Definite Quadratic Forms

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Definition *Quadratic form. Let A be a square matrix of size $n \in \mathbb{P}$. In addition, the matrix A is symmetric, and all of its elements are real. Also let $x \in \mathbb{R}^n$ be a column vector. Then $Q(x) = x^T Ax$ is said to be a quadratic form.* □

Examples

1. Let $n = 1$, then $Q(x) = ax^2$.
2. Let $n = 2$, and
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
where $a_{12} = a_{21}$ for symmetry. Then
$$Q(x) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

□

Different Types of Quadratic Forms

1. *Positive definite*: The quadratic form $Q(x)$ is said to be positive definite, if $Q(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. The corresponding matrix A is also said to be positive definite.
2. *Negative definite*: The quadratic form $Q(x)$ is said to be negative definite, if $Q(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. The corresponding matrix A is also said to be negative definite.
3. *Positive semidefinite*: The quadratic form $Q(x)$ is said to be positive semidefinite, if $Q(x) \geq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. The corresponding matrix A is also said to be positive semidefinite.
4. *Negative semidefinite*: The quadratic form $Q(x)$ is said to be negative semidefinite, if $Q(x) \leq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. The corresponding matrix A is also said to be negative semidefinite.
5. *Indefinite*: The quadratic form $Q(x)$ is said to be indefinite, if $Q(x)$ is positive for some $x \in \mathbb{R}^n \setminus \{0\}$ and negative for others. The corresponding matrix A is also said to be indefinite. □

Eigenvalues

Let A be a real symmetric matrix. Then all its eigenvalues are real.

1. The matrix A is *positive definite*, if all its eigenvalues are positive.
2. The matrix A is *negative definite*, if all its eigenvalues are negative.

3. The matrix A is *positive semidefinite*, if all its eigenvalues are non-negative.
4. The matrix A is *negative semidefinite*, if all its eigenvalues are non-positive.
5. The matrix A is *indefinite*, if its eigenvalues are both positive and negative.

Facts Diagonal elements of positive definite matrices.

1. Let A be a positive definite matrix of size (order) n. Then

$$a_{ii} > 0, \quad i = 1, 2, \ldots, n$$

2. Let A be a positive semidefinite matrix of size (order) n. Then

$$a_{ii} \geq 0, \quad i = 1, 2, \ldots, n$$

□

Principal Minors

Definition The ith principal minor of a square matrix A is the matrix A_i formed by the first i rows and columns of the matrix A. □

So the first principal minor of A is the matrix $A_1 = [a_{11}]$. The second principal minor of A is the matrix

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Facts

1. The matrix A is *positive definite*, if all of its principal minors $A_1, A_2, \ldots, A_n$ have positive determinants.
2. If the determinants of all the principal minors are nonzero, and alternate in sign, starting with $\det A_1 < 0$, then the matrix A is *negative definite*.
3. The matrix A is *positive semidefinite*, if all of its principal minors $A_1, A_2, \ldots, A_n$ have nonnegative determinants.
4. If the determinants of all the principal minors, alternate in sign, starting with $\det A_1 \leq 0$, then the matrix A is *negative semidefinite*. □