ASSIGNMENT # 4

1. $x^T A x = 2x_1^2 + 3x_2^2 + 5x_3^2 - 2x_1x_2 + 4x_1x_3$

2. $y^T = (y_1 \; y_2 \; y_3)$; $y^T B y = f(y)$

$f(y) = y_1^2 - y_2^2 + 4y_3^2 - 2y_1y_2 + 4y_2y_3$

$$B = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & g \end{bmatrix}$$

$$y^T B y = y^T \left[ay_1 + by_2 + cy_3 \quad by_1 + dy_2 + ey_3 \quad cy_1 + ey_2 + gy_3 \right]^T$$

$$= \left(ay_1^2 + by_1y_2 + cy_3y_1\right) + \left(by_1y_2 + dy_2^2 + ey_2y_3\right) + \left(cy_1y_3 + ey_2y_3 + gy_3^2\right)$$

$$= ay_1^2 + dy_2^2 + gy_3^2 + 2by_1y_2 + 2cy_1y_3 + 2ey_2y_3$$

$\therefore a = 1$; $d = -1$; $g = 4$; $b = -1$; $c = 0$; $e = 2$

$$B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix}$$

3. MINIMIZE $f(x_1, x_2) = x_1^2 + x_2^2 - x_1x_2$

$\nabla f(x_1, x_2) = [2x_1 - x_2 \quad 2x_2 - x_1] = 0$; $x_2 = 2x_1$; $x_1 = 2x_2$

$x_2 = 2(2x_2) \Rightarrow x_2 = x_1 = 0$

$$Hf(x_1, x_2) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \; ; \; \Delta_1 = (Hf)_{11} \; ; \; \Delta_2 = |Hf|$$

$\Delta_1 = 2 > 0$; $\Delta_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 2^2 - 1 = 3 > 0$.

$\Rightarrow Hf(x_1, x_2)$ IS POSITIVE DEFINITE EVERYWHERE ON $\mathbb{R}^2$

THUS $(0, 0)$ IS A GLOBAL MINIMIZER

4. GLOBAL MINIMIZER OF $f(x, y) = e^{x-y} + e^{y-x} + e^{x^2}$

$$\nabla f(x, y) = \left[e^{x-y} - e^{y-x} + 2x e^{x^2}, \; -e^{x-y} + e^{y-x} \right]^T$$

$$Hf(x,y) = \begin{bmatrix} e^{x-y} + e^{y-x} + (4x^2+2)e^{x^2} & -e^{x-y} - e^{y-x} \\ -e^{x-y} - e^{y-x} & e^{x-y} + e^{y-x} \end{bmatrix}$$

i) $\Delta_1 = e^{x-y} + e^{y-x} + (4x^2+2)e^{x^2} > 0 \quad \forall\, x, y$

BECAUSE ALL TERMS ARE POSITIVE.

ii) $\Delta_2 = \det H(f(x,y))$; $e^{x-y} + e^{y-x} \triangleq a$; $(4x^2+2)e^{x^2} \triangleq b$

$$\Delta_2 = \begin{vmatrix} a+b & -a \\ -a & a \end{vmatrix} = a(a+b) - a^2 = ab > 0 \quad \text{AS } a, b > 0$$

$\therefore Hf(x,y)$ IS POSITIVE DEFINITE $\forall\, x, y, z$

$\therefore f(x,y)$ IS GLOBALLY MINIMIZED BY LETTING $\nabla f(x,y) = 0$

$\Rightarrow -e^{x-y} + e^{y-x} = 0$ AND $e^{x-y} - e^{y-x} + 2xe^{x^2} = 0$

$\therefore e^{x-y} = e^{y-x}$, AND $2xe^{x^2} = 0 \Rightarrow x = 0$; $y = x = 0$

$\therefore (x^*, y^*) = (0,0)$ IS A GLOBAL MINIMIZER

5. MIN $10x + 2y^2 + z^2 + 8z$

SUBJECT TO $(x+y+z) = 100$

SOLUTION: $L = 10x + 2y^2 + z^2 + 8z + \lambda(100 - x - y - z)$

$$\frac{\partial L}{\partial x} = 10 - \lambda = 0$$

$$\frac{\partial L}{\partial y} = 4y - \lambda = 0$$

$$\frac{\partial L}{\partial z} = 2z + 8 - \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 100 - x - y - z = 0$$

$\lambda = 10$; $y = 10/4 = 2{\cdot}5$;

$2z + 8 - 10 = 0 \Rightarrow z = 1$

$100 - x - 2{\cdot}5 - 1 = 0 \Rightarrow x = 96{\cdot}5$

$(x, y, z) = (96{\cdot}5, 2{\cdot}5, 1)$