JENSEN'S INEQUALITY

PRELUDE :

LET $A = (x_1, y_1)$ AND $B = (x_2, y_2)$ BE POINTS IN AN x-y PLANE.

ALSO LET $x_1 < x_2$

$\lambda \in [0, 1]$

$u = \lambda x_1 + (1-\lambda)x_2$; $v = \lambda y_1 + (1-\lambda) y_2$

EQUATION OF LINE JOINING POINTS A AND B IS:

$$(y - y_1) = \frac{(y_2 - y_1)}{(x_2 - x_1)} (x - x_1) \quad : \quad L$$

THEN; $x_1 \leq u \leq x_2$

POINT (u, v) LIES ON THE LINE L.

CONVEX FUNCTIONS

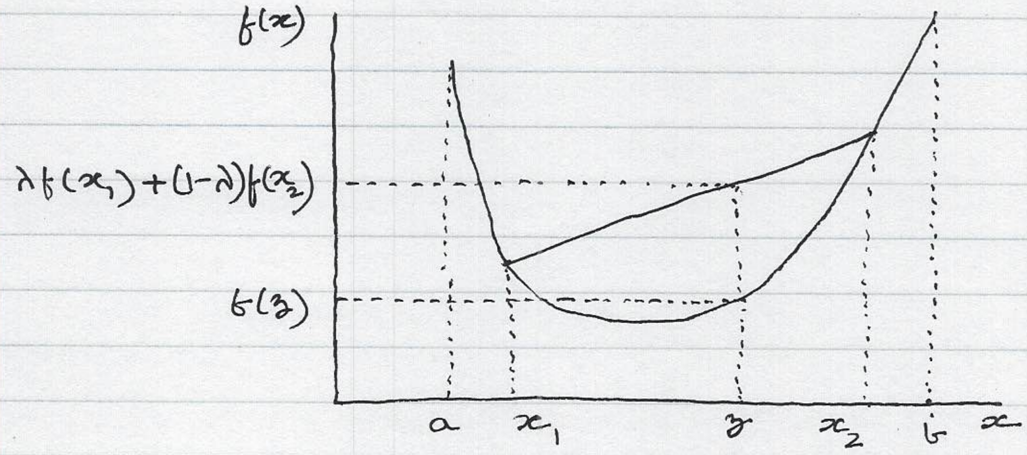
DEFINITION : LET f BE A REAL-VALUED FUNCTION DEFINED ON AN INTERVAL $I = [a, b]$

f IS SAID TO BE CONVEX ON I IF $\forall x_1, x_2 \in I$; $\lambda \in [0,1]$

$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

f IS SAID TO BE STRICTLY CONVEX IF THE INEQUALITY IS STRICT.

THAT IS, FUNCTION FALLS BELOW (STRICTLY CONVEX) OR IS NEVER ABOVE (CONVEX) THE STRAIGHT LINE JOINING POINTS $(x_1, f(x_1))$ AND $(x_2, f(x_2))$ □



$z = \lambda x_1 + (1-\lambda) x_2$

DEFINITION f IS CONCAVE (STRICTLY CONCAVE) IF $-f$ IS CONVEX (STRICTLY CONVEX). $\square$

THEOREM IF $f(x)$ IS TWICE DIFFERENTIABLE ON $[a, b]$ AND $f''(x) \geq 0$ ON $[a, b]$ THEN $f(x)$ IS CONVEX ON $[a, b]$. $\square$

FACT $-\ln x$ IS STRICTLY CONVEX ON $(0, \infty)$

PROOF: $-\ln x \triangleq f(x)$. THEN $f''(x) = 1/x^2 > 0$ FOR $\forall x \in (0, \infty)$.
BY THEOREM, $-\ln(x)$ IS STRICTLY CONVEX ON $(0, \infty)$.
ALSO BY DEFINITION $\ln x$ IS STRICTLY CONCAVE ON $(0, \infty)$. $\square$

JENSEN'S INEQUALITY LET $f(\cdot)$ BE A CONVEX FUNCTION DEFINED ON AN INTERVAL I. IF $x_1, x_2, \ldots, x_n \in I$, AND $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ WITH $\sum_{i=1}^{n} \lambda_i = 1$, THEN

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

PROOF: USE INDUCTION AND DEFINITION OF CONVEXITY. $\square$

THUS JENSEN'S INEQUALITY IS THE NOTION OF CONVEXITY EXTENDED TO n POINTS.

JENSEN'S INEQUALITY IN PROBABILITY THEORY

LET X BE A RANDOM VARIABLE, $E(\cdot)$ BE THE EXPECTATION OPERATOR, AND $g(\cdot)$ BE A CONVEX FUNCTION, THEN

$$g(E(X)) \leq E(g(X))$$

PROVIDED THE EXPECTATIONS EXIST. $\square$