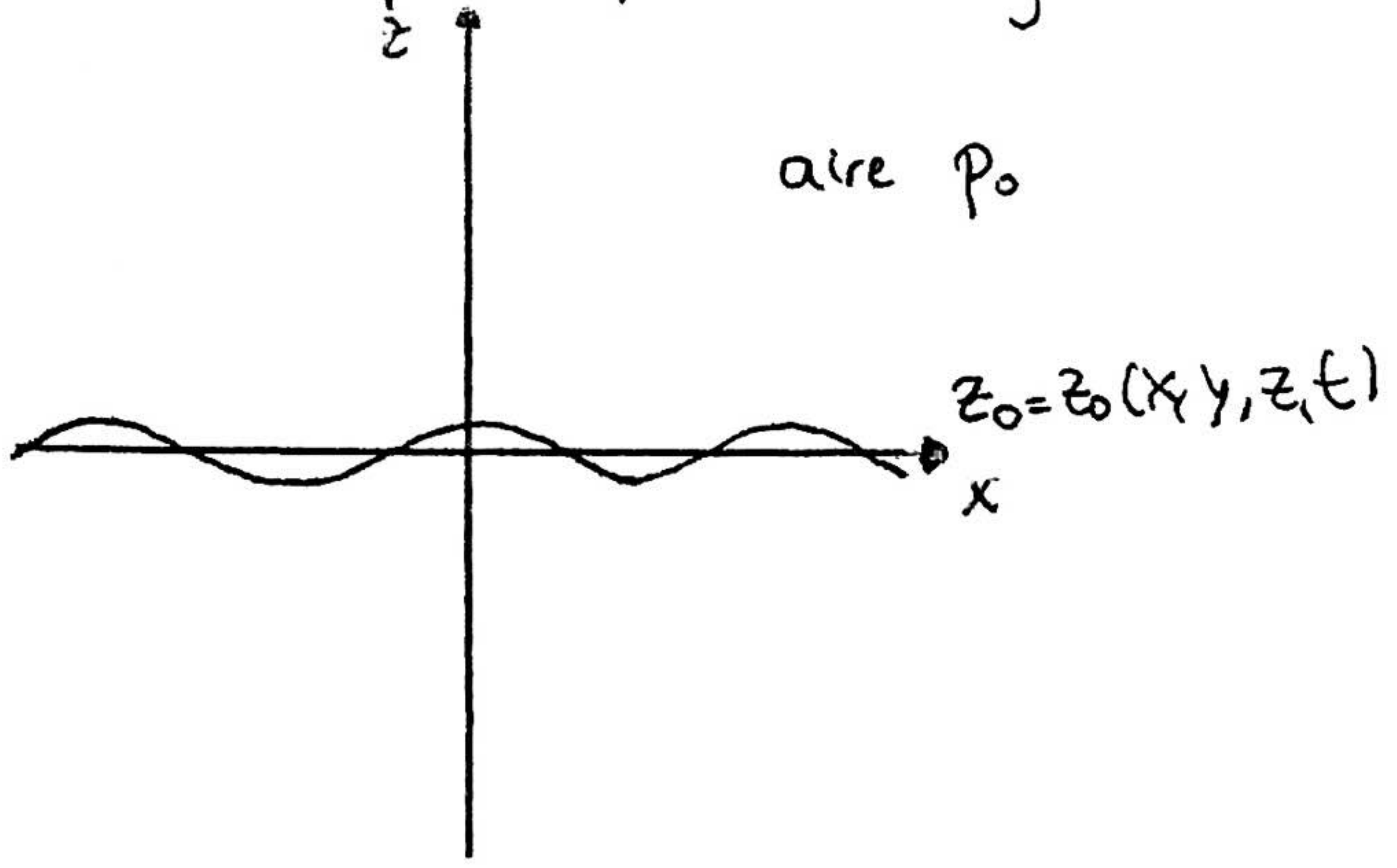
Ejercicio 7.1

Fluido incompresible, solo actúa g.

Se produce una perturbación externa sobre la superficie

Velocidades pequeñas



a) ¿Qué implica $|(\underline{u}\cdot\underline{\nabla})\underline{u}| << \left|\frac{\partial u}{\partial t}\right|$? $\quad \underline{u} \sim a f(kx-\omega t)$

$$\frac{\partial u}{\partial t} \sim -\omega a \frac{\partial f}{\partial t}(kx-\omega t)$$

$$\frac{\partial u}{\partial x} \sim k a \frac{\partial f}{\partial x}(kx-\omega t) \Rightarrow \frac{\partial u}{\partial t} + (\underline{u}\cdot\underline{\nabla})\underline{u} \sim -af'\omega + ka\frac{\partial f}{\partial x}(af)$$

$$\Rightarrow ka\frac{\partial f}{\partial x}(af) << 1 \Rightarrow ak << 1 \Rightarrow \underline{a << \lambda}$$

b) Al despreciar el término convectivo

$$\frac{\partial u}{\partial t} = -\frac{\underline{\nabla} p}{\rho} - g\hat{z} = -\underline{\nabla}\left(\frac{p}{\rho} + gz\right)$$

Si tomo $\underline{u}$ irrotacional $\Rightarrow \underline{u = \nabla\phi}$

Que puedo pues $\underline{\nabla}\times\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(\underline{\nabla}\times\underline{u}) = \underline{\nabla}\times\left(-\underline{\nabla}\left(\frac{p}{\rho}+gz\right)\right) = 0$ $\quad \underline{\nabla}\times\underline{\nabla}.$

incompresible

$$\Rightarrow \nabla^2\phi = 0 \overset{!}{=} \underline{\nabla}\cdot\underline{u}$$

$$\Rightarrow \underline{\nabla}\left(\frac{\partial\phi}{\partial t} + \frac{p}{\rho} + gz\right) = 0 \Leftrightarrow \frac{\partial\phi}{\partial t} + \frac{p}{\rho} + gz = C(t)$$

Redefino $\phi' = \phi + \int C(t) \Rightarrow \frac{\partial\phi'}{\partial t} = \frac{\partial\phi}{\partial t} + C(t)$ $\quad \left(\begin{matrix}\underline{u} = \nabla\phi' \checkmark \\ u = \nabla\phi' \checkmark\end{matrix}\right)$

$$\therefore \frac{\partial\phi'}{\partial t} + \frac{p}{\rho} + gz = 0$$

$\gamma(t)$ $\quad \Rightarrow \phi'' = \phi' + \int\gamma(t)$

Sea $z = z_0$ superficie $\Rightarrow p = p_0 \Rightarrow \frac{\partial\phi'}{\partial t} + \frac{p_0}{\rho} + gz_0 = 0$

$$\Rightarrow \frac{\partial\phi''}{\partial t} + gz_0 = 0$$

Por comodidad con abuso de notación $\frac{\partial \phi}{\partial t} + g z_0 = 0$

$$\Rightarrow \boxed{z_0 = -\frac{1}{g}\left(\frac{\partial \phi}{\partial t}\right)\Bigg|_{z=z_0}}$$

ii) $\underline{u} = \underline{\nabla}\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$ $\qquad$ $\underline{u} = \frac{\partial \underline{x}}{\partial t} = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right)$

$$\Rightarrow \frac{\partial \phi}{\partial z} \sim \frac{\partial z}{\partial t}$$

$$\therefore \quad z \cong -\frac{1}{g}\frac{\partial \phi}{\partial t} \Rightarrow \left(\frac{\partial z}{\partial t}\right)_{z=z_0} \sim \left(\frac{\partial \phi}{\partial z}\right)_{z=z_0} = -\frac{1}{g}\left(\frac{\partial^2 \phi}{\partial t^2}\right)_{z=z_0}$$

$$\therefore \quad \boxed{\begin{array}{l} \nabla^2 \phi = 0 \\ \left(\frac{\partial \phi}{\partial z} + \frac{1}{g}\frac{\partial^2 \phi}{\partial t^2}\right)\Bigg|_{z=z_0} = 0 \end{array}}$$

$$\phi = Z(z)\tau(t) \Big|_{z=0} \Rightarrow \dot{Z}\tau\Big|_{z=0} + \frac{1}{g} Z\ddot{\tau}\Big|_{z=0} = 0 \quad \Rightarrow \quad \underbrace{\frac{\dot{Z}}{Z}}_{\lambda^2} = -\frac{\ddot{\tau}}{\tau}\frac{1}{g}$$

$$\frac{\ddot{\tau}}{\tau}\frac{1}{g} = -\lambda^2 \qquad\qquad \dot{Z} - \lambda^2 Z = 0 \qquad e^{\pm\lambda^2 z}$$

$$Z(z) = e^{\lambda^2 z}$$

profundidad infinita
mata el término negativo

$$\ddot{\tau} + \lambda^2 g \tau = 0$$

$$\tau(t) = e^{\pm i\lambda\sqrt{g}t}$$

$$\phi = e^{\lambda^2 z}\left[A\cos(\lambda\sqrt{g}t) + B\sin(\underbrace{\lambda\sqrt{g}}_{\omega}t)\right] A(x,y)$$

$$\lambda^2 = k$$

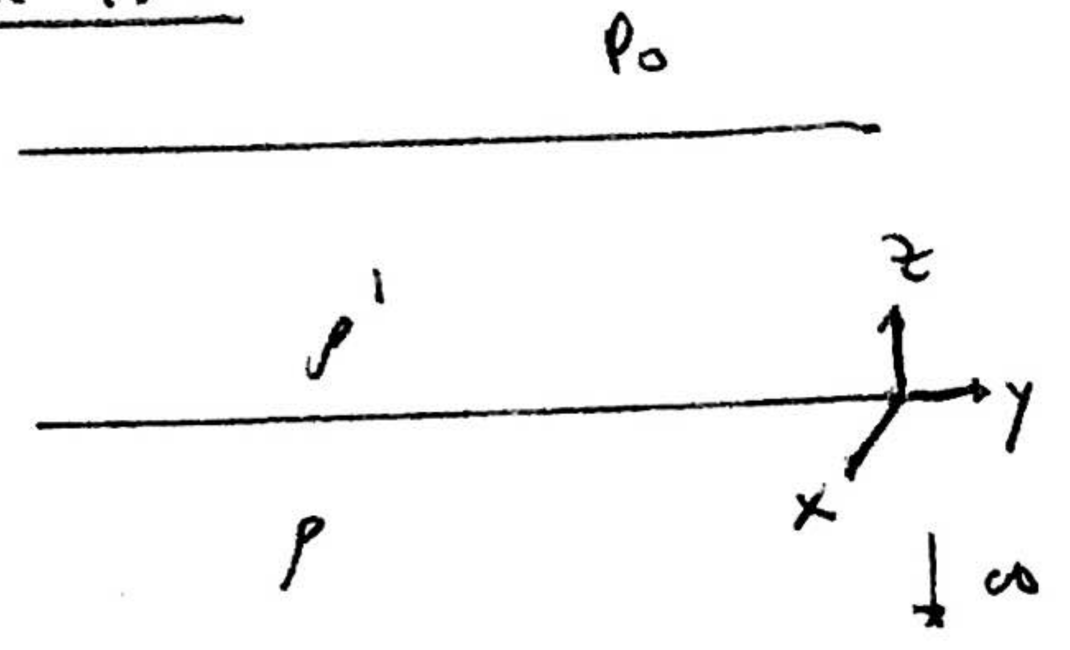
$$\lambda\sqrt{g} = \omega \quad \Rightarrow \boxed{k = \frac{\omega^2}{g}} \qquad\qquad \phi = X Z \tau$$

$$\phi = e^{\lambda^2 z}\left[A(x)\cos(\omega t) + B(x)\sin(\omega t)\right]$$

MAL, PRIMERO SE PLANTEABA

QUE $\nabla^2\phi = 0$

Ejercicio 7.4



$$\nabla^2 \phi' = 0 \qquad \left(\frac{\partial \phi'}{\partial z} + \frac{1}{g}\frac{\partial^2 \phi'}{\partial t^2}\right)\Bigg|_{z=h'} = 0$$

$$\nabla^2 \phi = 0$$

$$2)\ \left(g(\rho-\rho')\frac{\partial \phi}{\partial z} + \rho\frac{\partial^2 \phi}{\partial t^2} - \rho'\frac{\partial^2 \phi'}{\partial t^2}\right)\Bigg|_{z=0} = 0$$

$\nabla^2 \phi = 0 \qquad \phi = f(z)\cos(kx-\omega t)$

$\Rightarrow f''\cos(kx-\omega t) - k^2 f\cos(kx-\omega t) = 0 \quad \forall x\ \forall t$

$\Leftrightarrow f'' = k^2 f \qquad \underset{\text{propongo}}{\Rightarrow} f = e^{\lambda z}$

$\Rightarrow \lambda^2 = k^2 \quad \Leftrightarrow |\lambda| = k$

$\Rightarrow f = Ae^{kz} + Be^{-kz}$ como $h \to -\infty$

$\Rightarrow B = 0$, o la solución divergería, así

$$\phi = Ae^{kz}\cos(kx-\omega t)$$

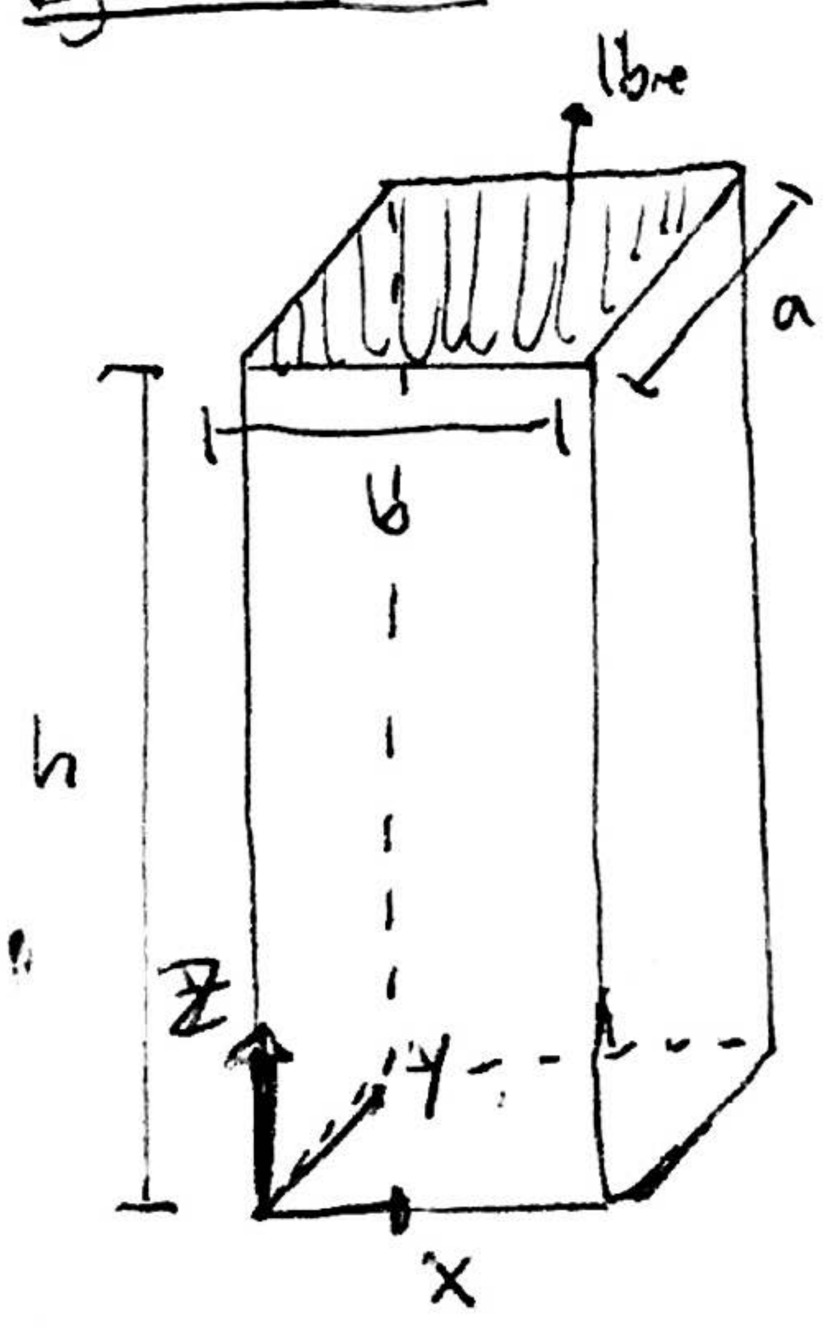
$$\phi' = \left(A'\,ch(k(z-h')) + B'\,sh(k(z-h'))\right)\cos(kx-\omega t)$$

$\Rightarrow$ contorno en h'

$$kA'\,ch(k(h'-h')) + B'k\,sh(k(h'-h')))\cos(kx-\omega t) - \frac{\omega^2}{g}\phi'\Big|_{h'} = 0$$

$$\boxed{\text{plantear } 2) \ \wedge \ \left.\frac{\partial \phi'}{\partial z} = \frac{\partial \phi}{\partial z}\right|_{z=0} \ ✓}$$

Ejercicio 7.6



$$\phi = X(x)Y(y)Z(z)T(t) \qquad \left(\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z}\right)\Bigg|_{z=h} = 0$$

$$\nabla^2 \phi = 0$$

$$u\cdot\hat{n} = 0 \Leftrightarrow \left.\frac{\partial \phi}{\partial \hat{n}}\right|_{\Sigma} = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$\Leftrightarrow \frac{X''}{X} = -\lambda_1^2 \qquad \frac{Y''}{Y} = -\lambda_2^2 \qquad \frac{Z''}{Z} = +(\lambda_1^2 + \lambda_2^2)$$

$$\Rightarrow \quad X = A\cos(\lambda_1 x) + B\sin(\lambda_1 x)$$

$$\left.\frac{\partial X}{\partial x}\right|_{x=0} = 0 = \left[-A\sin(0) + B\cos(0)\right]\lambda_1 \Leftrightarrow B = 0$$

$$\therefore \quad X = A\cos(\lambda_1 x)$$

$$\left.\frac{\partial X}{\partial x}\right|_{x=a} = 0 \Leftrightarrow \sin(\lambda_1 a) = 0 \Leftrightarrow \lambda_1 a = n\pi$$

$$\therefore \quad X = A\cos\left(\frac{n\pi}{a}x\right)$$

análogo $Y = B\cos\left(\frac{m\pi}{b}y\right)$

$$\Rightarrow \phi = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A\cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right) Z(z)\, \mathcal{T}(t)$$

$$Z'' = \underbrace{\left[\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right]}_{k^2} Z \quad \Rightarrow Z = A'\cosh(kz) + B'\sinh(kz)$$

$$\mathcal{T}(t) = e^{i\omega t}$$

$$\Rightarrow \quad \left.\frac{\partial Z}{\partial z}\right|_{z=0} = 0 \Leftrightarrow k\left[\underbrace{A'\sinh(0)}_{=0} + B'\cosh(0)\right] = 0$$

$$\Rightarrow B' = 0$$

$$\therefore \quad \phi(\underline{r},t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right)\cosh(kz)\, e^{i\omega t}$$

Por último

$$\left.\left|\frac{\partial^2 (Z\mathcal{T})}{\partial t^2} + g\frac{\partial (Z\mathcal{T})}{\partial z}\right|\right|_{z=h} = 0$$

$$\Leftrightarrow \mathcal{T}'' Z(h) + g\mathcal{T} Z'(h) = 0$$

$$\Leftrightarrow A_{nm}\mathcal{T}''\, \mathrm{ch}(kh) + gA_{nm}\, \mathrm{sh}(kh)\, \mathcal{T} = 0 \quad \forall t$$

$$\Leftrightarrow A_{nm}\left[-\omega^2 + g\, \mathrm{tgh}(kh)\right] e^{i\omega t} = 0$$

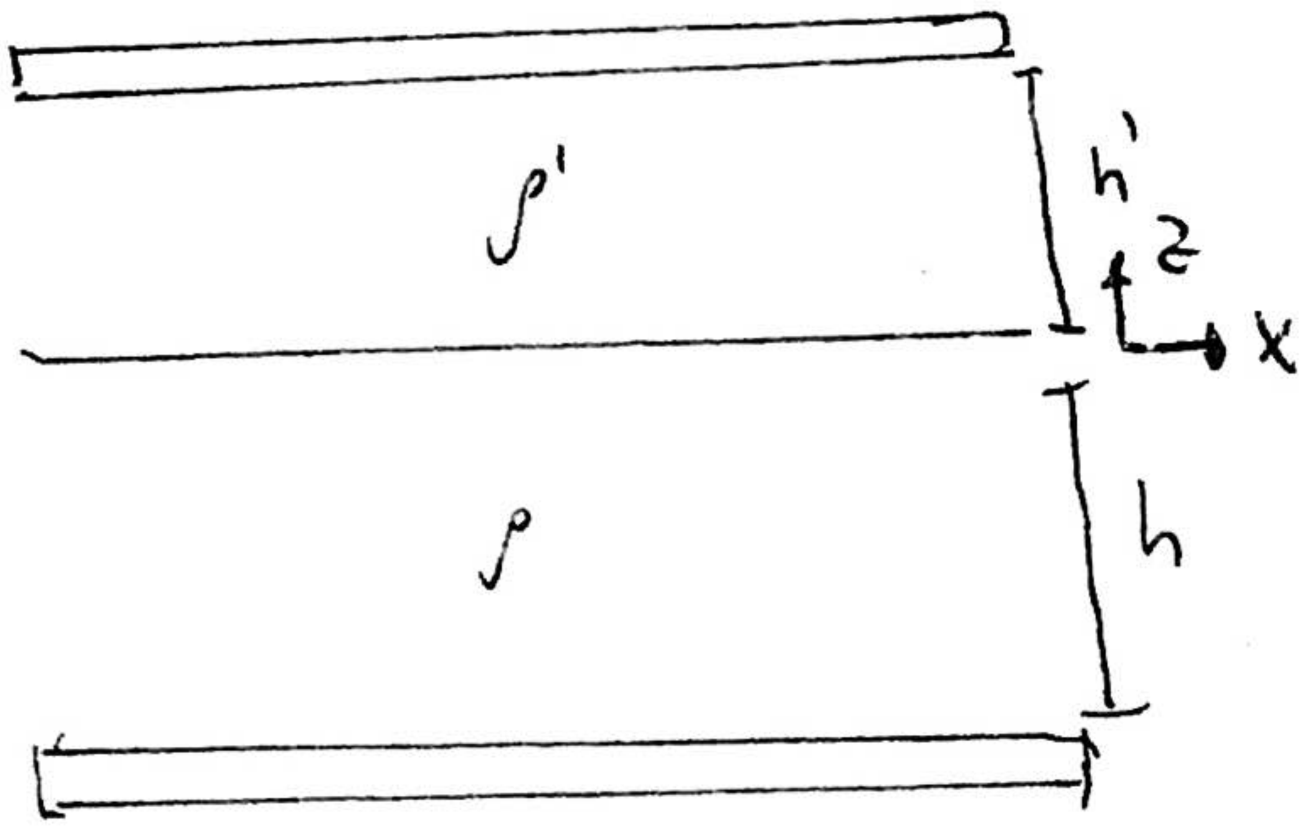
$$\Rightarrow \omega^2 = g\, \mathrm{tgh}(kh)$$

Finalmente

$$\phi(\underline{r},t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\cos\left(\frac{n\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right)\cosh\left(\sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}\, z\right) e^{i\left(\sqrt{g\, \mathrm{tgh}\left(\sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}}\right)}\right)t}$$

$+$ CI $\phi(x,y,0,0)$

Ejercicio 7.3



¿$w(k)$?

$$\nabla^2\phi' = 0 \qquad \frac{\partial\phi'}{\partial z} + \frac{1}{g}\frac{\partial^2\phi'}{\partial t^2} = 0$$

$$\nabla^2\phi = 0 \qquad \left(g(\rho-\rho')\frac{\partial\phi}{\partial z} + \rho\frac{\partial^2\phi}{\partial t^2} - \rho'\frac{\partial^2\phi'}{\partial t^2}\right)\Bigg|_{z=0} = 0$$

$$g(\rho-\rho')\dot{z}\tau + \rho z\ddot{\tau} - \rho z'\ddot{\tau}' = 0$$

$$\phi(x,z,t) = f(z)F(x,t)$$

$$\phi'(x,z,t) = f'(z)F'(x,t)$$

$$\Rightarrow \quad f''F + fF'' = 0 \quad \Rightarrow \quad \underbrace{\frac{f''}{f}}_{\lambda^2} = \underbrace{-\frac{F''}{F}}_{-\lambda^2}$$

$$A\,ch(-kh) + B\,sh(-kh) = 0$$

$$A\,ch(k(h-z)) + B\,sh(k(h-z)) =$$

$$A = 0$$

$\Rightarrow$ Propongo $\phi' = f'(z)\cos(kx-\omega t) \Rightarrow \phi = [A\,ch(kz) + B\,sh(kz)]\,c(kx-\omega t)$

$\phi = f(z)\cos(kx-\omega t) \qquad \phi' = [A'ch(kz) + B'sh(kz)]\,c(kx-\omega t)$

$$\Rightarrow g(\rho-\rho')\dot{f}(z)\cos(kx-\omega t) + \rho(-\omega)f(z)\sin(kx-\omega t) - \rho'(-\omega)\dot{f}(z)\sin(kx-\omega t) = 0$$