LAGRANGE MULTIPLIERS MOTIVATION

PROBLEM: $\max f(x_1, x_2)$; $x_1, x_2 \in \mathbb{R}$

$g(x_1, x_2) = 0$

f & g ARE SUFFICIENTLY SMOOTH

ESTABLISH THAT: MAXIMIZATION OCCURS WHEN

$\nabla f = \lambda \nabla g$; $\lambda \in \mathbb{R}$

DISCUSSION:

STEP 1: IT IS ESTABLISHED THAT ∇g IS ORTHOGONAL TO THE SURFACE $g(x) = 0$, WHERE $x = (x_1, x_2)$

PROOF: x IS A POINT ON THE SURFACE

$(x + \epsilon)$ IS A NEARBY POINT ON THE SURFACE

TAYLOR EXPANSION AROUND x GIVES:

$g(x+\epsilon) \simeq g(x) + \epsilon^T \nabla g(x)$

AS $g(x) \simeq g(x+\epsilon) = 0$; WE HAVE $\epsilon^T \nabla g(x) = 0$

$\Rightarrow \nabla g(x)$ IS ORTHOGONAL TO ϵ

AS ϵ IS PARALLEL TO THE SURFACE $g(x) = 0$

$\Rightarrow \nabla g(x)$ IS $\perp$r TO THE SURFACE $g(x) = 0$. $\square$

STEP 2: IT IS ESTABLISHED THAT ∇f IS PARALLEL TO ∇g. THAT IS: $\nabla f = \lambda \nabla g$; $\lambda \in \mathbb{R}$

PROOF: SUPPOSE P IS A POINT ON THE SURFACE $g(x) = 0$, AND $f(x)$ HAS A LOCAL MAXIMA (OR MINIMA) RELATIVE TO ITS OTHER VALUES ON THE SURFACE.

LET $r(t)$ BE AN ARBITRARY PARAMETERIZED CURVE WHICH LIES ON THE CONSTRAINT SURFACE $g(x) = 0$, AND HAS $\overset{r}{R}(0) = P$.

LET $r(t) = (x_1(t), x_2(t))$; AND $h(t) = f(x_1(t), x_2(t))$.

THIS SET UP GUARANTEES MAXIMUM AT $t=0$

$\therefore\ h'(t) = \nabla f \big|_{r(t)} \cdot r'(t)$

SINCE $t=0$ IS A LOCAL MAXIMUM, WE HAVE

$h'(0) = \nabla f \big|_{P} \cdot r'(0)$

$\Rightarrow \nabla f \big|_P$ IS ORTHOGONAL (PERPENDICULAR) TO ANY CURVE ON THE CONSTRAINT SURFACE THROUGH POINT P.

$\Rightarrow \nabla f \big|_P$ IS PERPENDICULAR TO THE SURFACE

AS $\nabla g \big|_P$ IS ALSO PERPENDICULAR TO THE SURFACE

$\nabla f \big|_P$ IS PARALLEL TO $\nabla g \big|_P \Rightarrow \nabla f = \lambda \nabla g$

$\square$