Implicit Formulae for Deferred-Payment Rebate and American Cash-or-Nothing Binary Option Pricing

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June 2019

1 Introduction

1.1 Problem Addressing

This is a brief presentation on the derivation and result of implicit (closed-form) pricing formulae of deferred-payment American binary option as well as the rebate part of standard barrier options.

As a common knowledge, the implicit pricing formulae of immediate-payment American binary options are more easily available than those of deferred ones. However, the deferred ones are more frequently used in practice, whose implicit formulae will be derived in risk-neutral space by transferring probability measures with Girsanov Theorem in this report.

Also by seeing standard barrier options with deferred-payment rebate as a portfolio of zero-rebate standard barrier together with its rebate part as an American binary, we could easily achieve their implicit formulae as well.

2 Derivation

2.1 Joint Probability Density Function

First achieve the joint density function for a Brownian motion with a non-zero drift and its maximum or minimum to date. Let $\tilde{W}(t)$ be a Brownian motion on $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}),\ 0 \le t \le T$.

Let α be a given number, that

$$\hat{W}(t) := \tilde{W}(t) + \alpha t, \quad 0 \le t \le T \tag{1}$$

Also define:

$$\hat{M}(T) := \max_{0 \le t \le T} \hat{W}(t) \tag{2}$$

$$\hat{N}(T) := \min_{0 \le t \le T} \hat{W}(t) \tag{3}$$

where $(m,w) \in \{w \le m, m \ge 0\}, \quad (n,w) \in \{w \ge n, n \le 0\}$.

By Girsanov's Theorem:

Such a $\hat{\mathbb{P}}(A)$ can be constructed that $\hat{W}(t)$ is a Brownian motion with zero drift under $\hat{\mathbb{P}}$, where the joint density of $\begin{cases} \hat{M}(T), \hat{W}(T) \\ \hat{N}(T), \hat{W}(T) \end{cases}$ under $\hat{\mathbb{P}}$ can be achieved as :

$$\hat{f}_{\hat{M},\hat{W}}(m,w) = \frac{2(2m-2w)}{T\sqrt{2\pi T}} e^{-\frac{1}{2T}(2m-w)^2}, w \le m, m \ge 0. \tag{4}$$

$$\hat{f}_{\hat{N},\hat{W}}(n,w) = -\frac{2(2n-2w)}{T\sqrt{2\pi T}} e^{-\frac{1}{2T}(2n-w)^2}, n \le w, n \le 0. \tag{5}$$

Then to achieve joint density under $\tilde{\mathbb{P}}$:

$$\tilde{f}_{\hat{M},\hat{W}}(m,w) = \frac{2(2m-2w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}, w \le m, m \ge 0. \tag{6}$$

$$\tilde{f}_{\hat{N},\hat{W}}(n,w) = -\frac{2(2n-2w)}{T\sqrt{2\pi T}}\, e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2n-w)^2}, n \le w, n \le 0. \tag{7}$$

2.2 Joint Cumulative Density Function

Then integrate to gain $\begin{cases} \tilde{\mathbb{P}}\{\hat{M}(T) \le m\} \\ \tilde{\mathbb{P}}\{\hat{N}(T) \le \ n\} \end{cases}$:

$$\begin{aligned} \tilde{\mathbb{P}}\{\hat{M}(T) \le m\} &= \int_0^m \int_w^m \frac{2(2u-2w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2u-w)^2} \mathrm{d}u\,\mathrm{d}w \\ &+ \int_{-\infty}^0 \int_0^m \frac{2(2u-2w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2u-w)^2} \mathrm{d}u\,\mathrm{d}w \end{aligned} \tag{8}$$

$$\begin{aligned} \tilde{\mathbb{P}}\{\hat{N}(T) \le \ n\} &= -\int_n^0 \int_n^w \frac{2(2u-2w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2u-w)^2} \mathrm{d}u\,\mathrm{d}w \\ &- \int_0^\infty \int_n^0 \frac{2(2u-2w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2u-w)^2} \mathrm{d}u\,\mathrm{d}w \end{aligned} \tag{9}$$

Finally gain:

$$\tilde{\mathbb{P}}\{\hat{M}(T) \le m\} = -e^{2\alpha m} N\left(\frac{-m-\alpha T}{\sqrt{T}}\right) + N\left(\frac{m-\alpha T}{\sqrt{T}}\right) \tag{10}$$

$$\tilde{\mathbb{P}}\{\hat{N}(T) \ge \ n\} = -e^{2\alpha n} N\left(\frac{n+\alpha T}{\sqrt{T}}\right) + N\left(\frac{-n+\alpha T}{\sqrt{T}}\right) \tag{11}$$

3 Result

The probability of triggering the upper or lower barrier is finally presented below. Hence the fair value of deferred-payment American binary options should be calculated by multiplying probability of triggering and cash (rebate) payment value.

$$
\begin{aligned}
&\textit{Probability}\ \{\textit{Call American Binary or Upwards Rebate Payment Triggered}\}\\
=&\tilde{\mathbb{P}}\{\textit{Price of Asset Maximum Larger Than Higher Barrier}\}\\
=&\tilde{\mathbb{P}}\{\hat{M}(T) \geq b\} = 1 + e^{2\alpha b} N\left(\frac{-b-\alpha T}{\sqrt{T}}\right) - N\left(\frac{b-\alpha T}{\sqrt{T}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\textit{Probability}\ \{\textit{Put American Binary or Downwards Rebate Payment Triggered}\}\\
=&\tilde{\mathbb{P}}\{\textit{Price of Asset Minimum Smaller Than Lower Barrier}\}\\
=&\tilde{\mathbb{P}}\{\hat{N}(T) \leq b\} = 1 + e^{2\alpha b} N\left(\frac{b+\alpha T}{\sqrt{T}}\right) - N\left(\frac{-b+\alpha T}{\sqrt{T}}\right)
\end{aligned}
$$

$$
\text{where} \begin{cases} b = \dfrac{1}{\sigma} log\left(\dfrac{B}{S_0}\right) \\ \alpha = \dfrac{1}{\sigma}(r - \dfrac{1}{2}\sigma^2) \\ T:\ TTM \\ N(*):\ Standard\ Normal\ CDF \end{cases}
$$