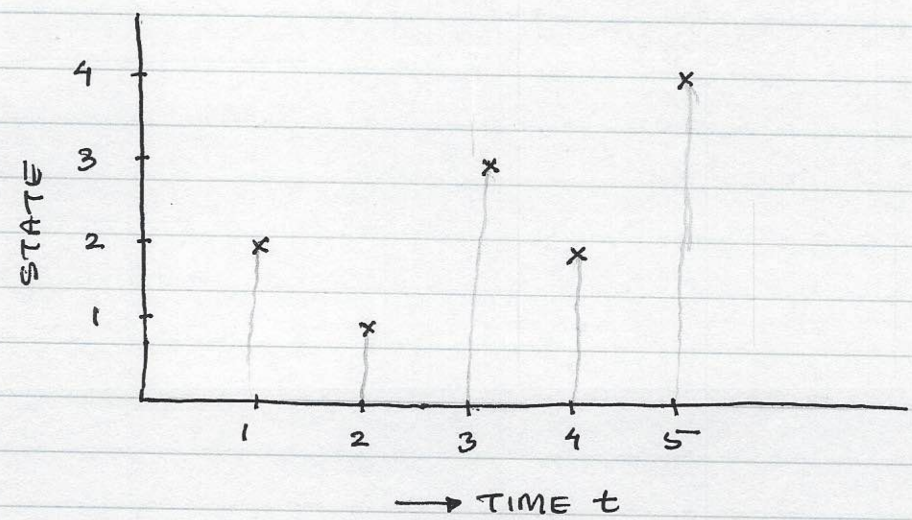
DISCRETE-TIME MARKOV PROCESS

- A SYSTEM IS ASSUMED TO EXIST IN A TOTAL OF N NUMBER OF STATES. AT REGULARLY SPACED DISCRETE TIMES.
  THE SYSTEM UNDERGOES A CHANGE OF STATE (POSSIBLY BACK TO THE SAME STATE, ACCORDING TO PROBABILITIES ASSOCIATED WITH THE STATE.

- DENOTE TIME INSTANTS BY $t = 1, 2, 3, \ldots$
  THE STATE AT TIME t IS DENOTED BY q_t



LET THE TOTAL NUMBER OF POSSIBLE STATES BE $N = 4$

STATE AT TIME $t = 1$ IS $q_1 = 2$

STATE AT TIME $t = 2$ IS $q_2 = 1$

STATE AT TIME $t = 3$ IS $q_3 = 3$

STATE AT TIME $t = 4$ IS $q_4 = 2$

- PROBABILITY THAT A SYSTEM AT TIME t IS IN STATE $q_t = j$ MIGHT DEPEND UPON THE STATES OF THE SYSTEM AT TIMES $1, 2, \ldots, (t-1)$.

- IN A MARKOV PROCESS, WE ASSUME THAT

$$P(q_t = j \mid q_{t-1} = i, q_{t-2} = k, \ldots) = P(q_t = j \mid q_{t-1} = i)$$

- FOR SIMPLICITY ASSUME THAT THE PROBABILITY ON THE RIGHT-HAND SIDE IS INDEPENDENT OF TIME. THAT IS,

$$P(q_t = j \mid q_{t-1} = i) = a_{ij} \quad ; \quad 1 \le i, j \le N$$

$$= \text{STATE TRANSITION PROBABILITY.}$$

- $a_{ij} \geq 0 \; ; \quad \forall \, i, j$

$$\sum_{j=1}^{N} a_{ij} = 1$$

- THE ABOVE PROCESS IS CALLED <u>OBSERVABLE MARKOV MODEL</u>, WHERE EACH STATE CORRESPONDS TO AN OBSERVABLE EVENT.

EXAMPLE MARKOV MODEL OF WEATHER

STATE 1: PRECIPITATION (RAIN / SNOW)

STATE 2: CLOUDY

STATE 3: SUNNY

ON DAY t, WEATHER IS IN ANY ONE OF THE THREE STATES.

$A = [a_{ij}]$ = STATE TRANSITION MATRIX

$$A = \begin{array}{c} \\ \text{PPT.} \\ \text{CLOUDY} \\ \text{SUNNY} \end{array} \begin{array}{c} \begin{array}{ccc} \text{PPT} & \text{CLOUDY} & \text{SUNNY} \end{array} \\ \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{bmatrix} \end{array} = [a_{ij}]$$

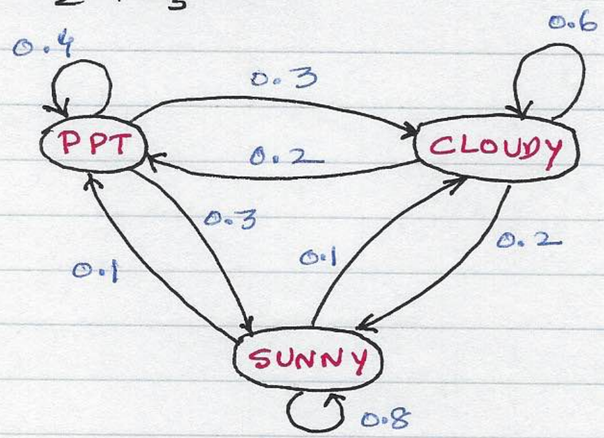
QUESTION: WHAT IS THE PROBABILITY, THAT THE WEATHER FOR EIGHT CONSECUTIVE DAYS IS:

SUNNY – SUNNY – SUNNY – RAIN – RAIN – SUNNY – CLOUDY – SUNNY

LET $\pi_i = P(q_1 = i)$; $1 \le i \le N$

DENOTE THE INITIAL PROBABILITIES BY π_1, π_2, AND π_3;

WHERE $\pi_1 + \pi_2 + \pi_3 = 1$



TRANSITION DIAGRAM

OBSERVABLE SEQUENCE = O

= (SUNNY, SUNNY, SUNNY, RAIN, RAIN, SUNNY, CLOUDY, SUNNY)

= (3 , 3 , 3 , 1 , 1 , 3 , 2 , 3) ← STATES

= 1 2 3 4 5 6 7 8 ← DAY

ASSUME : $\pi_1 = \pi_2 = 0$; AND $\pi_3 = 1$.

P(O) = PROBABILITY OF THE GIVEN SEQUENCE OF EVENTS

$$= P(3)\,P(3|3)\,P(3|3)\,P(1|3)\,P(1|1)\,P(3|1)\,P(2|3)\,P(3|2)$$

$$= \pi_3\, a_{33}\, a_{33}\, a_{31}\, a_{11}\, a_{13}\, a_{32}\, a_{23}$$

$$= 1\,(0.8)^2(0.1)(0.4)(0.3)(0.1)(0.2)$$

$$= 1.536 \times 10^{-4}$$

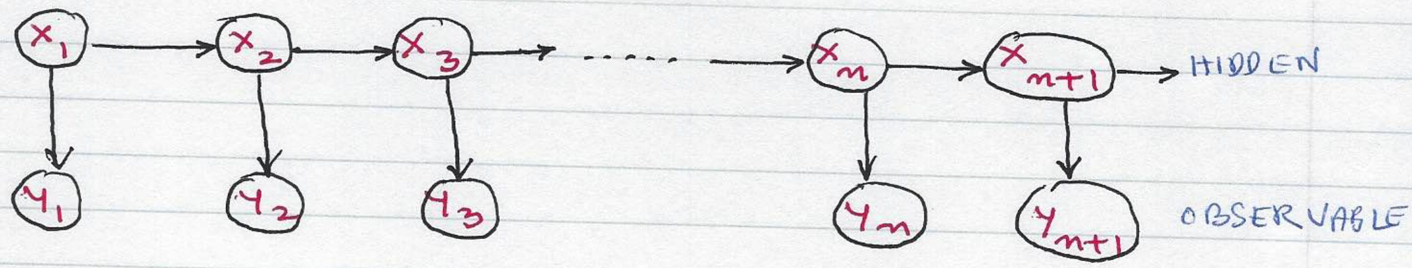
□

HIDDEN MARKOV MODEL (HMM)

$\{X_1, X_2, \ldots\}$ = HIDDEN STATE SEQUENCE

$\{Y_1, Y_2, \ldots\}$ = OBSERVABLE SEQUENCE

$\{X_i \mid i \geq 1\}$ IS A MARKOV PROCESS

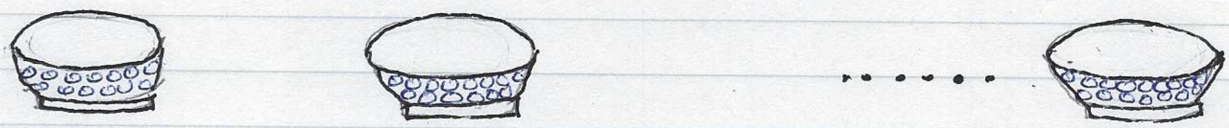


EXAMPLE URN-AND-BALL MODEL

N = # OF LARGE URNS IN A ROOM.

EACH URN HAS A LARGE QUANTITY OF COLORED BALLS

M = TOTAL # OF DIFFERENT COLORS.



URN-1 URN-2 URN-N

URN-1	URN-2	URN-N
$P(RED) = b_1(1)$	$P(RED) = b_2(1)$	$P(RED) = b_N(1)$
$P(BLUE) = b_1(2)$	$P(BLUE) = b_2(2)$	$P(BLUE) = b_N(2)$
$P(GREEN) = b_1(3)$	$P(GREEN) = b_2(3)$	$P(GREEN) = b_N(3)$
$P(YELLOW) = b_1(4)$	$P(YELLOW) = b_2(4)$	$P(YELLOW) = b_N(4)$
$\vdots$	$\vdots$	$\vdots$
$P(ORANGE) = b_1(M)$	$P(ORANGE) = b_2(M)$	$P(ORANGE) = b_N(M)$

N-STATE URN & BALL MODEL

– THE PHYSICAL PROCESS FOR OBTAINING OBSERVATIONS IS AS FOLLOWS.

i) A GENIE IS IN THE ROOM, AND ACCORDING TO SOME RANDOM PROCEDURE, IT CHOOSES AN INITIAL URN.

ii) FROM THE URN, A BALL IS CHOSEN AT RANDOM, AND ITS COLOR IS RECORDED AS THE OBSERVATION.
THE BALL IS THEN REPLACED IN THE URN FROM WHICH IT WAS SELECTED.

iii) A NEW URN IS THEN SELECTED. THE CHOICE OF URN IS DICTATED BY THE STATE-TRANSITION MATRIX OF THE HMM. FROM THIS NEW URN, A BALL IS SELECTED. ITS COLOR IS RECORDED AS AN OBSERVATION.

iv) STEP iii) IS REPEATED.

- NOTE THAT THE BALL COLOR DOES NOT TELL US ABOUT THE URN FROM WHICH IT WAS SELECTED. THAT IS, URN IS <u>HIDDEN</u>, BALL COLOR IS THE <u>OBSERVABLE</u>.

ELEMENT OF A HMM

1. N = # OF STATES IN THE MARKOV MODEL

   STATES ARE LABELLED AS $\{1, 2, 3, \ldots, N\}$

   q_t = STATE AT TIME t; WHERE $t = 1, 2, \ldots, T$

2. STATE TRANSITIONS FOLLOW MARKOV PROCESS.

   STATE TRANSITION MATRIX $= A = [a_{ij}]$

   $a_{ij} = P[q_{t+1} = j \mid q_t = i]$; $1 \le i, j \le N$

3. INITIAL STATE DISTRIBUTION : $\pi = \{\pi_i \mid 1 \le i \le N\}$

   $\pi_i = P(q_1 = i)$; $1 \le i \le N$

4. M = NUMBER OF OBSERVATION SYMBOLS

   OBSERVATION SYMBOLS CORRESPOND TO THE PHYSICAL OUTPUT OF THE SYSTEM BEING MODELED.

   SET OF INDIVIDUAL SYMBOLS $= V = \{v_1, v_2, \ldots, v_M\}$

5. OBSERVATION SYMBOL PROBABILITY DISTRIBUTION $= B$.

   $$B = \left\{ b_j(k) \;\middle|\; b_j(k) = P(O_t = v_k \mid q_t = j) ; 1 \le k \le M ; 1 \le j \le N \right\}$$

   O_t = OUTPUT SYMBOL AT TIME t, WHERE $t = 1, 2, \ldots, T$

NOTE THAT A HMM IS SPECIFIED BY A, B, AND π.

LET $\lambda = (A, B, \pi)$.

□

HMM GENERATOR

GIVEN: N, M, A, B, AND π

OUTPUT: OBSERVATION SEQUENCE $O = (o_1, o_2, \ldots, o_T)$

$o_t \in V$; $1 \leq t \leq T$

T = NUMBER OF OBSERVATIONS

STEP 1: CHOOSE AN INITIAL STATE $q_1 = i$ ACCORDING TO THE INITIAL STATE DISTRIBUTION π

STEP 2: SET $t \leftarrow 1$

STEP 3: CHOOSE $o_t = v_k$ ACCORDING TO SYMBOL PROBABILITY DISTRIBUTION IN STATE j. THAT IS, $b_j(k)$

STEP 4: TRANSIT TO A NEW STATE $q_{t+1} = j$ ACCORDING TO THE STATE TRANSITION PROBABILITY DISTRIBUTION FOR STATE i. THAT IS a_{ij}.

STEP 5: SET $t \leftarrow (t+1)$. RETURN TO STEP 3, IF $t < T$. OTHERWISE TERMINATE THE PROCEDURE. $\square$

FOLLOWING TABLE SHOWS THE SEQUENCE OF STATES AND OBSERVATIONS GENERATED BY THE ABOVE PROCEDURE.

TIME	1	2	3	4	5	6	.....	T
STATE	q_1	q_2	q_3	q_4	q_5	q_6	......	q_T
OBSERVATION	o_1	o_2	o_3	o_4	o_5	o_6	....	o_T

THREE BASIC PROBLEMS FOR HMM

GIVEN THE FORM OF HMM, THREE BASIC PROBLEMS OF INTEREST MUST BE SOLVED FOR THE MODEL TO BE USEFUL IN THE REAL WORLD.

NOTATION

$\lambda = (A, B, \pi)$, HMM

$O = (O_1, O_2, \dots, O_T)$ = OBSERVATION SEQUENCE

$q = (q_1, q_2, \dots, q_T)$ = CORRESPONDING STATE SEQUENCE

$P(O)$ = PROBABILITY OF OBSERVATION SEQUENCE O

PROBLEM 1: GIVEN λ (HMM), AND O (OBSERVATION SEQUENCE); HOW DO WE COMPUTE EFFICIENTLY $P(O)$

PROBLEM 2: GIVEN λ (HMM), AND O (OBSERVATION SEQUENCE); HOW DO WE SELECT THE CORRESPONDING STATE SEQUENCE q, THAT IS OPTIMAL IN SOME SENSE (THAT IS, BEST EXPLAINS THE OBSERVATIONS)?

PROBLEM 3: GIVEN THE OBSERVATION SEQUENCE O, DETERMINE THE MODEL PARAMETERS λ THAT MAXIMIZES $P(O)$.

EXAMPLE

THIS EXAMPLE INVESTIGATES THE CORRELATION BETWEEN THE GROWTH OF A TREE, AND THE TEMPERATURE.

- TEMPERATURES ARE : H (HOT), AND C (COLD)
- TREE GROWTH SIZE IS SPECIFIED BY THREE DIFFERENT TREE RING SIZES : S (SMALL), M (MEDIUM), AND L (LARGE)

HIDDEN STATES : $\{H, C\}$

OUTPUT STATES : $\{S, M, L\}$

- TRANSITION MATRIX = A
  EMISSION MATRIX = B

$$A = \begin{array}{c c} & \begin{array}{cc} H & C \end{array} \\ \begin{array}{c} H \\ C \end{array} & \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array} \qquad B = \begin{array}{c c} & \begin{array}{ccc} S & M & L \end{array} \\ \begin{array}{c} H \\ C \end{array} & \begin{bmatrix} 0.1 & 0.4 & 0.5 \\ 0.7 & 0.2 & 0.1 \end{bmatrix} \end{array}$$

- INITIAL STATE DISTRIBUTION : $\pi = \begin{array}{c} \begin{array}{cc} H & C \end{array} \\ \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \end{array}$

THAT IS, AT TIME $t = 1$; PROB. THAT TEMPERATURE IS EQUAL TO H IS $\pi_H = 0.6$.

PROB. THAT TEMPERATURE IS EQUAL TO C IS $\pi_C = 0.4$.

- LET T = LENGTH OF OUTPUT SEQUENCE
  O = OUPUT SEQUENCE = $(O_1, O_2, \ldots, O_T)$
  q = HIDDEN-STATES SEQUENCE = $(q_1, q_2, \ldots, q_T)$

$q = (H, H, C, C)$

$O = (S, M, S, L)$

AT TIME $t = 1$; TEMPERATURE = H $\Rightarrow \pi_H = 0.6$

$$P(q, O) = \underbrace{\pi_H\, b_H(S)}_{t=1}\ \Big|\ \underbrace{a_{HH}\, b_H(M)}_{t=2}\ \Big|\ \underbrace{a_{HC}\, b_C(S)}_{t=3}\ \Big|\ \underbrace{a_{CC}\, b_C(L)}_{t=4}$$

$$\therefore\ P(q, O) = (0.6)(0.1)\ \Big|\ (0.7)(0.4)\ \Big|\ (0.3)(0.7)\ \Big|\ (0.6)(0.1)$$

$$= 2.1168 \times 10^{-4}$$

$\square$