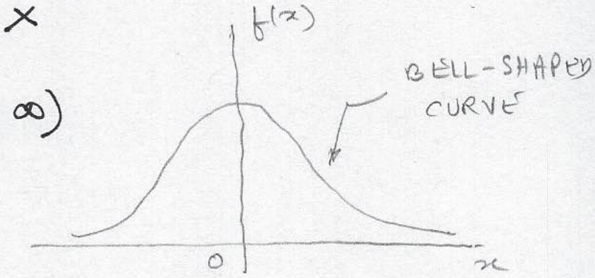
STANDARD NORMAL DISTRIBUTION

$X \sim N(0,1) \iff X$ HAS STANDARD NORMAL DISTRIBUTION

RV X HAS STANDARD NORMAL DISTRIBUTION

i) PROBABILITY DENSITY FUNCTION OF RV X



$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad ; \quad x \in (-\infty, +\infty)$$

a) $f(x) \geq 0$

b) $\int_{-\infty}^{\infty} f(x)\,dx \stackrel{?}{=} 1$

PROOF:

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx$$

$$I^2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy = \frac{1}{2\pi} \underset{\uparrow x}{\int_{-\infty}^{\infty}} \underset{\uparrow y}{\int_{-\infty}^{\infty}} e^{-(x^2+y^2)/2}\,dy\,dx$$

$x = r\cos\theta \quad ; \quad y = r\sin\theta \quad ; \quad dx\,dy = r\,d\theta\,dr$

$$I^2 = \frac{1}{2\pi} \underset{\uparrow r}{\int_{0}^{\infty}} \underset{\uparrow \theta}{\int_{0}^{2\pi}} e^{-r^2/2}\, r\,dr\,d\theta \qquad \int_0^{2\pi} d\theta = 2\pi$$

$$= \int_0^{\infty} r e^{-r^2/2}\,dr = \int_0^{\infty} e^{-t}\,dt = 1 \qquad \frac{r^2}{2} = t \; ; \; \frac{2r}{2}\,dr = dt$$

$$\therefore r\,dr = dt$$

$\therefore I^2 = 1 \Rightarrow I = +1$

$\square$

ii) $E(X) \stackrel{?}{=} 0$

PROOF: $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2}\,dx$

$\frac{x^2}{2} = t$

$\frac{2x\,dx}{2} = dt$

$= \frac{1}{\sqrt{2\pi}} \int_{+\infty}^{\infty} e^{-t}\,dt = 0$

□

iii) $E(X^2) \stackrel{?}{=} 1$

PROOF TWO STEPS

STEP 1 $\Gamma(\cdot) =$ GAMMA FUNCTION

$\uparrow$ U.C. GAMMA

$\Gamma(\alpha) = \int_{0}^{\infty} e^{-y} y^{\alpha-1}\,dy; \alpha > 0$ (DEFINITION) ;

PROPERTIES:

1. $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1); \alpha > 1 \rightarrow$ USE INTEGRATION BY PARTS
2. $\alpha = n =$ INTEGER VALUE ; THEN $\Gamma(\alpha) = (\alpha-1)!$
3. $\Gamma(1/2) = \sqrt{\pi}$ □

PROOF OF PROPERTY 3

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = 1 \Leftrightarrow \int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi} \Leftrightarrow 2\int_{0}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$$

$$\Leftrightarrow \int_{0}^{\infty} e^{-x^2/2}\,dx = \sqrt{\frac{\pi}{2}}$$

LET $\frac{x^2}{2} = t \Rightarrow \frac{2x\,dx}{2} = dt ; dx = \frac{dt}{x} = \frac{dt}{\sqrt{2t}}$

$$\Leftrightarrow \int_{0}^{\infty} \frac{1}{\sqrt{2}} t^{-1/2} e^{-t}\,dt = \sqrt{\pi/2} \Leftrightarrow \int_{0}^{\infty} t^{-1/2} e^{-t}\,dt = \sqrt{\pi}$$

$$\Leftrightarrow \int_{0}^{\infty} t^{1/2-1} e^{-t}\,dt = \sqrt{\pi}$$

$$\Leftrightarrow \Gamma(1/2) = \sqrt{\pi}$$

□

STEP 2

EVEN FUNCTION

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\,dx$$

$\frac{x^2}{2} = t$

$\frac{2x\,dx}{2} = dt$

$\therefore x\,dx = dt$

$dx = \frac{dt}{\sqrt{2t}}$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} x^2 e^{-x^2/2}\,dx = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{2t\,e^{-t}\,dt}{\sqrt{2t}}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{1/2} e^{-t}\,dt = \frac{2}{\sqrt{\pi}} \int_0^{\infty} t^{3/2-1} e^{-t}\,dt$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\tfrac{3}{2}\right) = \frac{2}{\sqrt{\pi}} \frac{1}{2} \Gamma(1/2) = \frac{2}{\sqrt{\pi}\,2} \sqrt{\pi} = 1$$

$\square$

iv) MOMENT GENERATING FUNCTION

MGF IS COMPUTED BY THE METHOD OF "COMPLETING THE SQUARE"

$$M_X(t) = e^{t^2/2}$$

PROOF $\quad M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - x^2/2}\,dx$

$$tx - x^2/2 = -\frac{1}{2}\left[x^2 - 2tx\right] = -\frac{1}{2}\left[x^2 - 2tx + t^2 - t^2\right]$$

$$= -\frac{1}{2}\left[(x-t)^2 - t^2\right]$$

$$\therefore M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left\{(x-t)^2 - t^2\right\}\right] dx = \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}(x-t)^2\right] dx$$

$$= e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy$$

$= 1$

$x - t = y$

$dx = dy$

$$= e^{t^2/2}$$

$\square$

v) $M_X'(t) = \frac{1}{2} \cdot 2t\, e^{t^2/2} = t\, e^{t^2/2}$

$M_X''(t) = t^2 e^{t^2/2} + e^{t^2/2}$

$E(X) = M_X'(t)\Big|_{t=0} = 0 \triangleq \mu$

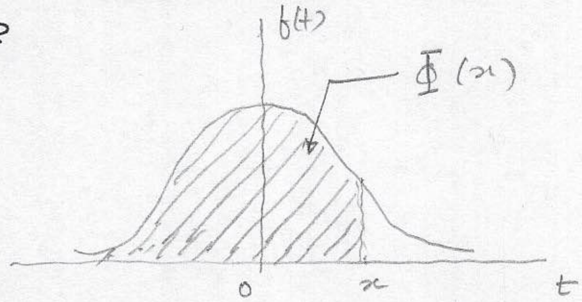
$E(X^2) = M_X''(t)\Big|_{t=0} = 1$

$VAR(X) = E(X^2) - \mu^2 = 1$

vi) CUMULATIVE DISTRIBUTION FUNCTION OF RV X

$F_X(x) = \int_{-\infty}^{x} f(t)\,dt \quad ; \quad x \in \mathbb{R}$

$\triangleq \Phi(x)$



$F_X(x)$ IS NUMERICALLY COMPUTED

$f(x) = f(-x); \quad \forall\, x \in \mathbb{R} \iff f(\cdot)$ IS AN EVEN FUNCTION

$\Phi(x) + \Phi(-x) = 1 \quad \forall\, x \in \mathbb{R}$