Today!

Choose one:
1) Snakes + Mayer-Vietoris
2) Persistence

Last time:

[for now, finite dimensional vector spaces]

A chain complex (of vector spaces) is a sequence of vec. spaces + linear maps

$$\cdots \leftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \leftarrow \cdots$$

such that (for all n...)

$\partial^2 = 0$ $(\partial_n \circ \partial_{n+1} = 0)$

equivalently,

$\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$

A chain complex is called exact if $\operatorname{im} \partial_n = \ker \partial_n$ for all n.

The homology of a chain complex is

$$H_n = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$

for $n = 0, 1, \ldots$

A short exact sequence is an exact chain complex w/ 5 elements, the first and last being 0.

Observation: if $C_\bullet$ is exact, then $H_\bullet = 0$

If $\partial_n = 0$ for all n, $H_\bullet = C_\bullet$

$$0 \xrightarrow{s} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{z} 0$$

1) exactness means that $0 = \operatorname{im} s = \ker f$
so f is injective

2) But, $\operatorname{im}(g) = \ker z = C$
so g is surjective

3) Because f is injective,
$\ker g = \operatorname{im} f \cong A$
By 1st isomorphism thm,
$\operatorname{im} g \cong B/\ker g$
$\operatorname{im} g = C$, $C \cong B/A$

For vector spaces, this implies
$\dim(A) + \dim(C) = \dim(B)$
i.e. $A \oplus C \cong B$

In groups, it only tells you $B \cong A \rtimes C$

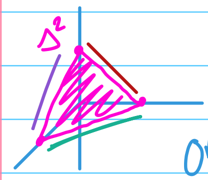
[now, infinite is fair game!]

Reminder: A singular n-simplex in X is a map $\Delta^n \to X$

$C_n^{sing}(X)$ = vector space whose basis is singular n-simp. at X

This is ghastly. Typically, X is uncountable, so $\{\Delta^n \to X\}$ is uncountable, so $C_n^{sing}(X)$ is uncountably ∞-dimensional.

The boundary faces of a simplex



$\{(t_0,t_1,t_2) \mid {t_i \geq 0 \atop \sum t_i = 1}\}$

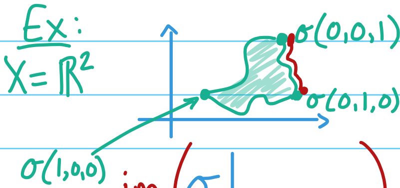
0^{th} boundary face of Δ^2 is
$\{(0,t_1,t_2) \mid {t_1+t_2=1 \atop 0 \leq t_i}\}$

1st boundary face:
$(t_0, 0, t_2)$

2nd: $(t_0, t_1, 0)$

Note: each boundary face "is" Δ^1
So, let $\sigma : \Delta^2 \to X$

Ex: $X = \mathbb{R}^2$



$\text{im}\left(\sigma|_{0^{th} \text{ boundary face of } \Delta^2}\right)$

$\sigma : \Delta^2 \to X$
$\sigma|_A : A \to X$

notice: $\Delta^1 = \{(s_0,s_1) \mid {s_0+s_1=1 \atop 0 \leq s_i}\}$

define $f_0(s_0,s_1) := (0,s_0,s_1)$

$f_0 : \Delta^1 \to \Delta^2$ $\quad \text{im}(f_0) = 0^{th}$ boundary face

So, we can really say:

$\sigma|_{0^{th} \text{ boundary face}} \circ f_0 : \Delta^1 \to X$ In other words, $\sigma|_{0^{th} \text{ boundary face}}$ can be thought of as a singular 1-simplex.
(explicitly, $\sigma \circ f_0$ is a singular 1-simplex)

Punchline:

define: $f_i : \Delta^{n-1} \to \Delta^n$
$(s_0, \dots, s_{n-1}) \to (s_0, \dots, \underset{\text{ith coordinate}}{0}, \dots, s_n)$

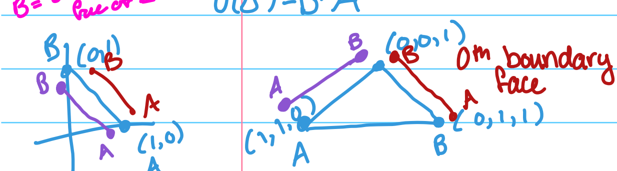
This is the ith boundary inclusion of Δ^n.
If $\sigma : \Delta^n \to X$ is a singular n-simplex, then
$\sigma \circ f_i : \Delta^{n-1} \to X$ is a singular $(n-1)$-simplex.

$$\partial([\sigma]) = \sum_{i=0}^{n} (-1)^i [\sigma \circ f_i]$$

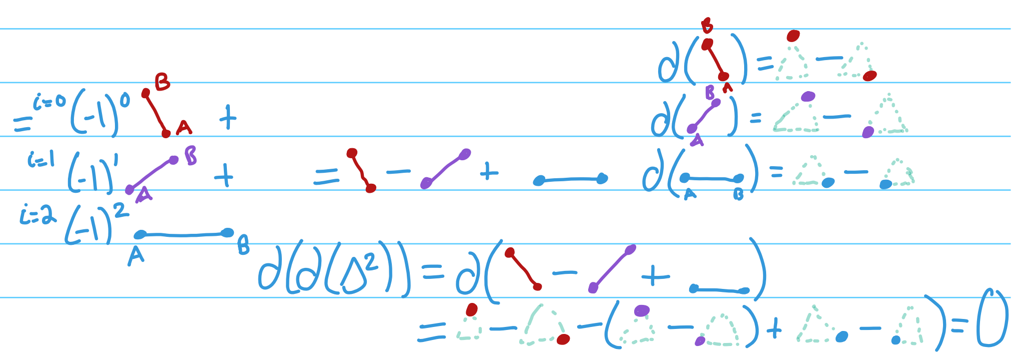
notation: $[\sigma]$ is the vector in C_n corresponding to the function σ.

$B = 0^{th}$ boundary face of Δ^1

$\partial(\Delta^1) = B - A$



$$\partial([\Delta^2]) = \sum_{i=0}^{2} (-1)^i \sigma \circ f_i$$



Lemma:
∂ from topology really is a chain map, meaning $\operatorname{im}\partial \subseteq \ker\partial$

Pf: :) exercise.

Let's prove some Theorems!

$$\begin{array}{ll} \underline{\text{Top}} & X \longrightarrow Y \\ & \quad \Downarrow \\ \underline{\text{Vec}} & H_k^{\text{sing}}(X) \to H_k^{\text{sing}}(Y) \ \forall k \end{array}$$

Question: $X \xrightarrow{f} Y$

idea: $\Delta^n \to X \to Y$ gives us $\Delta^n \to Y$
so $C_n(X) \to C_n(Y)$

$$\begin{tikzcd} \cdots \arrow[r] & C_n(X) \arrow[r] \arrow[d, "f^\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f^\#"] & \cdots \\ \cdots \arrow[r] & C_n(Y) \arrow[r, "\partial_n"] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

Does this commute?
meaning: $f^\# \circ \partial_n \overset{?}{=} \partial_n \circ f^\#$
Answer: Yes.

What we really need to verify, though:
I claim that this gets us $f^*: H_n(X) \to H_n(Y)$
so we should check...

$$\frac{\ker \partial_n^{(X)}}{\operatorname{im} \partial_{n+1}^{(X)}} \xrightarrow{?} \frac{\ker \partial_n^{(Y)}}{\operatorname{im} \partial_{n+1}^{(Y)}}$$

Answer: Yes
Pf: Exercise