Machine Learning

ASSIGNMENT 7

1. (30 points) **Question**: Consider the matrix

$$A = \begin{bmatrix} 1 & -2 \\ -4 & 3 \end{bmatrix}$$

   (a) (3 points) What is the sum of the eigenvalues of matrix A.

   (b) (3 points) What is the product of the eigenvalues of matrix A.

   (c) (6 points) Find the eigenvalues of the matrix A.

   (d) (6 points) Find the corresponding eigenvectors of the following matrix A. Normalize the length of the eigenvectors to unity.

   Answer:

   (a) Sum of eigenvalues $= 4$.

   (b) Product of eigenvalues $= -5$.

   (c) Eigenvalues of the matrix are: $\lambda_1 = -1$, and $\lambda_2 = 5$.

   (d) Eigenvector (normalized) corresponding to λ_i is X_i, where $i = 1, 2$; and

$$\begin{aligned} X_1 &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^T \\ X_2 &= \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}^T \end{aligned}$$

2. (50 points) We are given data points which belong to two different classes: C_1 and C_2. Let the set of two-dimensional data points be

$$X = \{x_1, x_2, x_3, x_4\}$$

   The data points x_1 and x_2, belong to the class C_1, and data points x_3 and x_4 belong to the class C_2 respectively, where

$$\begin{aligned} x_1 &= \begin{bmatrix} 1 & 2 \end{bmatrix}^T, \quad x_2 = \begin{bmatrix} 3 & 8 \end{bmatrix}^T \\ x_3 &= \begin{bmatrix} -2 & -3 \end{bmatrix}^T, \quad x_4 = \begin{bmatrix} -4 & -5 \end{bmatrix}^T \end{aligned}$$

   The goal is to determine the direction of the best linear discriminating vector.

   Hint:

   Step 1: Compute $m_1 = (x_1 + x_2)/2$, and $m_2 = (x_3 + x_4)/2$ as

$$m_1 = \begin{bmatrix} 2 & 5 \end{bmatrix}^T, \quad m_2 = \begin{bmatrix} -3 & -4 \end{bmatrix}^T$$

   Note that

$$(m_1 - m_2) = \begin{bmatrix} 5 & 9 \end{bmatrix}^T$$

Step 2: We have

$$\begin{aligned}(x_1 - m_1) &= \begin{bmatrix} -1 & -3 \end{bmatrix}^T, \quad (x_2 - m_1) = \begin{bmatrix} 1 & 3 \end{bmatrix}^T \\ (x_3 - m_2) &= \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \quad (x_4 - m_2) = \begin{bmatrix} -1 & -1 \end{bmatrix}^T\end{aligned}$$

Step 3: Compute

$$\begin{aligned}A_1 &= (x_1 - m_1)(x_1 - m_1)^T = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \\ B_1 &= (x_2 - m_1)(x_2 - m_1)^T = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \\ A_2 &= (x_3 - m_2)(x_3 - m_2)^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ B_2 &= (x_4 - m_2)(x_4 - m_2)^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\end{aligned}$$

Scatter matrix for class C_1 data points is equal to S_1, and scatter matrix for class C_2 data points is equal to S_2, where

$$\begin{aligned}S_1 &= A_1 + B_1 = \begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix} \\ S_2 &= A_2 + B_2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\end{aligned}$$

The total within-class scatter matrix is equal to S_W, where

$$S_W = S_1 + S_2 = \begin{bmatrix} 4 & 8 \\ 8 & 20 \end{bmatrix}$$

Step 4: Between-class scatter matrix is equal to S_B, where

$$S_B = (m_1 - m_2)(m_1 - m_2)^T = \begin{bmatrix} 25 & 45 \\ 45 & 81 \end{bmatrix}$$

Step 5: The direction of best linear discriminating vector is the direction of the eigenvector corresponding to the dominant eigenvalue of the matrix $D \triangleq S_W^{-1} S_B$. We obtain

$$\begin{aligned}S_W^{-1} &= \frac{1}{4}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \\ D &= S_W^{-1} S_B = a \begin{bmatrix} 35 & 63 \\ -5 & -9 \end{bmatrix}, \quad \text{where } a = \frac{1}{4}\end{aligned}$$

The eigenvalues of the matrix D are $\lambda = 0, 13/2$. Therefore the dominant eigenvalue is $13/2$. The corresponding eigenvectors of unit length are

$$w = \pm \frac{1}{\sqrt{50}} \begin{bmatrix} -7 & 1 \end{bmatrix}^T$$

Step 6: The vector w is alternately computed as follows. Compute $v = S_W^{-1}\left(m_1 - m_2\right)$. We have

$$w = \frac{v}{\|v\|}$$

Note that

$$v = a \begin{bmatrix} 7 & -1 \end{bmatrix}^T, \quad \text{where } a = \frac{1}{4}$$

which yields the w vector as in step 5.