{x - (\mu + t\sigma^2)\}^2 + \mu^2 - (\mu + t\sigma^2)^2\right]$$

$$= -\frac{1}{2\sigma^2}\left[\{x - (\mu + t\sigma^2)\}^2 + (2\mu + t\sigma^2)(-t\sigma^2)\right]$$

$$= -\frac{1}{2\sigma^2}\left[\{x - (\mu + t\sigma^2)\}^2\right] + \left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$$

$$\therefore M_X(t) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right] \underbrace{\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left\{\frac{x - (\mu + \sigma^2 t)}{\sigma}\right\}^2\right] dx}_{\sqrt{2\pi}\,\sigma}$$

$$= \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]$$

∎

③ $E(X) = M_X'(t)\Big|_{t=0} = \mu$; $E(X^2) = M_X''(t)\Big|_{t=0} = \sigma^2 + \mu^2$

1st DERIVATIVE 2nd DERIVATIVE

④ $Z = \dfrac{X-\mu}{\sigma}$; $Z \sim N(0,1)$

PROOF: $X \sim N(\mu, \sigma^2) \Rightarrow M_X(t) = \exp\left[\mu t + \dfrac{\sigma^2 t^2}{2}\right] = E(e^{tX})$

LET $Z = \dfrac{X-\mu}{\sigma} \Rightarrow X = \mu + \sigma Z$

$$E(e^{tX}) = E(e^{t(\mu + \sigma Z)}) = E(e^{\mu t + t\sigma Z})$$

$$= e^{\mu t} E(e^{t\sigma Z}) = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]$$

$\Rightarrow E(e^{t\sigma Z}) = \exp\left(\dfrac{\sigma^2 t^2}{2}\right)$ LET $\sigma t = \tau$

$\therefore E(e^{\tau Z}) = \exp\left(\dfrac{\tau^2}{2}\right) \Rightarrow Z \sim N(0,1)$ $\square$

APPROXIMATION OF BINOMIAL DISTRIBUTION

X IS BINOMIALLY DISTRIBUTED WITH PARAMETERS n & p;

WHERE $n = 1, 2, 3, \dots$; $0 < p < 1$

THEN $E(X) = np$; $VAR(X) = npq$; $q = 1-p$

AS $n \to \infty$; FOR $a < b$

$$P\left(a \leq \frac{X - np}{\sqrt{npq}} \leq b\right) \simeq \Phi(b) - \Phi(a)$$

DE MOIVRE - LAPLACE LIMIT THEOREM

PROOF: THIS RESULT IS A SPECIAL CASE OF CENTRAL LIMIT THEOREM