Implicit Formulae for Rebate Part of Double Barrier Option

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July 2019

1 Introduction

1.1 Problem Addressing

This is a brief presentation on the derivation and result of implicit (closed-form) pricing formulae of the rebate part of double barrier options as well as an exotic American cash-or-nothing binary option with two strike prices.

This report will derive the closed-form pricing formulae in a Fourier series form, by calculating the probability of hitting the lower and upper barrier with Laplace transform techniques and Cauchy's Residue Theorem.

Also by seeing double barrier options with rebate as a portfolio of zero-rebate double barrier together with its rebate part as an exotic American binary with two strike prices, we could easily achieve their implicit formulae as well.

2 Derivation

2.1 Fourier Series Representation of Density Function

By the assumption of geometric Brownian Motion:

$$\mathrm{d}z = \mu \mathrm{d}t + \sigma \mathrm{d}W, \tag{1}$$

where μ and σ are constants.

Then suppose there are two absorbing barriers without loss of generality:

$$\begin{cases} lower\ barrier\ 0 \\ upper\ barrier\ \ell \end{cases}$$

Then consider a $p(t,x;s,y)$ as the probability of avoiding the barrier from time t at $z(t)=x$ to time s at $z(s)=y$, where $t \leq s\ and\ 0 \leq x,y \leq \ell$.

Thereafter we attain the Kolmogorov backward and forward equations of the transition density function $p(t, x; s, y)$ as:

$$\begin{cases} p_t + \mu p_x + \frac{1}{2}\sigma^2 p_{xx} = 0 \\ -p_s - \mu p_y + \frac{1}{2}\sigma^2 p_{yy} = 0 \end{cases} \tag{2}$$

subject to:

$$\begin{cases} p(t, 0; *, *) = p(t, \ell; *, *) = 0,\ p(s, x; s, y) = \delta(x - y) \\ p(*, *; s, 0) = p(*, *; s, \ell) = 0,\ p(t, x; t, y) = \delta(x - y) \end{cases} \tag{3}$$

Where $\begin{cases} p_x = \dfrac{\mathrm{d}p}{\mathrm{d}x} \\ p_{xx} = \dfrac{\mathrm{d}^2 p}{\mathrm{d}x^2} \end{cases}$, and δ is the Dirac delta function.

To achieve an analytical form of $p(t, x; s, y)$, we represent it in a Fourier series according to Cox and Miller, Theory of Stochastic Processes (1965) Chapter 5.7, as:

$$\begin{aligned} p(t, x; s, y) =& e^{\frac{\mu}{\sigma^2}(y-x)} \frac{2}{\ell} \sum_{k=1}^{\infty} e^{-\lambda_k (s-t)} sin(k\pi \frac{x}{\ell}) sin(k\pi \frac{y}{\ell}) \\ ,\ where\ \lambda_k =& \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{k^2\pi^2\sigma^2}{\ell^2}\right). \end{aligned} \tag{4}$$

Substituting Kolmogorov equations (2) back into it can help to check its correctness easily. Then we seek to find an analytical solution of equation (4) term-wisely. In fact, this infinite series converges in a very fast manner, which will be discussed later.

2.2 Hitting Probability Density Functions

Let $g^+(t, x; s)$ denote the probability density function of the process starting at (t,x), hitting upper barrier for the first time while not hitting the lower one; $g^+(t, x; s)$ denotes the opposite.

Therefore $\forall\ t < T$,

$$\int_t^T g^+(t, x; s)\mathrm{d}s + \int_t^T g^-(t, x; s)\mathrm{d}s + \int_0^\ell p(t, x; T, y)\mathrm{d}y \equiv 1. \tag{5}$$

Taking equation (5) into equation (4) we gain:

$$\begin{aligned} g^+(t, x; s) + g^-(t, x; s) = & e^{\frac{\mu}{\sigma^2}(\ell - x)} \frac{\sigma^2}{\ell^2} \sum_{k=1}^{\infty} e^{-\lambda_k (s-t)} k\pi sin(k\pi \frac{\ell - x}{\ell}) \\ & + e^{-\frac{\mu}{\sigma^2}x} \frac{\sigma^2}{\ell^2} \sum_{k=1}^{\infty} e^{-\lambda_k (s-t)} k\pi sin(k\pi \frac{x}{\ell}). \end{aligned} \tag{6}$$

Set $\tau = s-t$, we have $g^+(t,x;s) = g^+(\tau,x)$, then we construct the kolmogorov backward equation of g^+ as a PDE:

$$-g_\tau^+ + \mu g_x^+ + \frac{1}{2}\sigma^2 g_{xx}^+ = 0, \tag{7}$$

subject to:

$$g^+(\tau,\ell) = \delta(\tau),\ g^+(0,x) = \delta(\ell - x)\ and\ g^+(\tau,0) = 0. \tag{8}$$

2.3 Solution to this PDE by Laplace Transformation

Consider its Laplace Transformation $\gamma^+(x)$ to solve $g^+(\tau,x)$:

$$\gamma^+(x;v) = \int_0^\infty e^{-v\tau} g^+(\tau,x)\mathrm{d}\tau, \quad \forall\ v \geq 0. \tag{9}$$

Therefore,

$$-v\gamma^+ + \mu\gamma_x^+ + \frac{1}{2}\sigma^2\gamma_{xx}^+ = 0, \tag{10}$$

subject to:

$$\gamma^+(0) = 0\ and\ \gamma^+(\ell) = 1. \tag{11}$$

Solve to gain:

$$\begin{gathered}\gamma^+(x;v) = e^{\frac{\mu}{\sigma^2}(\ell - x)}\frac{sinh(\theta(v)x)}{sinh(\theta(v)\ell)},\\ where\ \theta(v) = \frac{1}{\sigma^2}\sqrt{\mu^2 + 2\sigma^2 v}.\end{gathered} \tag{12}$$

Then invert its Laplace Transformation:

$$g^+(\tau,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\tau z}\gamma^+(x;z)\mathrm{d}z. \tag{13}$$

Then by Cauchy's Residue Theorem, we solve this equation (13) to gain the final answer $g^+(t,x;s)$, the density function of hitting the upper barrier, as:

$$g^+(t,x;s) = e^{\frac{\mu}{\sigma^2}(\ell - x)}\frac{\sigma^2}{\ell^2}\sum_{k=1}^{\infty} e^{z_k(s-t)}k\pi sin(k\pi\frac{\ell - x}{\ell}). \tag{14}$$

Similarly, $\gamma^-(x;v)$ and $g^-(t,x;s)$ can be calculated as:

$$\begin{gathered}\gamma^-(x;v) = e^{-\frac{\mu}{\sigma^2}x}\frac{sinh(\theta(v)(\ell - x))}{sinh(\theta(x)\ell)},\\ g^-(t,x;s) = e^{-\frac{\mu}{\sigma^2}x}\frac{\sigma^2}{\ell^2}\sum_{k=1}^{\infty} e^{z_k(s-t)}k\pi sin(k\pi\frac{x}{\ell}).\end{gathered} \tag{15}$$

Taking equations (14) and (15) back to equation (6) can easily help us to check their correctness.

3 Result

3.1 Deferred Payment Rebate

Let $P^{\pm}(T)$ denote the probability of hitting the upper or the lower barrier at time T, the value of the rebate payment (deferred) should be $V_d^{\pm}(T) = e^{-r_d(T-t)}(Rebate * P^{\pm}(T))$.

For $P^{\pm}(T)$:

$$P^{\pm}(T) = \int_t^T g^{\pm}(t,x;s)\mathrm{d}s = \gamma^{\pm}(x;0) - \int_T^{\infty} g^{\pm}(t,x;s)\mathrm{d}s. \tag{16}$$

Hence with equations (12) and (14):

$$\begin{aligned} P^{+}(T) &= e^{\frac{\mu}{\sigma^2}(\ell-x)}\left(\frac{sinh(\frac{\mu}{\sigma^2}x)}{sinh(\frac{\mu}{\sigma^2}\ell)} - \frac{\sigma^2}{\ell^2}\sum_{k=1}^{\infty}\frac{e^{-\lambda_k(T-t)}}{\lambda_k}k\pi sin(k\pi\frac{\ell-x}{\ell})\right); \\ P^{-}(T) &= e^{-\frac{\mu}{\sigma^2}x}\left(\frac{sinh(\frac{\mu}{\sigma^2}(\ell-x))}{sinh(\frac{\mu}{\sigma^2}\ell)} - \frac{\sigma^2}{\ell^2}\sum_{k=1}^{\infty}\frac{e^{-\lambda_k(T-t)}}{\lambda_k}k\pi sin(k\pi\frac{x}{\ell})\right); \end{aligned} \tag{17}$$

which means

$$\begin{aligned} V_d^{+}(T) &= e^{-r_d(T-t)}Rebate_{up} * e^{\frac{\mu}{\sigma^2}(\ell-x)}\left(\frac{sinh(\frac{\mu}{\sigma^2}x)}{sinh(\frac{\mu}{\sigma^2}\ell)} - \frac{\sigma^2}{\ell^2}\sum_{k=1}^{\infty}\frac{e^{-\lambda_k(T-t)}}{\lambda_k}k\pi sin(k\pi\frac{\ell-x}{\ell})\right); \\ V_d^{-}(T) &= e^{-r_d(T-t)}Rebate_{low} * e^{-\frac{\mu}{\sigma^2}x}\left(\frac{sinh(\frac{\mu}{\sigma^2}(\ell-x))}{sinh(\frac{\mu}{\sigma^2}\ell)} - \frac{\sigma^2}{\ell^2}\sum_{k=1}^{\infty}\frac{e^{-\lambda_k(T-t)}}{\lambda_k}k\pi sin(k\pi\frac{x}{\ell})\right); \end{aligned} \tag{18}$$

where $\lambda_k = \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{k^2\pi^2\sigma^2}{\ell^2}\right)$, and T is the maturity time while t is the start time which often taken as 0.

3.2 Immediate Payment Rebate

The value of immediate payment rebate is given by:

$$V_i^{\pm}(T) = Rebate * \int_t^T e^{-r_d(s-t)}g^{+}(t,x;s)\mathrm{d}s. \tag{19}$$

Let

$$\begin{aligned} \lambda_k^{'} &= \frac{1}{2}\left(\frac{\mu^{'2}}{\sigma^2} + \frac{k^2\pi^2\sigma^2}{\ell^2}\right), \\ \mu^{'} &= \sqrt{\mu^2 + 2\sigma^2 r_d} \end{aligned} \tag{20}$$

Then we can gain the value of immediate payment rebate as:

$$
\begin{aligned}
V_i^{+}(T) &= Rebate_{up} * e^{\frac{\mu}{\sigma^2}(\ell - x)} \left(\frac{sinh(\frac{\mu^{'}}{\sigma^2}x)}{sinh(\frac{\mu^{'}}{\sigma^2}\ell)} - \frac{\sigma^2}{\ell^2} \sum_{k=1}^{\infty} \frac{e^{-\lambda^{'}_k(T-t)}}{\lambda^{'}_k} k\pi sin(k\pi \frac{\ell - x}{\ell}) \right); \\
V_i^{-}(T) &= Rebate_{low} * e^{-\frac{\mu}{\sigma^2}x} \left(\frac{sinh(\frac{\mu^{'}}{\sigma^2}(\ell - x))}{sinh(\frac{\mu^{'}}{\sigma^2}\ell)} - \frac{\sigma^2}{\ell^2} \sum_{k=1}^{\infty} \frac{e^{-\lambda^{'}_k(T-t)}}{\lambda^{'}_k} k\pi sin(k\pi \frac{x}{\ell}) \right);
\end{aligned}
\tag{21}
$$

3.3 Convergence

Now we need to make sure that infinite series $e^{-\lambda_k(T-t)}$ converges in a rather fast manner. Set an error term ϵ, the number of terms k shall meet:

$$
k > \sqrt{\frac{-2\frac{log\epsilon}{T-t} - \frac{\mu^2}{\sigma^2}}{\frac{\pi^2\sigma^2}{\ell^2}}}\ , \ for\ k = 0, 1, 2, \dots\ . \tag{22}
$$

Take an example where $T - t = 1(yrs), \mu = 0, \sigma = 20\%, \ell = 0.9$, when $\epsilon = 10^{-6}$, k = 5; when $\epsilon = 10^{-10}$, k = 7. This is in fact a relatively fast convergence and can be viewed as an analytical formulae for practitioners in the most cases.