Today!

Preliminaries

Singular Homology?

Preliminaries

For us: Vector spaces are $\mathbb{R}^n$ (specified if ∞)

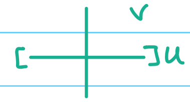
$\hookrightarrow V$ is a set of vectors that can be added, sub, and scaled by $\mathbb{R}^1$

eg: $\mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} x,y,z \in \mathbb{R} \right\}$

Subspaces

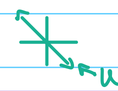
subset which is a subspace if it is also a subspace

eg: $V = \mathbb{R}^2$, $U = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$



- $V = \mathbb{R}^3$, $U = \{x+y+z=0\}$

  in $\mathbb{R}^2$, $U = \{x+y=0\} \Rightarrow$



Linear Transformation

$T : V \to W$ $(\mathbb{R}^n \to \mathbb{R}^m)$

$T(\alpha\vec{x} + \vec{y}) = \alpha T(\vec{x}) + T(\vec{y})$

Typically, T can be written as a matrix; $T = {}^{m}\overset{n}{\begin{bmatrix} & & \end{bmatrix}}$

(i.e., $T(v) = Tv$)

image: $\operatorname{im}(T) = \{Tv \mid v \in V\} \subseteq W$ (column space)

kernel: $\ker(T) = \{v \in V \mid Tv = 0\}$

Rank-nullity Theorem

$\dim(\operatorname{im} T) + \dim(\ker T) = \dim V$

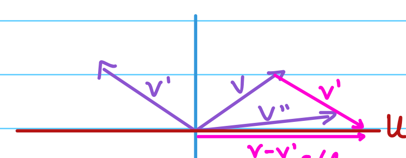
rank of T + nullity of T

Quotient Spaces

given $U \subseteq V$,

$V/U = \{[v] \mid [v] = [v'] \text{ if } v - v' \in U\}$

Ex: $V = \mathbb{R}^2$ $U = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

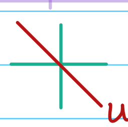


$v - v' \in U$, so $[v] = [v']$

$v - v'' \notin U$, so $[v] \neq [v'']$

so, $V/U = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \middle| y \in \mathbb{R} \right\} \cong \operatorname{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \cong \mathbb{R}$

Ex: $V = \mathbb{R}^2$, $U = \{x,y \mid x+y=0\}$



In V/U, eqv classes are lines parallel to $x+y=0$.

each class hits the y-axis at a unique point.

so, $\frac{V}{U} = \left\{ \left[\begin{bmatrix} 0 \\ y \end{bmatrix}\right] \right\}$ or $\left[\begin{bmatrix} x \\ 0 \end{bmatrix}\right]$ works! but,

$V/U = \left\{ \left[\begin{bmatrix} x \\ x \end{bmatrix}\right] \middle| x \in \mathbb{R} \right\}$

"Extend by linearity"

Given arbitrary set S, we'll denote $\mathbb{R}^S$ the vector space whose basis is S. ($\mathbb{R}^S \cong \mathbb{R}^{|S|}$)

Ex: $S=\{a,b,c\}$; $5a+3b-\sqrt{2}c$ same as $\begin{bmatrix} 5 \\ 3 \\ -\sqrt{2} \end{bmatrix}$

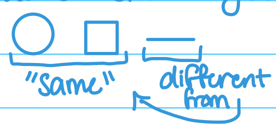
Important Point: any $f:S\to T$ extends to $f:\mathbb{R}^S\to\mathbb{R}^T$

"extend by linearity"

also, generally, to define a linear map, need only where the basis goes

Topology (Briefly)

"Rubber sheet geometry"



[Key Idea]: continuity.

Dramatis Personae:

- Topological Spaces (Tp Sp)
- Continuous Maps (cts)

Def: A topological space is a set X equipped w/ a topology $\mathcal{T}$

$\mathcal{T}$ is a collection of subsets of X such that:

1) $\emptyset, X \in \mathcal{T}$
2) arbitrary unions of sets in $\mathcal{T}$ are in $\mathcal{T}$
3) finite intersections of sets in $\mathcal{T}$ is in $\mathcal{T}$

Ex: $(\mathbb{R},\mathcal{T})$ where $\mathcal{T}$ is generated by $(a,b):\{x\in\mathbb{R} \mid a<x<b\}\ \forall a<b\in\mathbb{R}$

Is "usual" real numbers

$\mathcal{T}$ is called a topology on X or the open sets of X.

Def: A function $f:X\to Y$, really $(X,\mathcal{T})\to(Y,\mathcal{T}')$, is called continuous if the pre-image of each open set is open.

preImage? for $U\subseteq Y$

$f^{-1}(U)=\{x\in X \mid f(x)\in U\}\subseteq X$

Ex: $f(x)=x^2$

$f:\mathbb{R}\to\mathbb{R}$

$f^{-1}([0,1])=[-1,1]$

Defn: $X \cong Y$, X is homeomorphic to Y if there are continuous maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t. $f \circ g = id_Y$
$g \circ f = id_X$

Valid Maps	Equivalence	Subject
Continuous	homeomorphism	topology
linear transformation	(vector space) isomorphism	linear algebra
group homomorphism	group isomorphism	algebra
graph homomorphism	graph isomorphism	graph theory