PROVE THAT THE EIGENVALUES OF A SYMMETRIC MATRIX ARE REAL.

PROOF i) LET A BE A SYMMETRIX MATRIX, OF SIZE n

THAT IS, $A^T = A$

STEP ii) LET λ BE AN EIGENVALUE OF THE MATRIX A

LET x BE THE CORRESPONDING EIGENVECTOR $= [x_1, x_2, \dots, x_n]^T$

THAT IS, $Ax = \lambda x$

STEP iii) ASSUME THAT λ IS COMPLEX.

STEP iv) THEREFORE x IS ALSO COMPLEX

STEP v) $\overline{x^T} Ax = \overline{x^T} \lambda x = \lambda \overline{x^T} x = \lambda \sum_{i=1}^{n} |x_i|^2$

STEP vi) $\underbrace{\overline{x^T}(Ax)}_{\text{COMPLEX \#}} = (Ax)^T (\overline{x^T})^T$

$= (x^T A^T)\,\overline{x}$

$= (x^T A)\,\overline{x}$

LET $a \in \mathbb{C}$

COMPLEX CONJUGATE OF a IS $\overline{a}$

STEP vii) $\overline{\overline{x^T} A x}$

$= \overline{x^T A \overline{x}} = \overline{x^T} A x$

$\Rightarrow \overline{x^T} A x$ IS REAL

STEP viii) FROM (v) AND (vii) $\lambda \sum_{i=1}^{n} |x_i|^2$ IS REAL

$\Rightarrow \lambda$ IS REAL $\square$

EXAMPLE: SHOW THAT THE EIGENVALUES OF THE SYMMETRIX MATRIX A ARE REAL, WHERE

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$$

HINT: FIND THE EIGENVALUES FROM THE EQUATION

$\det(A - \lambda I) = 0$

2x2 IDENTITY MATRIX $\square$

Q IS A SYMMETRIC MATRIX
λ_1 & λ_2 ARE TWO DIFFERENT EIGENVALUES OF Q
THE CORRESPONDING EIGENVECTORS ARE x_1 & x_2
RESPECTIVELY
PROVE THAT x_1 IS ORTHOGONAL TO x_2

PROOF: $\quad Qx_1 = \lambda_1 x_1 \quad \& \quad Qx_2 = \lambda_2 x_2 \; ; \; \lambda_1 \neq \lambda_2$

TO PROVE THAT $x_1^T x_2 = 0$ (ORTHOGONALITY OF x_1 & x_2)

STEP i) $\quad x_2^T Q x_1 = x_2^T \lambda_1 x_1 = \lambda_1 x_2^T x_1$

STEP ii) $\quad x_1^T Q x_2 = x_1^T \lambda_2 x_2 = \lambda_2 x_1^T x_2$

STEP iii) IN STEP (i) TAKE TRANSPOSE BOTH SIDES

$[(ABC)^T = C^T B^T A^T]$

$$(x_2^T Q x_1)^T = (\lambda_1 x_2^T x_1)^T$$
$$x_1^T Q^T x_2 = \lambda_1 x_1^T x_2 \quad \Rightarrow \quad x_1^T Q x_2 = \lambda_1 x_1^T x_2$$

STEP iv) FROM STEPS (ii) & (iii)

$$x_1^T Q x_2 = \lambda_1 x_1^T x_2 = \lambda_2 x_1^T x_2$$
$$\Rightarrow (\lambda_1 - \lambda_2) x_1^T x_2 = 0$$

$\lambda_1 \neq \lambda_2$ THEREFORE $x_1^T x_2 = 0$ $\square$

NOTE: AS AN EIGENVECTOR IS NOT UNIQUE, IT IS POSSIBLE TO NORMALIZE ITS ~~VALUE~~ LENGTH TO UNITY.

THEREFORE, IF $\|x_1\| = 1$ & $\|x_2\| = 1$; & $x_1^T x_2 = 0$
THEN x_1 & x_2 ARE SAID TO BE ORTHONORMAL VECTORS.

LET A BE A SYMMETRIC MATRIX OF SIZE n, THEN

$A = \phi \Lambda \phi^T$

WHERE Λ = DIAGONAL MATRIX

THE DIAGONAL ELEMENTS ARE THE EIGENVALUES OF MATRIX A

ϕ = ORTHONORMAL MATRIX. THAT IS, $\phi^T \phi = I$

i) THE COLUMNS OF ϕ ARE THE EIGENVECTORS OF MATRIX A.

ii) THE LENGTH OF THESE COLUMN VECTORS ARE UNITY

iii) THE COLUMN VECTORS ARE ORTHOGONAL TO EACH OTHER

PROOF: CONSIDER TWO CASES

CASE i) MATRIX A HAS ALL DISTINCT EIGEN VALUES.

LET THESE BE $\lambda_1, \lambda_2, \ldots, \lambda_n$

LET THE CORRESPONDING EIGENVECTORS OF LENGTH UNITY BE $\phi_1, \phi_2, \ldots, \phi_n$

THAT IS $A\phi_i = \lambda_i \phi_i$, $1 \le i \le n$

IT IS KNOWN THA $\phi_i^T \phi_j = 0$; $i \ne j$; $1 \le i, j \le n$

(THAT IS, ~~OR~~ EIGENVECTORS ARE ORTHOGONAL TO EACH OTHER.); NOTE THAT

$$A \underbrace{[\phi_1 \; \phi_2 \cdots \phi_n]}_{\phi} = \underbrace{[\phi_1 \; \phi_2 \cdots \phi_n]}_{\phi} \underbrace{\begin{bmatrix} \lambda_1 & & \phi \\ & \lambda_2 & \\ \phi & & \ddots \; \lambda_n \end{bmatrix}}_{\Lambda}$$

$\therefore A\phi = \phi\Lambda \Rightarrow A = \phi \Lambda \phi^T$ (AS $\phi \phi^T = I$)

CASE ii) IF EIGENVALUES ARE REPEATED, THEN THE STATED RESULT FOLLOWS BY USING GRAM-SCHMIDT ORTHOGONALIZATION PROCEDURE.

SEE TEXTBOOK ON LINEAR ALGEBRA.

□

LET $B = A^T A$; WHERE A IS A REAL-VALUED SQUARE MATRIX OF SIZE n.

PROVE THAT B IS A POSITIVE SEMIDEFINITE MATRIX

PROOF NOTE THAT B IS A SYMMETRIC MATRIX

THAT IS, $B^T = B$

B IS A POSITIVE SEMIDEFINITE MATRIX, IF

$$x^T B x \geq 0, \quad \forall \, x \in \mathbb{R}^n$$

THIS OBSERVATION READILY FOLLOWS:

$$x^T B x = x^T A^T A x = (Ax)^T (Ax) \geq 0 \quad \forall \, x \in \mathbb{R}^n$$

$\square$

LET B BE A POSITIVE SEMIDEFINITE MATRIX. THEN ALL OF ITS EIGENVALUES ARE NONNEGATIVE.

$\square$