EIGENVALUES OF SYMMETRIC MATRICES

1. A IS AN $m \times n$ MATRIX

   $x \in \mathbb{R}^n \setminus \{0\}$

   $Ax = \lambda x$; $\lambda \in \mathbb{C}$ IS ~~THE~~ AN EIGENVALUE OF A

   x = EIGENVECTOR OF MATRIX A CORRESPONDING TO λ.

2. $(A - \lambda I)x = 0$; I IS AN IDENTITY MATRIX

   $x \neq 0$, IF $\det(A - \lambda I) = 0$

   THIS IS THE CHARACTERISTIC EQUATION OF MATRIX A

3. $\det(A - \lambda I) \neq 0$ IS A POLYNOMIAL OF DEGREE n IN λ

   $\therefore$ IT HAS n REAL OR COMPLEX ROOTS (INCLUDING MULTIPLI-
   -CITIES)

4. PROPERTIES OF SYMMETRIC MATRICES

   (a) ALL OF THE EIGENVALUES OF A SYMMETRIC MATRIX ARE REAL NUMBERS

   (b) EIGENVECTORS CORRESPONDING TO DISTINCT EIGENVALUES OF A SYMMETRIC MATRIX ARE ORTHOGONAL

5. IF A IS A SYMMETRIC MATRIX

(a) THE MATRIX A IS POSITIVE (NEGATIVE) DEFINITE IF AND ONLY IF ALL THE EIGENVALUES OF A ARE POSITIVE (NEGATIVE)

(b) THE MATRIX A IS POSITIVE (NEGATIVE) SEMIDEFINITE IF AND ONLY IF ALL THE EIGENVALUES OF A ARE NONNEGATIVE (NONPOSITIVE).

(c) THE MATRIX A IS INDEFINITE IF AND ONLY IF A HAS AT LEAST ON POSITIVE EIGENVALUE AND ONE NEGATIVE EIGENVALUE.

STATEMENT

THE EIGENVALUES OF A REAL SYMMETRIC MATRIX ARE ALL REAL

PROOF

LET A BE A SYMMETRIC MATRIX. $\Rightarrow A = A^T$

LET λ BE AN EIGENVALUE OF MATRIX A.

LET THE CORRESPONDING EIGENVECTOR BE Y.

THEREFORE $\quad Ax = \lambda x \; ; \quad x \neq 0$

AND $\quad \det(A - \lambda I) = 0 . \quad \Rightarrow (A - \lambda I)$ IS SINGULAR MATRIX

ASSUME THAT $\lambda = (h + ik) \; ; \quad i = \sqrt{-1}$

CONSIDER A MATRIX B

$$B = \{A - (h+ik)I\}\{A - (h-ik)I\} = (A - hI)^2 + k^2 I$$

MATRIX B IS REAL AND SINGULAR, BECAUSE $\{A - (h+ik)I\}$ IS SINGULAR

$\Rightarrow$ THERE EXISTS A NON-ZERO REAL VECTOR X SUCH THAT $BX = 0$

$$\therefore \quad 0 = x^T B x = x^T\{(A - hI)^2 + k^2 I\}x$$

$$= x^T (A - hI)^T (A - hI) x + x^T k^2 x$$

$$= \{(A - hI)x\}^T \{(A - hI)x\} + k^2 x^T x$$

NOTE THAT $\{(A - hI)x\}^T\{(A - hI)x\} \geq 0$.

ALSO $\quad x^T x > 0 \quad \Rightarrow \quad k = 0$

$\Rightarrow$ ALL EIGENVALUES OF A ARE REAL $\square$

STATEMENT

THE EIGENVECTORS ASSOCIATED WITH DISTINCT EIGENVALUES OF A REAL SYMMETRIC MATRIX ARE MUTUALLY ORTHOGONAL

PROOF

LET x_1 AND x_2 BE EIGENVECTORS ASSOCIATED WITH DISTINCT EIGENVALUES λ_1 AND λ_2 RESPECTIVELY. THEN

$$A x_1 = \lambda_1 x_1 \quad , \quad \text{AND} \quad A x_2 = \lambda_2 x_2$$

FURTHER

$$x_2^T A x_1 = x_2^T \lambda_1 x_1 \quad \text{— ①}$$

$$x_1^T A x_2 = x_1^T \lambda_2 x_2 \quad \text{— ②}$$

TAKE TRANSPOSE ON BOTH SIDES OF EQN. ①

$$(x_2^T A x_1)^T = (x_2^T \lambda_1 x_1)^T \qquad (A = A^T \text{ SYMMETRICITY})$$

$$x_1^T A x_2 = x_1^T \lambda_1 x_2 \quad \text{— ③}$$

EQNS ① & ③ $\Rightarrow$ $x_1^T \lambda_2 x_2 = x_1^T \lambda_1 x_2$

$$\left.\begin{array}{l} \lambda_2 x_1^T x_2 = \lambda_1 x_1^T x_2 \\ \text{AND } \lambda_1 \neq \lambda_2 \end{array}\right\}$$

$\Rightarrow x_1^T x_2 = 0$

$\Rightarrow x_1$ AND x_2 ARE ORTHOGONAL ($\perp^r$) $\square$

STATEMENT

LET A BE A REAL SYMMETRIC MATRIX. THEN ALL OF ITS EIGENVALUES ARE REAL (EARLIER RESULT).

IF ALL THE EIGENVALUES ~~ARE~~ ALSO DIFFERENT, THEN THERE EXISTS AN (ORTHOGONAL) UNITARY MATRIX P SUCH THAT $P^T A P = \Lambda$, WHERE Λ IS A DIAGONAL MATRIX, WITH ALL THE EIGENVALUES OF MATRIX A ON IT.

PROOF A REAL MATRIX P IS UNITARY, IF $PP^T = P^T P = I$ (I IS AN IDENTITY MATRIX.)

THE CASE OF A TWO BY TWO REAL SYMMETRIC MATRIX PROVIDES AN IMMEDIATE INSIGHT INTO THE STATED RESULT.

LET THE EIGENVALUES OF A BE λ_1 & λ_2, WHERE $\lambda_1 \neq \lambda_2$

LET THE CORRESPONDING EIGENVECTORS BE p_1 & p_2 RESPECTIVELY.

LET THE LENGTH OF THESE EIGENVECTORS BE UNITY. THEN

$$Ap_1 = \lambda_1 p_1 ; \quad Ap_2 = \lambda_2 p_2 ; \quad p_1^T p_1 = 1 ; \quad p_2^T p_2 = 1 ; \quad p_1^T p_2 = 0$$

$$[Ap_1 \quad Ap_2] = [\lambda_1 p_1 \quad \lambda_2 p_2]$$

$$A[p_1 \quad p_2] = [p_1 \quad p_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

LET $[p_1 \quad p_2] = P$; $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \Lambda$

THEN $AP = P\Lambda$ —— (1)

$$P^T = \begin{bmatrix} p_1^T \\ p_2^T \end{bmatrix} ; \quad P^T P = \begin{bmatrix} p_1^T \\ p_2^T \end{bmatrix} [p_1 \quad p_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

THAT IS, $P^T P = I$ —— (2)

FROM (1) & (2) $P^T A P = P^T P \Lambda = I\Lambda = \Lambda \Rightarrow P^T A P = \Lambda$ □

PROBLEM LET A BE A REAL, 2 X 2 SYMMETRIC MATRIX. LET

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

THE LEADING PRINCIPAL MINORS ARE: $D_1 = a$; $D_2 = \det A = ac - b^2$. THE MATRIX A IS POSITIVE DEFINITE. THAT IS: $D_1 > 0$, AND $D_2 > 0$.

SHOW THAT IT'S TWO EIGENVALUES ARE POSITIVE.

SOLUTION: THE CHARACTERISTIC EQUATION OF MATRIX A IS: $|\lambda I - A| = 0$. THAT IS:

$$\begin{vmatrix} (\lambda - a) & -b \\ -b & (\lambda - c) \end{vmatrix} = 0 \Rightarrow (\lambda - a)(\lambda - c) - b^2 = 0$$

THAT IS: $\lambda^2 - \lambda(a+c) + ac - b^2 = 0$

$$\lambda = \frac{1}{2}\left\{ (a+c) \pm \sqrt{(a+c)^2 - 4(ac - b^2)} \right\}$$

NOTE THAT BOTH ROOTS ARE POSITIVE; AS

$$(a+c) > \left\{ (a+c)^2 - 4(ac - b^2) \right\}^{1/2} \quad \square$$