1) Consider 3x3 matrix

$$\begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & 0 \\ 2 & 0 & 5 \end{bmatrix}$$

If $x^T = [\, x_1 \quad x_2 \quad x_3 \,]$, find quadratic form $x^T Ax$ associated with the matrix A.

$$x^T Ax = [\, 2x_1 - x_2 + 2x_3 \quad -x_1 + 3x_2 \quad 2x_1 + 5x_3 \,] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= (2x_1 - x_2 + 2x_3)x_1 + (-x_1 + 3x_2)x_2 + (2x_1 + 5x_3)x_3$$
$$= 2x_1^2 - 2x_1x_2 + 3x_2^2 + 4x_1x_3 + 5x_3^2$$

2) Since B is symmetric matrix, Suppose B matrix is

$$B = \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & g \end{bmatrix}$$

Then we have

$$By = \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & g \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$By = \begin{bmatrix} y_1a + y_2b + y_3c \\ y_1b + y_2e + y_3f \\ y_1c + y_2f + y_3g \end{bmatrix}$$

$$y^T By = [\, y_1 \quad y_2 \quad y_3 \,] \begin{bmatrix} y_1a + y_2b + y_3c \\ y_1b + y_2e + y_3f \\ y_1c + y_2f + y_3g \end{bmatrix}$$

$$y^T By = y_1^2a + y_1y_2b + y_1y_3c + y_1y_2b + y_2^2e + y_2y_3f + y_1y_3c + y_2y_3f + y_3^2g$$
$$= ay_1^2 + ey_2^2 + gy_3^2 + 2by_1y_2 + 2cy_1y_3 + 2fy_2y_3 = y_1^2 - y_2^2 + 4y_3^2 - 2y_1y_2 + 4y_2y_3$$

So, $a = 1$,$b = -1$,$c = 0$,$e = -1$,$f = 2$,$g = 4$, and B is

$$B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

3) Find the minimizer of function $f(x_1, x_2) = x_1^2 + x_2^2 - x_1x_2$

$$\nabla f = \begin{bmatrix} \frac{\delta f}{\delta x_1} & \frac{\delta f}{\delta x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 & 2x_2 - x_1 \end{bmatrix}$$

Let $\nabla f = 0$, we get

$$\begin{bmatrix} 2x_1 - x_2 & 2x_2 - x_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$
$$x_1 = 2x_2, x_2 = 2x_1, x_1 = x_2 = 0$$

Hessian matrix of f is

$$Hf(x_1, x_2) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$
$$Hf_{(1,1)} = 2 > 0$$

The determinant of a matrix is a product of the eigenvalues:

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 2^2 - 1 = 3 > 0$$

Therefore Hf is positive definite matrix, $(0,0)$ is local minimizer

4) Find the minimizer of function $f(x_1, x_2) = e^{x-y} + e^{y-x} + e^{x^2}$

$$\nabla f = \begin{bmatrix} e^{x-y} - e^{y-x} + 2xe^{x^2} & -e^{x-y} + e^{y-x} \end{bmatrix}$$

Let $\nabla f = 0$, we get

$$\begin{bmatrix} e^{x-y} - e^{y-x} + 2xe^{x^2} & -e^{x-y} + e^{y-x} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$
$$e^{x-y} = e^{y-x}, 2xe^{x^2} = 0, x = 0, y = 0$$

Hessian matrix of f is

$$Hf(x_1, x_2) = \begin{bmatrix} e^{x-y} + e^{y-x} + (4x^2+2)e^{x^2} & -e^{x-y} - e^{y-x} \\ -e^{x-y} - e^{y-x} & e^{x-y} + e^{y-x} \end{bmatrix}$$
$$Hf_{(1,1)} = e^{x-y} + e^{y-x} + (4x^2+2)e^{x^2} > 0$$

Let $a = e^{x-y} + e^{y-x}$ and $b = (4x^2+2)e^{x^2}$ The determinant of a matrix is a product of the eigenvalues:

$$\begin{vmatrix} a+b & -a \\ -a & a \end{vmatrix} = a(a+b) - a^2 = ab = (e^{x-y} + e^{y-x}) \cdot (4x^2+2)e^{x^2} > 0$$

Therefore Hf is positive definite matrix, $(0,0)$ is local minimizer

5) compute $min(10x + 2y^2 + z^2 + 8z)$, subject to $x + y + z = 100$ The constraint can be re-write as $100 - x - y - z = 0$

The Lagrangian function is $L = 10x + 2y^2 + z^2 + 8z + \lambda(100 - x - y - z)$ At stationary points of L, all partial derivatives should be zero, including the partial derivative with respect to

λ
$\frac{\delta L}{\delta x} = 10 - \lambda = 0$
$\frac{\delta L}{\delta y} = 4y - \lambda = 0$
$\frac{\delta L}{\delta z} = 2z + 8 - \lambda = 0$
$\frac{\delta L}{\delta \lambda} = 100 - x - y - z = 0$
Solving the equations we get $\lambda = 10$

$y = 2.5$

$z = 1$

$x = 100 - 2.5 - 1 = 96.5$

The minimizer point $(x, y, z) = (96.5, 2.5, 1)$