Linear Discriminant Analysis

Example

We are given data points which belong to two different classes: C_1 and C_2. Let the set of two-dimensional data points be

$$X = \{x_1, x_2, x_3, x_4\}$$

The data points x_1 and x_2, belong to the class C_1, and data points x_3 and x_4 belong to the class C_2 respectively, where

$$\begin{aligned} x_1 &= \begin{bmatrix} 1 & 2 \end{bmatrix}^T, \quad x_2 = \begin{bmatrix} -1 & 0 \end{bmatrix}^T \\ x_3 &= \begin{bmatrix} -1 & -1 \end{bmatrix}^T, \quad x_4 = \begin{bmatrix} 1 & -1 \end{bmatrix}^T \end{aligned}$$

The goal is to determine the direction of the best linear discriminating vector.

Step 1: Compute $m_1 = (x_1 + x_2)/2$, and $m_2 = (x_3 + x_4)/2$ as

$$m_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \quad m_2 = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$$

Note that

$$(m_1 - m_2) = \begin{bmatrix} 0 & 2 \end{bmatrix}^T$$

Step 2: We have

$$\begin{aligned} (x_1 - m_1) &= \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \quad (x_2 - m_1) = \begin{bmatrix} -1 & -1 \end{bmatrix}^T \\ (x_3 - m_2) &= \begin{bmatrix} -1 & 0 \end{bmatrix}^T, \quad (x_4 - m_2) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \end{aligned}$$

Step 3: Compute

$$\begin{aligned} A_1 &= (x_1 - m_1)(x_1 - m_1)^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ B_1 &= (x_2 - m_1)(x_2 - m_1)^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ A_2 &= (x_3 - m_2)(x_3 - m_2)^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ B_2 &= (x_4 - m_2)(x_4 - m_2)^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Scatter matrix for class C_1 data points is equal to S_1, and scatter matrix for class C_2 data points is equal to S_2, where

$$\begin{aligned} S_1 &= A_1 + B_1 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \\ S_2 &= A_2 + B_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

The total within-class scatter matrix is equal to S_W, where

$$S_W = S_1 + S_2 = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

Step 4: Between-class scatter matrix is equal to S_B, where

$$S_B = (m_1 - m_2)(m_1 - m_2)^T = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

Step 5: The direction of best linear discriminating vector is the direction of the eigenvector corresponding to the dominant eigenvalue of the matrix $D \triangleq S_W^{-1} S_B$. We obtain

$$\begin{aligned} S_W^{-1} &= \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ D &= S_W^{-1} S_B = \begin{bmatrix} 0 & -2 \\ 0 & 4 \end{bmatrix} \end{aligned}$$

The eigenvalues of the matrix D are $\lambda = 0, 4$. Therefore the dominant eigenvalue is 4. The corresponding eigenvectors of unit length are

$$w = \pm \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 2 \end{bmatrix}^T$$

Step 6: The vector w is alternately computed as follows. Compute $v = S_W^{-1}(m_1 - m_2)$. We have

$$w = \frac{v}{\|v\|}$$

Note that

$$v = \begin{bmatrix} -1 & 2 \end{bmatrix}^T$$

which yields the w vector as in step 5.