Gaussian Random Variables

Gaussian Distribution

A random variable X has a normal (or Gaussian) distribution, if the probability density function of X is given by

$$f_X\left(x\right)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\},\quad x\in\mathbb{R}$$

where $\mu\in\mathbb{R}$ and $\sigma\in\mathbb{R}^+$ are its parameters. Also $\mathcal{E}\left(X\right)=\mu$, and $Var(X)=\sigma^2$. Its moment generating function is given by

$$\mathcal{M}_X\left(t\right)=\exp\left(\mu t+\frac{\sigma^2t^2}{2}\right)$$

Use of this moment generating function yields

$$\mathcal{E}\left\{\left(X-\mu\right)^r\right\}=\begin{cases}0, & r\text{ is an odd integer}\\ \dfrac{r!}{(r/2)!}\dfrac{\sigma^r}{2^{r/2}}, & r\text{ is an even integer}\end{cases}$$

A normally distributed random variable X with mean μ and variance σ^2 is generally denoted by

$$X\sim\mathcal{N}\left(\mu,\sigma^2\right)$$

If a normal random variable has $\mu=0$, then it is called a centered normal random variable. If in addition $\sigma=1$, then it is called a *standard normal random variable.* Its probability density function $\phi\left(\cdot\right)$, and cumulative distribution function $\Phi\left(\cdot\right)$ are given by

$$\begin{aligned}\phi\left(x\right)&=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, & x\in\mathbb{R}\\ \Phi\left(x\right)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{y^2}{2}}dy, & x\in\mathbb{R}\end{aligned}$$

Also note that $\Phi\left(x\right)=\left(1-\Phi\left(-x\right)\right)$. The function $\Phi\left(\cdot\right)$ is generally evaluated numerically.

Multivariate Gaussian Distribution

Definition and properties of multivariate Gaussian (or normal) distribution are given.

Definition . *Let $Y_1,Y_2,\ldots,Y_n$ be a set of n independent standard normal random variables. Define*

$$X_i=\eta_i+\sum_{j=1}^{n}a_{ij}Y_j,\quad 1\le i\le m$$

where $a_{ij} \in \mathbb{R}, 1 \leq i \leq m, 1 \leq j \leq n$ *and* $\eta_i \in \mathbb{R}, 1 \leq i \leq m$ *are constants. The random variables* $X_1, X_2, \ldots, X_m$ *are said to have a multivariate normal distribution. Its joint probability density function exists provided its covariance matrix* Ξ *has a nonzero determinant, where*

$$\Xi = \left[\xi_{ij}\right], \quad \xi_{ij} = Cov\left(X_i, X_j\right), \quad 1 \leq i, j \leq m$$

$\square$

The above definition is valid because the sum of independent normal random variables is also a normal random variable. Therefore, each X_i is a normal random variable.

Observations

1. $\mathcal{E}\left(X_i\right) = \eta_i$ and $Var\left(X_i\right) = \xi_{ii} = \sum_{j=1}^{n} a_{ij}^2$, for $1 \leq i \leq m$.
2. The covariance $\xi_{ij} = Cov\left(X_i, X_j\right)$ is given by

$$\xi_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk}, \quad \text{for} \quad 1 \leq i, j \leq m$$

3. $\xi_{ij} = \xi_{ji}$, for $1 \leq i, j \leq m$. That is, the covariance matrix Ξ is symmetric.
4. Let

$$\begin{aligned} x &= \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}^T \in \mathbb{R}^m \\ \eta &= \begin{bmatrix} \eta_1 & \eta_2 & \cdots & \eta_m \end{bmatrix}^T \in \mathbb{R}^m \end{aligned}$$

The joint probability density function of the random variables $X_1, X_2, \ldots, X_m$ is

$$f_{X_1, X_2, \ldots, X_m}(x) = \frac{1}{(2\pi)^{m/2} (\det \Xi)^{1/2}} \exp\left\{-\frac{1}{2}(x - \eta)^T \Xi^{-1} (x - \eta)\right\}$$

5. Let $t = \begin{bmatrix} t_1 & t_2 & \cdots & t_m \end{bmatrix}^T$, and $X = \begin{bmatrix} X_1 & X_2 & \cdots & X_m \end{bmatrix}^T$, the joint moment generating function of the random variables $X_1, X_2, \ldots, X_m$ is

$$\begin{aligned} \mathcal{M}_{X_1, X_2, \ldots, X_m}(t) &= E\left\{\exp\left(t^T X\right)\right\} \\ &= \exp\left\{\eta^T t + \frac{1}{2} t^T \Xi t\right\} \end{aligned}$$

$\square$