UNCONSTRAINED OPTIMIZATION: FUNCTIONS OF ONE VARIABLE

1. LET f BE A FUNCTION OF ONE VARIABLE $\forall x \in D$ (= DOMAIN). A GLOBAL MAXIMUM OF f IS A POINT $x_0 \in D$ SUCH THAT

$$f(x_0) \geq f(x) \quad \forall x \in D$$

2. A POINT $x_0 \in D$ IS A LOCAL MAXIMUM OF f IF $\exists\ \Delta > 0$ SUCH THAT $f(x_0) \geq f(x)$ $\forall x$ IN $[x_0 - \Delta, x_0 + \Delta]$

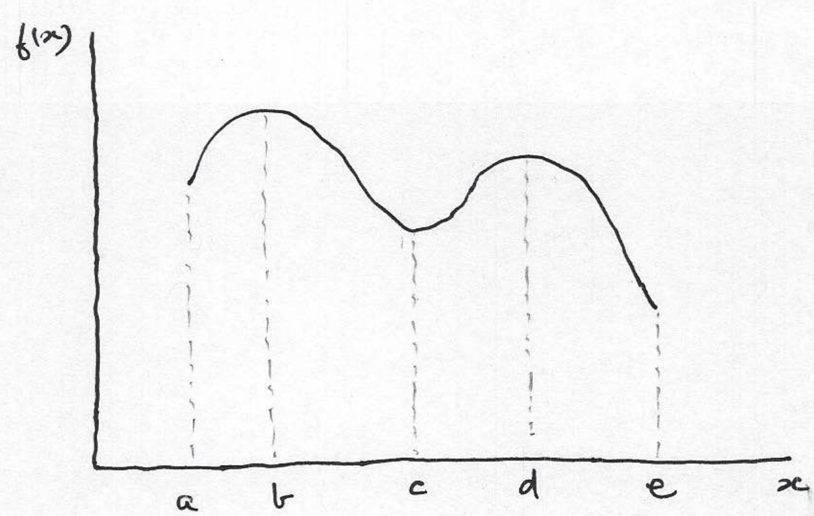
3. FINDING EXTREMA

EXTREMA, WHETHER THEY ARE LOCAL OR GLOBAL, CAN OCCUR IN THREE PLACES

i) AT THE BOUNDARY OF THE DOMAIN

ii) AT A POINT WITHOUT A DERIVATIVE, OR

iii) AT A POINT x_0 WITH $f'(x_0) = 0$



b, d	LOCAL MAXIMA
b	GLOBAL MAXIMUM
a, c, e	LOCAL MINIMA
e	GLOBAL MINIMUM

IF $f'(x_0) = 0$

i) x_0 IS A LOCAL MAXIMUM

ii) x_0 " " " MINIMUM

iii) x_0 " NEITHER

USE SECOND DERIVATIVE

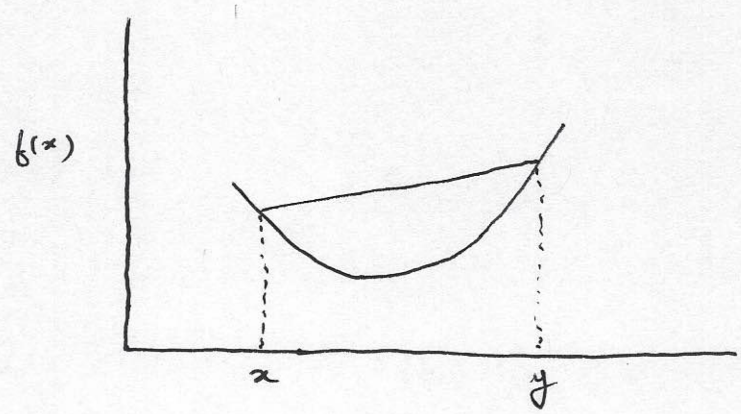
i) IF $f'(x_0) = 0$ AND $f''(x_0) > 0 \Rightarrow x_0$ IS A LOCAL MINIMUM

ii) " " " " $f''(x_0) < 0 \Rightarrow x_0$ " " " MAXIMUM

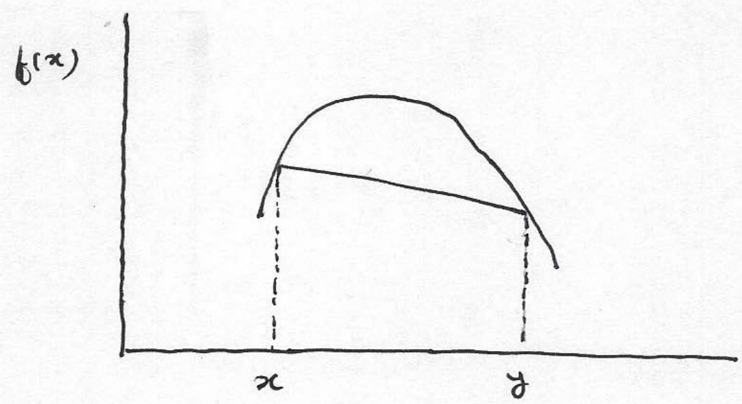
iii) " " " " $f''(x_0) = 0$ THEN x_0 MAY OR MAY NOT BE A LOCAL EXTREMUM

CONVEXITY

1. A CONVEX FUNCTION IS ONE WHERE THE LINE SEGMENT CONNECTING TWO POINTS $(x, f(x))$ AND $(y, f(y))$ LIES ABOVE THE FUNCTION.



CONVEX FUNCTION



CONCAVE FUNCTION

MATHEMATICALLY, A FUNCTION f IS CONVEX IF FOR ALL x, y AND ALL $0 < \alpha < 1$,

$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y)$$

THE FUNCTION IS CONCAVE, IF $-f$ IS CONVEX

THERE IS AN EASIER WAY TO CHECK FOR CONVEXITY WHEN

f IS TWICE DIFFERENTIABLE

i) FUNCTION f IS CONVEX ON SOME DOMAIN $[a, b]$

IF AND ONLY IF $f''(x) \geq 0$ $\forall x$ IN THE DOMAIN

ii) FUNCTION f IS CONCAVE ON SOME DOMAIN $[a, b]$

IF AND ONLY IF $f''(x) \leq 0$ $\forall x$ IN THE DOMAIN

IF $f(x)$ IS CONVEX, THEN ANY LOCAL MINIMUM IS ALSO A GLOBAL MINIMUM

IF $f(x)$ IS CONCAVE, THEN ANY LOCAL MAXIMUM IS ALSO A GLOBAL MAXIMUM.

PROBLEM. FIND WHETHER THE FOLLOWING FUNCTIONS ARE CONCAVE, CONVEX OR NEITHER

(a) $x^4 - 4x^3 + 6x^2 + 3x + 1$

(b) $-e^{x^2}$

SOLUTION

a) $f(x) = x^4 - 4x^3 + 6x^2 + 3x + 1$

$f'(x) = 4x^3 - 12x^2 + 12x + 3$

$f''(x) = 12x^2 - 24x + 12 = 12(x^2 - 2x + 1) = 12(x-1)^2 \geq 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow f(x)$ IS CONVEX $\forall x \in \mathbb{R}$

b) $f(x) = -e^{x^2}$

$f'(x) = -2xe^{x^2}$

$f''(x) = -2xe^{x^2} \cdot 2x - 2e^{x^2} = -2e^{x^2}(2x^2+1) \leq 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow f(x)$ IS CONCAVE $\forall x \in \mathbb{R}$ $\square$

PROBLEM. FIND A LOCAL EXTREMOM OF THE FUNCTION $f(x) = xe^{-x}$. INDICATE WHETHER IT IS A LOCAL MAXIMUM, A LOCAL MINIMUM OR NEITHER. IS IT A GLOBAL OPTIMUM ON THE DOMAIN $[-2, 2]$?

SOLUTION

$f(x) = xe^{-x}$

$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x)$

$f'(x) = 0$ WHEN $x = 1 \Rightarrow x = 1$ COULD BE A EXTREMUM

$$f''(x) = -e^{-x}(1-x) - e^{-x} = -e^{-x}(2-x)$$

$f''(1) = -e^{-1} < 0 \Rightarrow x = 1$ IS A MAXIMUM

□

PROBLEM: SHOW THAT $\ln x$ IS A CONCAVE FUNCTION

SOLUTION: $f(x) = \ln x \quad x > 0$

$$f'(x) = \frac{1}{x}$$

$f''(x) = -\frac{1}{x^2} < 0 \Rightarrow \ln x$ IS A CONCAVE FUNCTION

□