Today!
- Simplices
- Δ-complexes
- (simplicial) Chains, cycles, boundaries
- Homology!

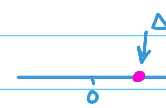
Reminder
- Linear Algebra
- Topology

Defn: The standard n-simplex is the set of points $\Delta^n = \{(t_0, \dots, t_n) \mid \sum_{i=1} t_i = 1, \text{each } t_i \geq 0\}$ in $\mathbb{R}^{n+1}$.

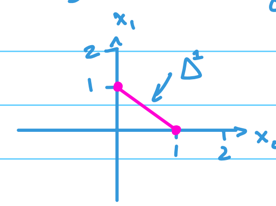
- Simplices are in some sense the "simplest" example of an n-dimensional object

Ex: $n=0$:

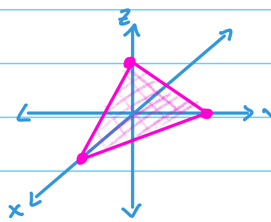
$\Delta^0 = \{t_0 \mid \sum t_i = 1\} = \{1\} \subseteq \mathbb{R}^1$



$\Delta^1 = \{(t_0, t_1) \mid \sum t_i = 1, t_i \geq 0\}$



$\Delta^2 = \{(t_0, t_1, t_2) \mid \sum t_i = 1, t_i \geq 0\}$



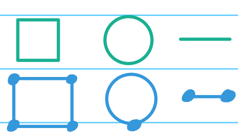
$\Delta^3 = \{(t_0, t_1, t_2, t_3) \mid \sum t_i = 1, t_i \geq 0\}$;
regular tetrahedron in $\mathbb{R}^4$
(filled in)

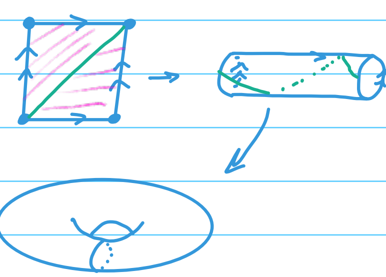
Building topological spaces from simplices

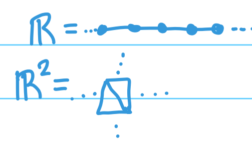
Informally: take a bunch of copies of simplices of various dimensions and "glue them together"

Rules: faces glue to faces at same dimension

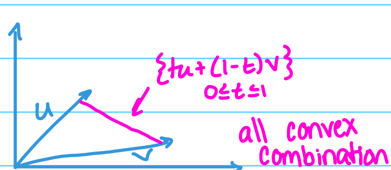
Ex:





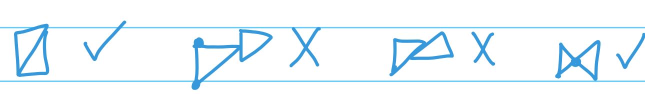


How many faces on Δ^n?
It's pascal's triangle!



... so the line is the convex hull of u and v ... so Δ^1 is convex hull of (1,0) and (0,1)

Gluing Rules:



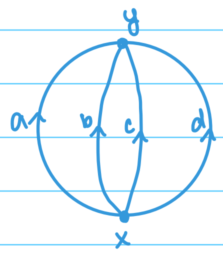
also allowed:

Question: How do we tell the difference between T^2 and S^2?

Ans: (linear) Algebra!

A first example of homology



This is a graph: 2v, 4e

How many cycles does this graph have?

Arrows are for tracking walks from either $x \to y$ or $y \to x$.

...whats a cycle? Parker says 0, 1, or 4.

Ex of cycles:

in $C^{-1} = b - c$ or $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

	bc^{-1}	
$a-b$	ab^{-1}	
$-a+b$	$a^{-1}b$	
$a-b+c-d$	$ab^{-1}cd^{-1}$	a, b, c, d
	0	don't go anywhere

$(a-b)+(c-d)$

$(a-d)+(c-b) = (a-d)-(b-c)$

So let's define...

$C_1(X) = \mathbb{R}^{\{a,b,c,d\}}$

(ie, $\mathbb{R}^4$)

Geometrically, a cycle is a sequence of edges such that they make a path that begins and ends at the same vertex

$a-b+c-d+d-b = a-2b+c$

$= a-b+(-b+c)$

Every geometric cycle can be written as a vector, but not vice versa

ex: $a+b$

So, C_1 is "too big". it has things in which are not cycles $\Rightarrow$ so lets define a **chain** as any linear combo of a, b, c, d (that is, $C_1(X)$ is the 1-chains of X)

A chain is a **cycle** if it is a linear combo of some geometric cycle.

Claim: This is "correct definition". In our example, any geom cycle can be written as linear combos of $(a-b), (b-c), (c-d)$*

* non-unique choice of small cycles

Geometrically, a cycle is a set without boundary.

So what does that mean???

$C_0(X) = \mathbb{R}^{\{x,y\}} = \mathbb{R}^2$

$C_1(X) = \mathbb{R}^{\{a,b,c,d\}} = \mathbb{R}^4$

we know what the boundary of a,b,c,d are...

$$C_1(X) \xrightarrow{\partial_1} C_0(X)$$

$a \longmapsto -x+y$

$b \longmapsto -x+y$

$c \longmapsto -x+y$

$d \longmapsto -x+y$

$$\partial_1 = \begin{bmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0-1+1-0 \\ 0+1-1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$