MATRIX ALGEBRA

1. A MATRIX IS A TWO-DIMENSIONAL ARRAY OF EITHER REAL OR COMPLEX NUMBERS

LET THE MARIX $A = [a_{ij}]$ BE OF SIZE $m \times n$

$m, n \in \mathbb{P}$

$a_{ij} \in \mathbb{R}$ OR $\mathbb{C}$

$\mathbb{R}$ = SET OF REAL NUMBERS = $(-\infty, +\infty)$

$\mathbb{C}$ = " " COMPLEX "

$\mathbb{P}$ = SET OF POSITIVE INTEGERS = $\{1, 2, 3, \ldots\}$

m = # OF ROWS IN MATRIX A

n = # " COLUMNS " " "

m & n ARE CALLED THE DIMENSIONS OF MATRIX A

2. TRANSPOSE OF A MATRIX A IS DENOTED BY A^T

IF $A = [a_{ij}]$, THEN $A^T = [a_{ji}]$

IF A IS $m \times n$ THEN A^T IS $n \times m$

3. ROW VECTOR AND COLUMN VECTOR

i) ROW VECTOR IS A MATRIX WITH ONE ROW

ii) COLUMN VECTOR IS A MATRIX WITH A SINGLE COLUMN.

4. SCALAR IS SIMPLY A SINGE NUMBER

5. SQUARE MATRIX: MATRIX A IS SQUARE IF THE NUMBER OF ROWS = NUMBER OF COLUMNS

6. MATRIX ADDITION & SUBTRACTION

LET $A = [a_{ij}]$ BE $m \times n$

$B = [b_{ij}]$ BE $m \times n$

$C = A \pm B = [c_{ij}]$

$c_{ij} = a_{ij} \pm b_{ij} \qquad 1 \leq i \leq m \; ; \; 1 \leq j \leq n$

7. MATRIX SCALAR MULTIPLICATION

$k \in \mathbb{R}$ OR $\mathbb{C}$ = SCALAR

$A = [a_{ij}]$

$B = kA \; ; \; B = [b_{ij}]$

WHERE $b_{ij} = k\,a_{ij}$

8. MATRIX & VECTOR MULTIPLICATION

$A = [a_{ij}] \; ; \; 1 \leq i \leq m \; ; \; 1 \leq j \leq n$

$R = [r_1 \;\; r_2 \; \ldots \; r_m]$ = ROW VECTOR

NOTE

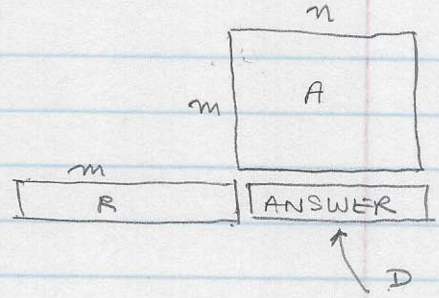
$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$ = COLUMN VECTOR

NOTE

$RA = D$

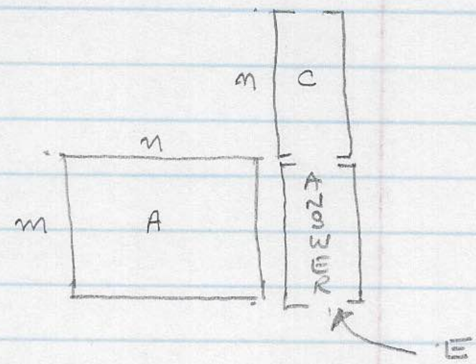
$D = [d_1 \; d_2 \; \ldots d_n]$

$d_j = \sum_{k=1}^{m} r_k a_{kj} \quad ; \quad 1 \le j \le n$



$AC = E = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}$

$e_i = \sum_{k=1}^{n} a_{ik} c_k \quad ; \quad 1 \le i \le m$



9. MATRIX MULTIPLICATION

$A = [a_{ij}] \qquad m \times n$

NOTE

$B = [b_{ij}] \qquad n \times p$

$C = [c_{ij}] \qquad m \times p$

$AB = C$

$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \qquad 1 \le i \le m \; ; \; 1 \le j \le p$

IN GENERAL $AB \neq BA$

IN ORDER TO MULTIPLY MATRICES A AND B :

NUMBER OF COLUMNS IN MATRIX A

= NUMBER OF ROWS IN MATRIX B

10. SPECIAL MATRICES

i) **SQUARE MATRIX**: $A = [a_{ij}]$ IS $m \times m$

NUMBER OF ROWS = NUMBER OF COLUMNS

ii) **DIAGONAL MATRIX** OF SIZE n

$$A = \begin{bmatrix} a_1 & & & \phi \\ & a_2 & & \\ & & \ddots & \\ \phi & & & a_n \end{bmatrix} = \text{DIAG}(a_1, a_2, \ldots, a_n)$$

OFF-DIAGONAL ELEMENTS ALL 0's

iii) **IDENTITY MATRIX** OF SIZE n

I = IDENTITY MATRIX, IS A DIAGONAL MATRIX, WITH ALL 1's ON THE MAIN DIAGONAL

EXAMPLES:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \phi \\ & 1 & & \\ & & \ddots & \\ \phi & & & 1 \end{bmatrix}$$

iv) **SYMMETRIC MATRIX**

A IS A SYMMETRIC MATRIX IF $A = A^T$

EXAMPLE:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & -4 & -7 \\ 1 & -7 & 6 \end{bmatrix}$$

10. DETERMINANT OF A SQUARE MATRIX

i) LET $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $a, b, c, d \in \mathbb{C}$ OR $\mathbb{R}$ 2×2 MATRIX

DETERMINANT OF MATRIX A = $\det A$ = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

$= ad - bc$

ii) LET

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$ 3×3 MATRIX

$$\det A = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

iii) $A = [a_{ij}]$ = SQUARE MATRIX

A_{ij} = MATRIX OBTAINED FROM MATRIX A BY DELETING $i^{\underline{th}}$ ROW AND $j^{\underline{th}}$ COLUMN

MINOR OF a_{ij} = $\det A_{ij}$

COFACTOR OF a_{ij} = M_{ij} = $(-1)^{i+j} \det A_{ij}$

iv) $\det(AB) = (\det A)(\det B)$

A AND B ARE SQUARE MATRICES OF THE SAME SIZE

v) $\det(A^T) = \det(A)$

12. INVERSE OF A SQUARE MATRIX

i) ALL SQUARE MATRICES DO NOT HAVE INVERSE

ii) IF A SQUARE MATRIX A HAS AN INVERSE, THEN DENOTE IT BY A^{-1}. THEN IN THIS CASE

$$A^{-1}A = AA^{-1} = I$$

iii) LET $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $\Delta = \det A = (ad - bc)$

A^{-1} EXISTS IF $\Delta \neq 0$. IN THIS CASE

$$A^{-1} = \frac{1}{\Delta}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$; CHECK THAT $A^{-1}A = AA^{-1} = I$

iv) ADJOINT OF A MATRIX A (A IS SQUARE)

$$= \operatorname{adj} A = [\beta_{ji}]$$

β_{ji} = COFACTOR OF a_{ji}

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A$$; $\det A \neq 0$; A IS SAID TO BE INVERTIBLE

13. TRACE OF A SQUARE MATRIX

$A = [a_{ij}]$ IS A $n \times n$ MATRIX

$$\text{TRACE}(A) = \sum_{i=1}^{n} a_{ii}$$ = SUM OF ALL DIAGONAL ELEMENTS OF A

1. LENGTH OF A ROW OR COLUMN VECTOR

LET $a = [a_1, a_2, \cdots, a_n]$ = ROW VECTOR

LENGTH OF VECTOR a IS $= \left\{ \sum_{i=1}^{n} a_i \bar{a}_i \right\}^{1/2} \triangleq \|a\|$

$\bar{a}_i$ = COMPLEX CONJUGATE OF a_i

DENOTES ABBREVIATION

NOTE THAT $\|\cdot\|$ IS CALLED THE EUCLIDEAN NORM

LENGTH OF A COLUMN VECTOR IS SIMILARLY DEFINED

2. VECTOR OF UNIT LENGTH

LET a BE A VECTOR, AND $\|a\| \neq 0$, THEN

$\frac{a}{\|a\|}$ IS A VECTOR OF UNIT LENGTH

3. ORTHOGONAL VECTORS.

LET $a = [a_1, a_2, \cdots, a_n]$; $a_i \in \mathbb{R}$; $1 \le i \le n$

$b = [b_1, b_2, \cdots, b_n]$; $b_i \in \mathbb{R}$; $1 \le i \le n$

$\sum_{i=1}^{n} a_i b_i \triangleq a \circ b = c$; $c \in \mathbb{R}$

IF $c = 0$, THEN a AND b ARE CALLED ORTHOGONAL (PERPENDICULAR VECTORS)

4. ORTHONORMAL VECTORS

LET a & b BE VECTORS, EACH OF UNIT LENGTH; AND $a \circ b = 0$ (NOTE THAT $\|a\| = \|b\| = 1$)

THEN VECTORS a AND b ARE SAID TO $\underbrace{\text{ORTHO}}_{\text{PERPENDICULAR}}\underbrace{\text{NORMAL}}_{\text{NORMALIZED}}$ VECTORS

1. LINEAR EQUATION

$ax = b$; $a, b \in \mathbb{R}$; $\mathbb{R} = (-\infty, +\infty)$

SOLVE FOR x (x IS UNKNOWN)

i) $a \neq 0$; $x = \frac{b}{a}$

ii) $a = 0$; $b = 0$ $\Rightarrow$ x IS ANY NUMBER (MANY SOLUTIONS)

$0 \times 5 = 0$; $0 \times 7 = 0$; $0 \times (-7) = 0$

SOLUTION SOLUTION SOLUTION

iii) $a = 0$; $b \neq 0$ $\Rightarrow$ INCONSISTENT EQUATION.

NO SOLUTION.

2. SYSTEM OF LINEAR EQUATIONS

$Ax = b$

A IS A $n \times n$ SQUARE MATRIX

x & b ARE COLUMN VECTORS OF SIZE n

A, b IS GIVEN; x IS UNKNOWN

i) A IS INVERTIBLE ; $x = A^{-1} b$ ($\det A \neq 0$)

ii) $\det A = 0$ $\Rightarrow$ A IS NOT INVERTIBLE

(a) IF $b = 0$ $\Rightarrow$ x HAS MANY SOLUTIONS

(b) IF $b \neq 0$ $\Rightarrow$ x MAY BE INCONSISTENT OR IT MAY HAVE MANY SOLUTIONS

EXAMPLES

EXAMPLE 1:

$$A = \begin{bmatrix} 2 & -5 & 6 \\ 7 & -9.1 & -8 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

MATRIX A HAS 2 ROWS & 3 COLUMNS

THAT IS, THIS MATRIX IS OF SIZE 2×3

2 ROWS 3 COLUMNS

$a_{11} = 2$; $a_{12} = -5$; $a_{13} = 6$

$a_{21} = 7$; $a_{22} = -9.1$; $a_{23} = -8$ □

EXAMPLE 2: TRANSPOSE OF THE MATRIX A IN THE LAST EXAMPLE IS:

$$A^T = \begin{bmatrix} 2 & 7 \\ -5 & -9.1 \\ 6 & -8 \end{bmatrix}$$

END OF EXAMPLE □

EXAMPLE 3:

$R = \begin{bmatrix} 2 & -5 & -6 & 7 \end{bmatrix}$ = ROW VECTOR

THIS IS A 1×4 MATRIX

$$C = \begin{bmatrix} 1 \\ 2 \\ -9 \\ 8 \end{bmatrix} = \text{COLUMN VECTOR}$$

THIS IS A 4×1 MATRIX

R^T IS A COLUMN VECTOR

C^T IS A ROW VECTOR □

EXAMPLE 4: MATRIX ADDITION / SUBTRACTION

$$A = \begin{bmatrix} -6 & 4 \\ -3 & -2 \end{bmatrix} \; ; \quad B = \begin{bmatrix} 2 & -6 \\ -7 & 8 \end{bmatrix}$$

$$C = A + B = \begin{bmatrix} -4 & -2 \\ -10 & 6 \end{bmatrix} \; ; \quad D = A - B = \begin{bmatrix} -8 & 10 \\ 4 & -10 \end{bmatrix} \qquad \square$$

EXAMPLE 5: MATRIX SCALAR MULTIPLICATION

$$A = \begin{bmatrix} -6 & 4 \\ -3 & -2 \end{bmatrix} \; ; \quad k = +2$$

$$kA = \begin{bmatrix} -12 & 8 \\ -6 & -4 \end{bmatrix} \qquad \square$$

EXAMPLE 6:

$$R = \begin{bmatrix} 1 & -3 & 2 \end{bmatrix} \; ; \quad C = \begin{bmatrix} 2 & 4 \end{bmatrix}^T$$

$$A = \begin{bmatrix} -1 & 4 \\ 2 & 5 \\ 3 & -2 \end{bmatrix}$$

. FIND RA AND AC

$$\overbrace{\begin{bmatrix} -1 & 4 \\ 2 & 5 \\ 3 & -2 \end{bmatrix}}^{A}$$

$$RA = \overbrace{\begin{bmatrix} 1 & -3 & 2 \end{bmatrix}}^{R} \underbrace{\begin{bmatrix} -1 & -15 \end{bmatrix}}_{RA}$$

$$\left.\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\} C$$

$$\overbrace{\begin{bmatrix} -1 & 4 \\ 2 & 5 \\ 3 & -2 \end{bmatrix}}^{A} \underbrace{\begin{bmatrix} 14 \\ 24 \\ -2 \end{bmatrix}}_{AC} \qquad \square$$

EXAMPLE 7: MATRIX MULTIPLICATION

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 0 & 4 \end{bmatrix} \; ; \quad B = \begin{bmatrix} -6 & 3 \\ 1 & 4 \\ 2 & 0 \end{bmatrix}$$

$C = AB$?

SOLUTION: A IS 2×3

B IS 3×2

$$\begin{array}{cc} & \underset{B}{\begin{bmatrix} -6 & 3 \\ 1 & 4 \\ 2 & 0 \end{bmatrix}} \\ \underset{A}{\begin{bmatrix} 2 & 3 & -1 \\ -1 & 0 & 4 \end{bmatrix}} & \underset{C}{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}} \end{array}$$

$c_{11} = 2(-6) + 3(1) + (-1)2 = -11$

$c_{12} = 2(3) + 3(4) + (-1)0 = 18$

$c_{21} = (-1)(-6) + 0(1) + 4(2) = 14$

$c_{22} = (-1)3 + 0(4) + 4(0) = -3$

$$C = \begin{bmatrix} -11 & 18 \\ 14 & -3 \end{bmatrix}$$

$\square$

EXAMPLE 8: PROVE THAT

i) $(AB)^T = B^T A^T$

ii) $(A^T)^T = A$

EXAMPLE 9. ON DETERMINANTS

i) $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$; $\det A = (1)(-4) - 3(-2) = 2$

ii) $A = \begin{bmatrix} 6 & 1 & 8 \\ -2 & 1 & 0 \\ 4 & 3 & -2 \end{bmatrix}$; $\det A = \begin{vmatrix} 6 & 1 & 8 \\ -2 & 1 & 0 \\ 4 & 3 & -2 \end{vmatrix}$

$$\det A = 6\begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix} - (1)\begin{vmatrix} -2 & 0 \\ 4 & -2 \end{vmatrix} + 8\begin{vmatrix} -2 & 1 \\ 4 & 3 \end{vmatrix}$$

$$= 6(-2) - (-2)(-2) + 8[(-2)3 - 4(1)]$$

$$= -12 - 4 - 80 = -96$$

$\square$

EXAMPLE 10. INVERSE OF A 2X2 MATRIX

$A = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$. FIND A^{-1}

SOLUTION: $A^{-1} = \frac{1}{17}\begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}$

CHECK IF: $AA^{-1} = I$

$$\overbrace{\begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix} \cdot \frac{1}{17}}^{A^{-1}}$$

$$\overbrace{\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}}^{A} \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix} \frac{1}{17} \longrightarrow = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\square$

EXAMPLE 11. TRACE OF A MATRIX

$$A = \begin{bmatrix} 1 & -2 & 6 & 3 \\ 4 & 5 & 2 & 0 \\ 6 & 7 & -9 & 2 \\ 1 & 3 & -6 & -7 \end{bmatrix}$$

MAIN DIAGONAL

TRACE (A) = SUM OF ELEMENTS ON THE MAIN DIAGONAL

$= 1+5-9-7 = -10$ □

EXAMPLE 12. LENGTH OF A VECTOR.

$a = [3, 0, -4]$; LENGTH OF VECTOR $a = \{3^2 + 0^2 + (-4)^2\}^{1/2}$

$\|a\| = 5$ □

EXAMPLE 13. VECTOR OF UNIT LENGTH

$b = \frac{a}{\|a\|} = \left[\frac{3}{5}, 0, -\frac{4}{5}\right]$; CHECK THAT $\|b\| = 1$ □

EXAMPLE 14. ORTHOGONAL VECTORS

$a = [1 \quad 1]$; $b = [1 \quad -1]$

$a \circ b = (1)(1) + (1)(-1) = 0 \Rightarrow$ a AND b ARE ORTHOGONAL VECTORS □

EXAMPLE 15. ORTHONORMAL VECTORS

$a = \left[\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right]$; $b = \left[\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}}\right]$

NOTE THAT $\|a\| = 1$; $\|b\| = 1$; $a \circ b = 0$

THEREFORE a AND b ARE ORTHONORMAL VECTORS. □

EIGENVALUES & EIGENVECTORS

1. $\mathbb{C}$ = SET OF COMPLEX NUMBERS

$\mathbb{C} = \{ a+ib \mid a,b \in \mathbb{R} \ ; \ i = \sqrt{-1} \}$

$a+ib$ = IMAGINARY NUMBER

SCALAR NUMBER = CONSTANT NUMBER

2. A IS A COMPLEX SQUARE MATRIX $(n \times n)$

X IS A NON-ZERO COMPLEX COLUMN VECTOR SATISFYING

$AX = \lambda X$; WHERE $\lambda \in \mathbb{C}$ IS A COMPLEX SCALAR.

i) λ = EIGEN-VALUE OF MATRIX A

ii) $X \neq 0$ IS THE EIGEN-VECTOR OF MATRIX A CORRESPONDING TO EIGEN-VALUE λ

EIGEN IS A GERMAN WORD MEANING CHARACTERISTIC

THE EIGEN-VECTOR CORRESPONDING TO THE EIGEN-VALUE λ IS NOT UNIQUE.

3. LET I BE AN IDENTITY MATRIX OF SIZE n

$AX = \lambda X = \lambda I X$

$\Rightarrow AX - \lambda I X = 0 \quad \Rightarrow (A - \lambda I) X = 0$

VECTOR X IS NON ZERO IF $\det(A - \lambda I) = 0$

$\Rightarrow$ THERE ARE AT MOST n DISTINCT EIGENVALUES OF A

i) EIGEN-VALUE = CHARACTERISTIC VALUE

ii) $\det(A - \lambda I) = 0$ = CHARACTERISTIC EQUATION OF MATRIX A

iii) $\det(A - \lambda I)$ = CHARACTERISTIC POLYNOMIAL OF MATRIX A

FACT: A IS A $m \times m$ MATRIX.

$\underbrace{\lambda_1, \lambda_2, \ldots, \lambda_m}$ ARE m EIGEN-VALUES OF MATRIX A

CAN BE REPEATED

i) $\text{TRACE}(A) = \sum_{i=1}^{m} \lambda_i$

ii) $\det A = \lambda_1 \lambda_2 \cdots \lambda_m = \prod_{i=1}^{m} \lambda_i$

iii) EIGENVECTOR IS NOT UNIQUE

IF X IS AN EIGENVECTOR, THAT IS $AX = \lambda X$, THEN $AY = \lambda Y$; WHERE $Y = aX$; $a \neq 0$

$\square$

FACT A IS A COMPLEX MATRIX OF SIZE m.

i) ITS EIGENVALUES $\lambda_1, \lambda_2, \ldots, \lambda_m$ ARE DISTINCT.

ii) $X_1, X_2, \ldots, X_m$ ARE THE CORRESPONDING EIGEN-VECTORS.

iii) P = MATRIX WHOSE COLUMNS ARE $X_1, X_2, \ldots, X_m$

iv) Λ = DIAGONAL MATRIX $= \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$

THEN $\boxed{P^{-1}AP = \Lambda}$

$\square$

EXAMPLE $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

i) TRACE (A) = SUM OF DIAGONAL ELEMENTS $= 2+2 = 4$

ii) $\det A = (2)(2) - (1)(1) = 3$

iii) FIND EIGENVALUES OF A

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$; $(\lambda I - A) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda-2)^2 - 1 = (\lambda-1)(\lambda-3)$

$\det(\lambda I - A) = 0 \Rightarrow \boxed{\lambda = 1, 3}$ = EIGEN-VALUES OF MATRIX A

CHECK: $\lambda_1 + \lambda_2 = 4 =$ TRACE (A)

$\lambda_1 \cdot \lambda_2 = (1)(3) = 3 = \det(A)$

iv) FIND EIGEN-VECTORS

a) $\lambda_1 = 1$; $AX_1 = \lambda_1 X_1$; FIND X_1

LET $X_1 = \begin{bmatrix} x \\ y \end{bmatrix}$

$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X_1} = \underbrace{1 \begin{bmatrix} x \\ y \end{bmatrix}}_{\lambda_1 X_1} \Rightarrow \begin{aligned} 2x + y &= x \Rightarrow x = -y \\ x + 2y &= y \Rightarrow x = -y \end{aligned}$

$X_1 = \begin{bmatrix} -y \\ y \end{bmatrix}$; $y \neq 0$; y CAN TAKE SEVERAL VALUES, AS THE EIGENVECTOR IS NOT UNIQUE

LENGTH OF $X_1 = \sqrt{(-y)^2 + y^2} = \sqrt{2}\, y$

AN EIGENVECTOR OF UNIT LENGTH $= X_1 = \begin{bmatrix} -y/(\sqrt{2}y) \\ y/(\sqrt{2}y) \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

b) $\lambda_2 = 3$; $AX_2 = \lambda_2 X_2$; FIND X_2

LET $X_2 = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{array}{l} 2x + y = 3x \Rightarrow x = y \\ x + 2y = 3y \Rightarrow y = x \end{array}$$

$$A \quad X_2 \quad \lambda_2 \; X_2$$

$$X_2 = \begin{bmatrix} y \\ y \end{bmatrix} ; \; y \neq 0 ;$$

LENGTH OF $X_2 = \sqrt{y^2 + y^2} = \sqrt{2}\, y$

AN EIGEN-VECTOR OF UNIT LENGTH $= X_2 = \begin{bmatrix} y/(\sqrt{2}y) \\ y/(\sqrt{2}y) \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

c) $P = [X_1 \quad X_2] = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$; WHERE $X_1 = [-1 \quad 1]^T$
$X_2 = [1 \quad 1]^T$

$$P^{-1} = -\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} A P = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \Lambda$$

□

EXAMPLE $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

i) TRACE (A) = SUM OF DIAGONAL ELEMENTS $= 1+4 = 5$

ii) $\det A = (1)(4) - (-2)(1) = 6$

iii) FIND EIGENVALUES OF A

$(\lambda I - A) = \begin{bmatrix} \lambda-1 & -1 \\ 2 & \lambda-4 \end{bmatrix}$; $\det(\lambda I - A) = (\lambda-1)(\lambda-4) - (2)(-1)$
$= (\lambda-3)(\lambda-2)$

$\det(\lambda I - A) = 0 \Rightarrow \boxed{\lambda = 2, 3}$ = EIGEN-VALUES OF MATRIX A

CHECK : $\lambda_1 + \lambda_2 = 5$ = TRACE (A)

$\lambda_1 \cdot \lambda_2 = 6 = \det(A)$

iv) FIND EIGEN-VECTORS

a) $\lambda_1 = 2$; $A x_1 = \lambda_1 x_1$; FIND x_1

LET $x_1 = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x_1} = \underbrace{2 \begin{bmatrix} x \\ y \end{bmatrix}}_{\lambda_1 x_1} \Rightarrow \begin{aligned} x+y = 2x &\Rightarrow x = y \\ -2x+4y = 2y &\Rightarrow x = y \end{aligned}$$

$x_1 = \begin{bmatrix} y \\ y \end{bmatrix}$; $y \neq 0$

b) $\lambda_2 = 3$; $A x_2 = \lambda_2 x_2$; FIND x_2

LET $x_2 = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{aligned} x+y = 3x &\Rightarrow y = 2x \\ -2x+4y = 3y &\Rightarrow y = 2x \end{aligned}$$

$\therefore x_2 = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix}$; $x \neq 0$

□