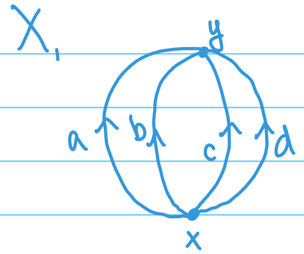
Today!

Singular homology is a functor



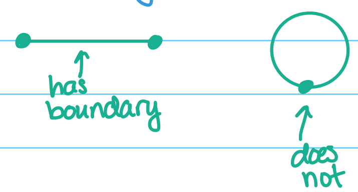
example: a - b (up a, down b)

chain groups (vector spaces):

$C_0(X_1) = \mathbb{R}^{\{x,y\}} \ (= \mathbb{R}^2)$

$C_1(X_1) = \mathbb{R}^{\{a,b,c,d\}} \ (= \mathbb{R}^4)$

Geometrically, a cycle is something with no boundary.



define $\partial_1 : C_1 \to C_0$

The first boundary map:

$a \mapsto y - x$

$b \mapsto y - x$

$c \mapsto y - x$

$d \mapsto y - x$

In matrix form:

$$\partial_1 = \begin{array}{c} \\ x \\ y \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

1-cycles correspond to $\ker(\partial_1)$

$$= \mathrm{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}\right)$$

0-boundaries are $\mathrm{im}(\partial_1) = \mathrm{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \subseteq C_0(X_1)$

When C_k is not defined, treat it as 0...

that is,

$C_{-1}(X_1) = 0$

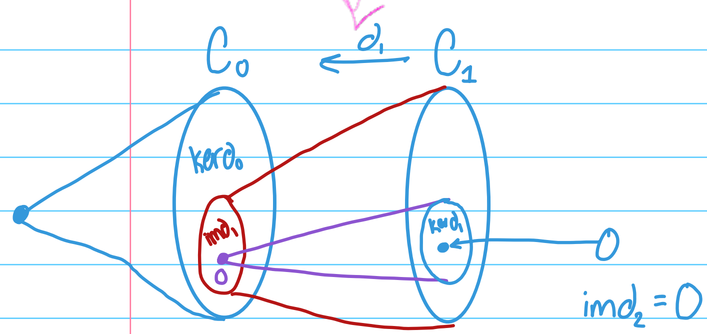
$C_2(X_1) = 0$ so we get the chain complex

$$0 \xleftarrow{\partial_0} C_0(X_1) \xleftarrow{\partial_1} C_1(X_1) \xleftarrow{\partial_2} 0$$

(so there are maps ∂_0, ∂_2 but they're both 0 maps)

$\ker(\partial_0) = \mathbb{R}^2$

$\mathrm{im}\, \partial_2 = 0$



define

$$H_0(X_1) \cong \mathbb{R} = \frac{\ker \partial_0}{\mathrm{im}\, \partial_1} = \frac{\mathbb{R}^2}{\mathbb{R}} \cong \mathbb{R}$$

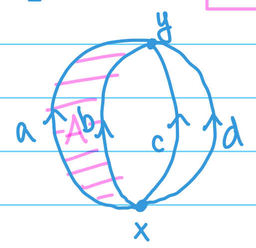
$$H_1(X_1) \cong \mathbb{R}^3 = \frac{\ker \partial_1}{\mathrm{im}\, \partial_2} = \ker \partial_1$$

$\dim H_1$ counts cycles... what does $\dim H_0$ say?

$C_\bullet(X): 0 \leftarrow C_0(X) \leftarrow C_1(X) \leftarrow C_2(X) \leftarrow \dots$

$C_\bullet(X_1): 0 \leftarrow \overset{x,y}{\mathbb{R}^2} \leftarrow \overset{a,b,c,d}{\mathbb{R}^4} \leftarrow 0$

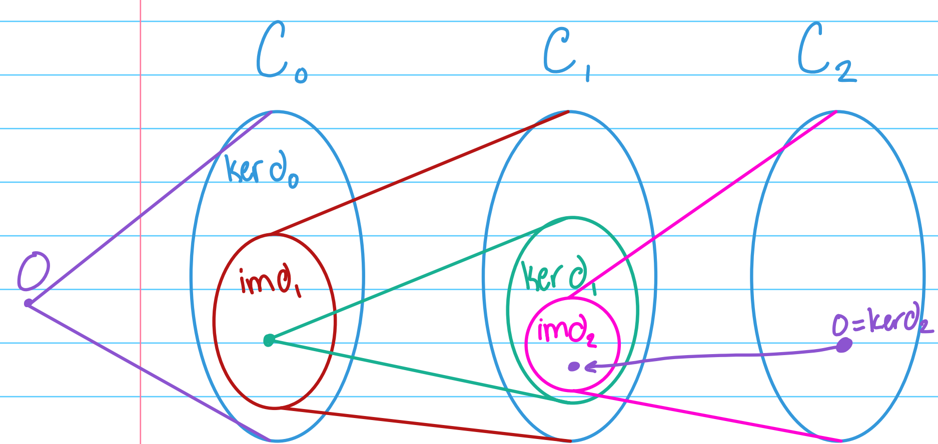
X_2: add in $\boxed{\text{2-cell } A}$. $C_\bullet(X_2): 0 \leftarrow \overset{x,y}{\mathbb{R}^2} \leftarrow \overset{abcd}{\mathbb{R}^4} \leftarrow \overset{\boxed{A}}{\mathbb{R}}$



For X_2, $\partial_1 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ same as X_1

But we now have $\partial_2 : C_2 \to C_1$

$\partial_2(A) = a - b \quad \partial_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$



$\ker \partial_0 = \mathbb{R}^2$ $\quad$ $\ker \partial_1 = \mathbb{R}^3$ $\quad$ $\ker \partial_2 = 0$

$\operatorname{im} \partial_1 = \mathbb{R}^4$ $\quad$ $\operatorname{im} \partial_2 = \mathbb{R}^1$ $\quad$ $\operatorname{im} \partial_3 = 0$

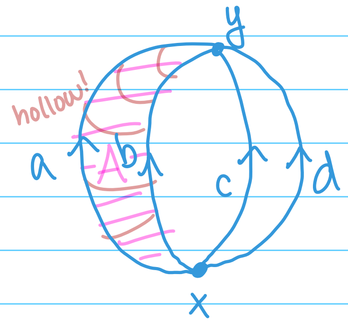
$\dfrac{\ker \partial_0}{\operatorname{im} \partial_1} = \mathbb{R} \qquad \dfrac{\ker \partial_1}{\operatorname{im} \partial_2} = \mathbb{R}^2 \qquad \dfrac{\ker \partial_2}{\operatorname{im} \partial_3} = 0$

$H_0(X_2) \cong \mathbb{R} \qquad H_1(X_2) \cong \mathbb{R}^2 \;\; \substack{b-c \\ c-d} \qquad H_2(X_2) = 0$

Compare:

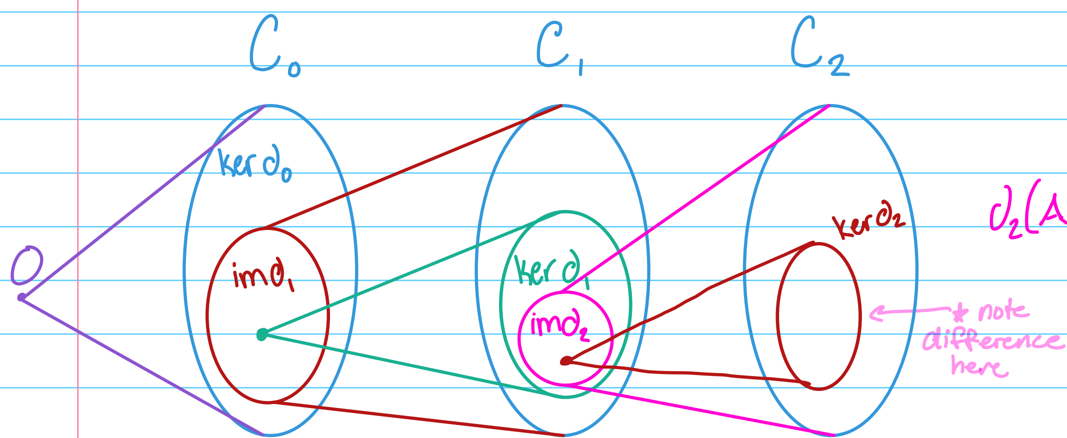
$H_0(X_1) = \mathbb{R} \qquad H_1(X_1) = \underset{\substack{a-b \\ b-c \\ c-d}}{\mathbb{R}^3}$

X_3: add in $\boxed{\text{2-cell } B}$ (along a,b just like A)



$$C_\bullet(X_3): 0 \leftarrow \overset{x\,y}{\mathbb{R}^2} \leftarrow \overset{a\,b\,c\,d}{\mathbb{R}^4} \leftarrow \overset{A\,\boxed{B}}{\mathbb{R}^2}$$

$$\partial_2(B) = a-b \qquad \partial_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$



$$\partial_2(A-B) = (a-b)-(a-b) = 0$$

$\ker\partial_0 = \mathbb{R}^2$ $\qquad$ $\ker\partial_1 = \mathbb{R}^3$ $\qquad$ $\ker\partial_2 = \cancel{0}\ \mathbb{R}^1$

$\operatorname{im}\partial_1 = \mathbb{R}^4$ $\qquad$ $\operatorname{im}\partial_2 = \mathbb{R}^1$ $\qquad$ $\operatorname{im}\partial_3 = 0$

$$\frac{\ker\partial_0}{\operatorname{im}\partial_1} = \mathbb{R} \qquad \frac{\ker\partial_1}{\operatorname{im}\partial_2} = \mathbb{R}^2 \qquad \frac{\ker\partial_2}{\operatorname{im}\partial_3} = 0$$

$$H_0(X_2) \cong \mathbb{R} \qquad H_1(X_2) \cong \mathbb{R}^2 \ \begin{smallmatrix} b-c \\ c-d \end{smallmatrix} \qquad H_2(X_2) = 0 \qquad H_3(X_3) \cong \mathbb{R}^1$$

Compare:

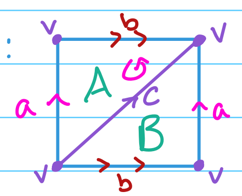
$$H_0(X_1) = \mathbb{R} \qquad H_1(X_1) = \underset{\begin{smallmatrix} a-b \\ b-c \\ c-d \end{smallmatrix}}{\mathbb{R}^3}$$

Ex

$H_n(S^n) \cong \mathbb{R}$ (what about lower dim? TBC'd)

Ex

Torus:



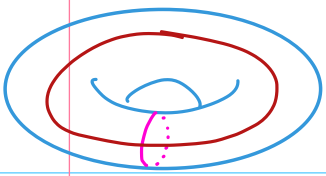
$$C_\bullet(T): 0 \leftarrow C_0(T) \leftarrow C_1(T) \leftarrow C_2(T) \leftarrow 0$$

$$0 \leftarrow \overset{v}{\mathbb{R}} \xleftarrow{\partial_1} \overset{a\,b\,c}{\mathbb{R}^3} \xleftarrow{\partial_2} \overset{A,B}{\mathbb{R}^2} \leftarrow 0$$

$\partial_1: [0\ 0\ 0]$

$$\partial_2: \begin{array}{c c} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} -1 & 1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} \end{array}$$

$\partial_2(A) = -a + c - b$



$\ker \partial_0 = \mathbb{R}$ $\quad$ $\ker \partial_1 = \mathbb{R}^3$ $\quad$ $\ker \partial_2 = \mathbb{R}$

$\operatorname{im} \partial_1 = 0$ $\quad$ $\operatorname{im} \partial_2 = \mathbb{R}$ $\quad$ $\operatorname{im} \partial_3 = 0$

$H_0(T) = \mathbb{R}^1$ $\quad$ $H_1(T) = \mathbb{R}^2$ $\quad$ $H_2(T) = \mathbb{R}$

Def: A *chain complex of vector spaces* is

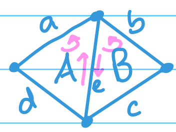
$$\cdots \leftarrow C_{n-1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \leftarrow \cdots$$

such that

$$\partial^2 = 0 \qquad (\partial_n \circ \partial_{n+1} = 0 \;\forall n)$$

Note: This means $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$

The homology of a chain cmplx is $\ker/\operatorname{im}$



Intuitively, $\partial(A+B)$ ought to be $a+b+c+d$.
But $\partial(A) = a+e+d$
$\partial(B) = b+e+c$
and we want ∂ to be linear.

But then we'd get
$\partial(A+B) = a+b+c+d+2e$

Sol'n: introduce "orientation" (or order) in a way that makes e's cancel

Def: A *singular n-simplex* of a space X is a map $\Delta^n \to X$

$$\left(\Delta^n = \left\{(t_0, \ldots, t_n) \,\middle|\, \begin{matrix} \sum t_i = 1 \\ t_i \geq 0 \;\forall i \end{matrix}\right\}\right)$$

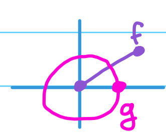
Ex: $\Delta^0 = \{1\}$, so any function $\Delta^0 \to X$ just picks a point out in X.

So, the set of singular 0-simplices of X is basically just X.

Ex: $\Delta^1 = \{(t, 1-t) \mid 0 \leq t \leq 1\} \subseteq \mathbb{R}^2$
A 1-simplex in X is a curve,
that is, Δ^1 is basically just $[0,1]$ an $f : [0,1] \to X$ is a curve.

Ex: $f : \Delta^1 \to \mathbb{R}^2$
$(t, 1-t) \longmapsto (t, t)$

$g(t, 1-t) = (\cos(2\pi t), \sin(2\pi t))$



Def: The n-singular chain group of X are the (really ∞-dimensional) vector spaces whose basis is the n-simplices of X.

(Note: in our examples so far, the simplicial chain group is a finite-dimensional subspace of the singular chain gp)