# Source/Vortex Panel Method

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## 1 Introduction

The Source/Vortex Panel Method is an efficient way to compute the invsicid velocity and pressure distributions around airfoils. The following sections will review the theory behind this method and present the significant equations needed in a Matlab implementation. This report would not have been possible without the excellent code resources written by jte0419 on GitHub: `https://github.com/jte0419/Panel_Methods`.

## 2 Problem Setup

Place an airfoil in the xy-plane so that the leading edge is at the origin and the chord line is perfectly aligned with the positive x-axis. Then, split the airfoil into $N \approx 200$ panels that are oriented clockwise, starting from the trailing edge. This configuration is shown in Fig. 1.

Now pick $i \in \{1, 2, ..., N\}$, and consider panel $i$. Let $\mathbf{n}_i$ be the unit normal vector to panel $i$ that points away from the airfoil, and let $\mathbf{t}_i$ be the unit tangent vector along panel $i$ that points toward panel $i + 1$. These vectors are placed at the center of the panel, denoted by $(x_i, y_i)$. Also define $\delta_i$ to be the angle from the positive x-axis to $\mathbf{n}_i$, and $\beta_i = \delta_i - \alpha$ to be the angle from the free stream velocity $U$ to $\mathbf{n}_i$. This geometry is shown in Fig. 2.

With these angles defined, put a source sheet on panel $i$ with uniform strength $\lambda_i$ and a vortex sheet on panel $i$ with uniform strength $\gamma$. While the source strengths will be allowed to change from panel to panel, all vortex sheets will have the same strength $\gamma$. This leaves us with $N + 1$ unknowns: $\lambda_1, \lambda_2, ..., \lambda_N, \gamma$. By carefully solving for the $\lambda_i$'s and $\gamma$, we can define the inviscid flow around the airfoil. The overall strategy of this process is displayed in Fig. 3.

## 3 System of Equations

To solve for the $N$ source strengths and the single vortex strength, let's consider the velocity potential $\phi$, whose gradient is the velocity field $\mathbf{u}(x, y)$. From theory outlined in [1], the velocity potential from

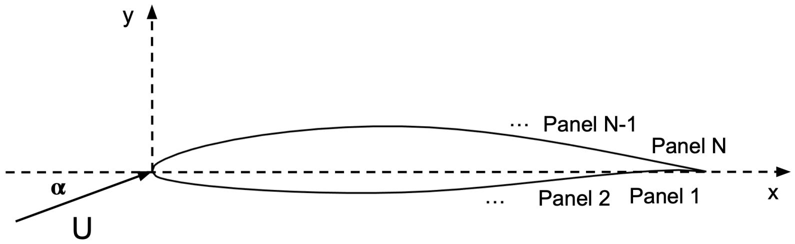


**Figure 1:** An airfoil is oriented so that its chord line coincides with the +x-axis, and the uniform free stream velocity $U$ approaches at angle $\alpha$.

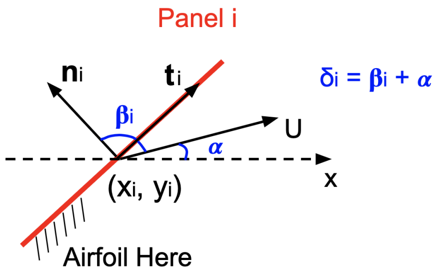


**Figure 2:** Relevant angles associated with panel $i$ in the Source/Vortex Panel Method.

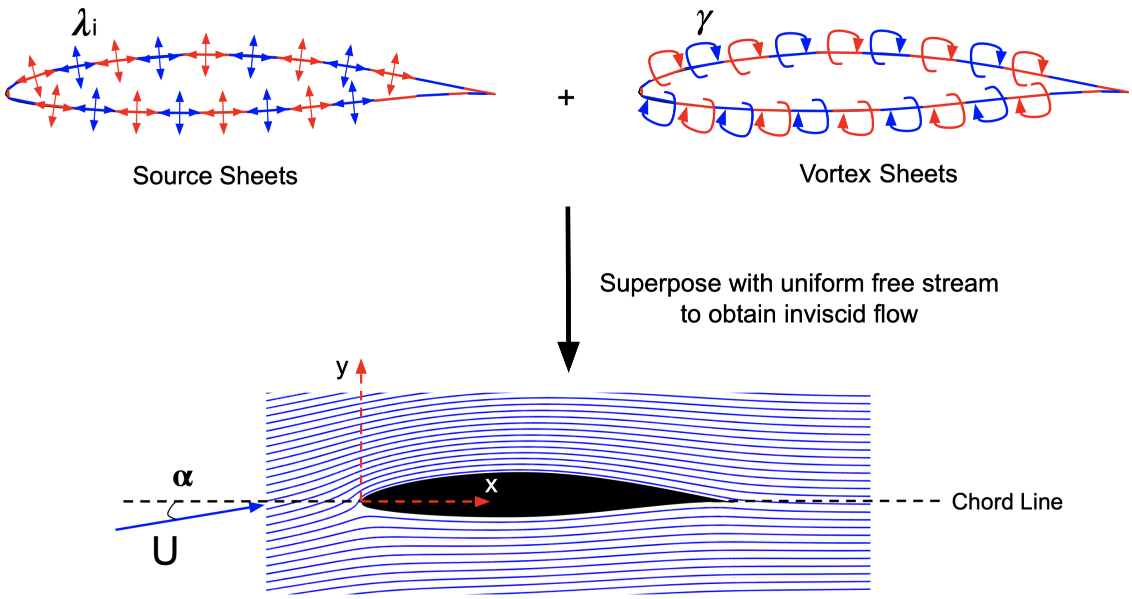


**Figure 3:** In the Source/Vortex Panel Method, an airfoil is discretized into $N$ panels. Along each of these panels, there is a source sheet of strength $\lambda_i$ and a vortex sheet of strength $\gamma$. If each $\lambda_i$ and $\gamma$ is chosen correctly, then they will induce an inviscid flow around the airfoil that matches real-world high Reynolds number flow. Individual panels are indicated by the blue and red lines that circle the entire airfoil. The alternating blue and red colors of the sources and vortices indicate that they are part of different source and vortex sheets.

the uniform flow $U$ at a point $(x_P, y_P)$ is

$$\phi_{\text{uniform}}(x_P, y_P) = U\cos(\alpha)\, x_P + U\sin(\alpha)\, y_P. \tag{1}$$

Meanwhile, the velocity potential at a point $(x_P, y_P)$ from the vortex sheet on panel $j$ is

$$\phi_{\text{vortex},\, j}(x_P, y_P) = -\frac{\gamma}{2\pi}\int_{\text{panel j}} \theta_{P,\, j}\, ds_j, \tag{2}$$

where $\theta_{P,j}$ is the angle between a line parallel to the x-axis passing through $(x_j, y_j)$ and a line passing through $(x_j, y_j)$ and $(x_P, y_P)$. Similarly, the velocity potential at a point $(x_P, y_P)$ due to the source sheet on panel $j \in \{1, 2..., N\}$ is given by

$$\phi_{\text{source},\, j}(x_P, y_P) = \frac{\lambda_j}{2\pi}\int_{\text{panel j}} \ln(r_{P,\, j})\, ds_j, \tag{3}$$

where $r_{P,j}$ is the distance from the center of panel $j$ to the point $(x_P, y_P)$. Since velocity potentials are governed by linear differential equations, the potential $\phi$ for the entire flow at an arbitrary point $(x_P, y_P)$ is the superposition of constituent potentials:

$$\phi(x_P, y_P) = \underbrace{U\text{cos}(\alpha)\, x_P + U\text{sin}(\alpha)\, y_P}_{\substack{\text{influence of uniform flow}\\ \text{on point } (x_P, y_P)}} + \underbrace{\sum_{j=1}^{N} \frac{\lambda_j}{2\pi}\int_{\text{panel j}} \ln(r_{P,j})\, ds_j}_{\substack{\text{influence of all source sheets}\\ \text{on point } (x_P, y_P)}} + \underbrace{\sum_{j=1}^{N} -\frac{\gamma}{2\pi}\int_{\text{panel j}} \theta_{P,j}\, ds_j}_{\substack{\text{influence of all vortex sheets}\\ \text{on point } (x_P, y_P)}}. \tag{4}$$

Now, if we take $(x_P, y_P)$ to be the center point of panel $i$, denoted by $(x_i, y_i)$, then we can write

$$\phi(x_i, y_i) = U\text{cos}(\alpha)\, x_i + U\text{sin}(\alpha)\, y_i + \sum_{j=1}^{N} \frac{\lambda_j}{2\pi}\int_{\text{panel j}} \ln(r_{i,j})\, ds_j + \sum_{j=1}^{N} -\frac{\gamma}{2\pi}\int_{\text{panel j}} \theta_{i,j}\, ds_j, \tag{5}$$

where $r_{i,j}$ is the distance from the center of panel $j$ to the center of panel $i$ and $\theta_{i,j}$ is the angle between the normal vector of panel $j$ and the normal vector of panel $i$. From the no-penetration boundary condition, fluid cannot pass through the airfoil. Therefore, we need the component of velocity normal to each panel $i$ to be zero [1]. This means that the derivative of $\phi$ in the direction of the unit normal $\mathbf{n}_i$ to panel $i$ has to vanish for all $N$ panels. After a little algebra, this amounts to writing

$$\underbrace{\nabla\phi \cdot \mathbf{n}_i|_{(x_i, y_i)}}_{\text{normal velocity}} = U\text{cos}(\beta_i) + \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \underbrace{\int_{\text{panel j}} \frac{\partial}{\partial \mathbf{n}_i} \ln(r_{i,j})\, ds_j}_{I_{i,j}} + \sum_{j=1}^{N} -\frac{\gamma}{2\pi} \underbrace{\int_{\text{panel j}} \frac{\partial}{\partial \mathbf{n}_i} \theta_{i,j}\, ds_j}_{K_{i,j}} = 0, \tag{6}$$

where $\beta_i$ is the angle between the free stream velocity and the normal vector to panel $i$, as shown in Fig. 2. Using formulae derived here and here, we can define and compute the quantities

$$I_{i,j} = \begin{cases} \int_{\text{panel j}} \frac{\partial}{\partial \mathbf{n}_i} \ln(r_{i,j})\, ds_j & i \neq j \\ \\ \pi & i = j \end{cases} \tag{7}$$

and

$$K_{i,j} = \begin{cases} \int_{\text{panel j}} \frac{\partial}{\partial \mathbf{n}_i} \theta_{i,j}\, ds_j & i \neq j \\ \\ 0 & i = j. \end{cases} \tag{8}$$

The $I_{i,j}$ and $K_{i,j}$ terms are called geometric integrals because they depend only on the geometry of the panels, and indicate how source and vortex sheets on one panel influence the velocity potential at another panel. With $I_{i,j}$ and $K_{i,j}$ defined, we can re-express the no-penetration condition on panel $i$ as

$$\sum_{j=1}^{N} \lambda_j I_{i,j} - \gamma \sum_{k=1}^{N} K_{i,j} = -2\pi\, U\text{cos}(\beta_i). \tag{9}$$

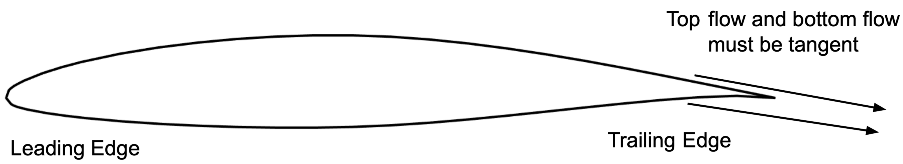


**Figure 4:** Diagram representing the Kutta condition, which requires flow from the upper surface of a wing to leave the trailing edge tangent to the flow from the lower surface of the wing.


**Figure 5:** Panel 1 is on the bottom surface of the wing below the trailing edge, and panel $N$ is on the top surface of the wing above the trailing edge.

Writing Eq. (9) for each of the $N$ panels gives $N$ equations in the $N+1$ unknowns $\lambda_1, ..., \lambda_N, \gamma$. While we have made progress, we still need one more equation to define the source strengths and the vortex strength.

To extract the last equation, we mandate that the flow coming off the trailing edge from the upper surface of the wing should be tangent to the flow coming off the trailing edge from the lower surface of the wing, as shown in Fig. 4. In aerodynamics, this is known as the Kutta condition, and it has produced consistently accurate results in inviscid flow theory. To write an equation for the Kutta condition, let's recall that panel 1 and panel $N$ are oriented as in Fig. 5, and that $\mathbf{t}_i$ is the unit tangent vector to panel $i$ that points in the direction of panel $i+1$. Then, the tangential component of velocity along panel 1 is

$$\nabla\phi \cdot \mathbf{t}_1|_{(x_1,y_1)} = U\sin(\beta_1) + \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} J_{1,j} + \sum_{j=1}^{N} -\frac{\gamma}{2\pi} L_{1,j}, \tag{10}$$

and the tangential component of velocity on panel N is

$$\nabla\phi \cdot \mathbf{t}_N|_{(x_N,y_N)} = U\sin(\beta_N) + \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} J_{N,j} + \sum_{j=1}^{N} -\frac{\gamma}{2\pi} L_{N,j}. \tag{11}$$

In these equations, $\beta_i$ is still the angle between the free stream velocity and the normal vector to panel $i$, and the geometric integrals $J_{i,j}$ and $L_{i,j}$ are defined to be

$$J_{i,j} = \begin{cases} \int_{\text{panel j}} \frac{\partial}{\partial \mathbf{t}_i} \ln(r_{i,j})\, ds_j & i \neq j \\ 0 & i = j \end{cases}, \tag{12}$$

$$L_{i,j} = \begin{cases} \int_{\text{panel j}} \frac{\partial}{\partial \mathbf{t}_i} \theta_{i,j}\, ds_j & i \neq j \\ -\pi & i = j \end{cases}. \tag{13}$$

A derivation of $J_{i,j}$ can be found here and a derivation of $L_{i,j}$ can be found here.

After setting

$$\underbrace{U\sin(\beta_1) + \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} J_{1,j} + \sum_{j=1}^{N} -\frac{\gamma}{2\pi} L_{1,j}}_{\text{tangential velocity along } \mathbf{t}_1} + \underbrace{U\sin(\beta_N) + \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} J_{N,j} + \sum_{j=1}^{N} -\frac{\gamma}{2\pi} L_{N,j}}_{\text{tangential velocity along } \mathbf{t}_N} = 0, \tag{14}$$

we can write a matrix equation to solve for the $N$ unknown source strengths and the 1 unknown vortex strength:

$$\begin{bmatrix} I_{1,1} & I_{1,2} & I_{1,3} & \cdots & I_{1,N} & -\sum_{j=1}^{N} K_{1,j} \\ I_{2,1} & I_{2,2} & I_{2,3} & \cdots & I_{2,N} & -\sum_{j=1}^{N} K_{2,j} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ I_{N,1} & I_{N,2} & I_{N,3} & \cdots & I_{N,N} & -\sum_{j=1}^{N} K_{N,j} \\ J_{1,1}+J_{N,1} & J_{1,2}+J_{N,2} & J_{1,3}+J_{N,3} & \cdots & J_{1,N}+J_{N,N} & -\sum_{j=1}^{N}(L_{1,j}+L_{N,j}) \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_N \\ \gamma \end{bmatrix} \tag{15}$$

$$= -2\pi U \begin{bmatrix} \cos(\beta_1) \\ \cos(\beta_2) \\ \cos(\beta_3) \\ \vdots \\ \cos(\beta_N) \\ \sin(\beta_1)+\sin(\beta_N) \end{bmatrix}.$$

# 4 Computing Streamlines

Once the source and vortex strengths are known, we can find the horizontal and vertical velocity at any point $(x_P, y_P)$ in the flow field by evaluating the gradient of the velocity potential $\phi$ given by Eq. (4). Specifically, the horizontal velocity $u$ is

$$u(x_P, y_P) = U\cos(\alpha) + \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \underbrace{\int_{\text{panel j}} \frac{\partial}{\partial x_P} \ln(r_{P,j})\, ds_j}_{M_{x,P,j}} + \sum_{j=1}^{N} -\frac{\gamma}{2\pi} \underbrace{\int_{\text{panel j}} \frac{\partial}{\partial x_P} \theta_{P,j}\, ds_j}_{N_{x,P,j}}, \tag{16}$$

and the vertical velocity $v$ is

$$v(x_P, y_P) = U\sin(\alpha) + \sum_{j=1}^{N} \frac{\lambda_j}{2\pi} \underbrace{\int_{\text{panel j}} \frac{\partial}{\partial y_P} \ln(r_{P,j})\, ds_j}_{M_{y,P,j}} + \sum_{j=1}^{N} -\frac{\gamma}{2\pi} \underbrace{\int_{\text{panel j}} \frac{\partial}{\partial y_P} \theta_{P,j}\, ds_j}_{N_{y,P,j}}. \tag{17}$$

Derivations of analytical formulae for $M_{x,P,j}$ and $M_{y,P,j}$ can be found here and analytical formulae for $N_{x,P,j}$ and $N_{y,P,j}$ can be found here.

# 5 Computing the Pressure Coefficient

After $u$ and $v$ are known at $(x_P, y_P)$, the pressure coefficient $c_p$ can be defined by

$$c_p(x_P, y_P) = 1 - \frac{u(x_P, y_P)^2 + v(x_P, y_P)^2}{U^2}. \tag{18}$$

This formula is derived from Bernoulli's equation.

## References

[1] J. D. Anderson. *Incompressible Inviscid Flows: Source and Vortex Panel Methods*, pages 52–74. Springer Berlin Heidelberg, Berlin, Heidelberg, 1992.