Práctica 1

① a) $z \in \mathbb{C}$. Probar:

i) $|Re(z)| \leq |z|$ , $|Im(z)| \leq |z|$

$z = Re(z) + i\, Im(z)$

$\Rightarrow |z| = \sqrt{(Re(z))^2 + (Im(z))^2}$

$z = (Re(z))^2 + (Im(z))^2 \geq (Re(z))^2 = |Re(z)|$

análogo para $|Im(z)|$

ii) $2|Re(z)||Im(z)| \leq |z|^2$

$z^2 = Re(z)^2 + 2i Re(z) Im(z) - Im(z)^2$ $\quad |z^2| = |z|^2$

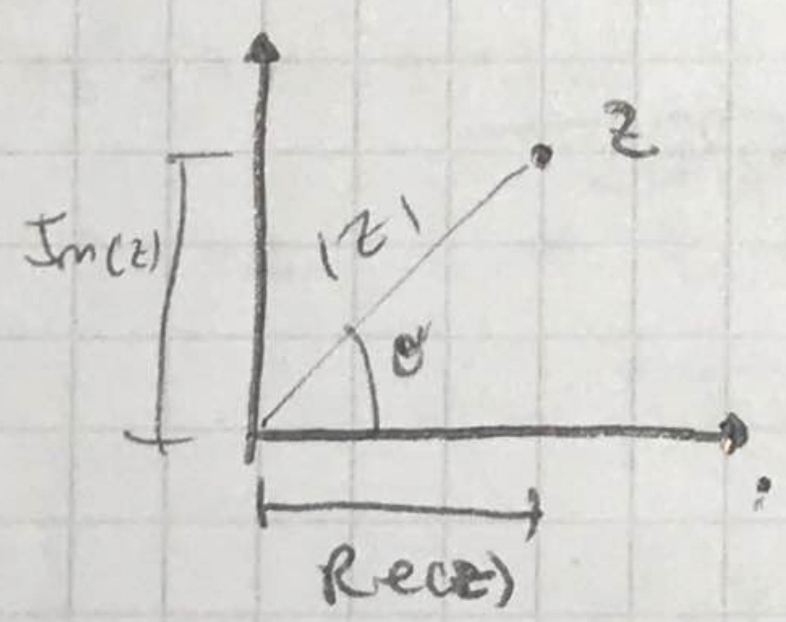
$|z|^2 \geq 2|Re(z)||Im(z)|$

iii) $|z| \leq |Re(z)| + |Im(z)| \leq \sqrt{2}|z|$

$|z| = |Re(z) + i\, Im(z)| \leq |Re(z)| + |i\, Im(z)| = |Re(z)| + |Im(z)|$

$z = |z| (\cos(\theta) + i \sin(\theta))$

$|z|^2 = |z|^2 . (\cos^2(\theta) - \sin^2(\theta) + 2i\cos(\theta)\sin(\theta)) = |z|^2 . ( \cos(\theta_2)$




$|z| =$

$|z||\cos(\theta)| + |z||\sin(\theta)| \leq 2|z|$ (with $|\cos(\theta)| \leq 1$, $|\sin(\theta)| \leq 1$)

vale, pero no sirve

$|z|(|\cos(\theta)| + |\sin(\theta)|)$ $\quad \leq \frac{\sqrt{2}}{2}$, $\frac{\sqrt{2}}{2}$

$\cos(\theta) + \sin(\theta)$

$\theta$ máx $= k\frac{\pi}{4} \Rightarrow$

$|z| \leq \sqrt{2}|z|$

iv) $z^{-1} = \dfrac{\bar{z}}{|z|^2}$ si $z \neq 0$

$z^{-1} = w \;/\; z \cdot w = 1$ $\quad \Rightarrow W = \dfrac{1}{z}$ $= \cancel{\dfrac{\bar{z}}{z^2} = \dfrac{\bar{z}}{|z|^2} = \dfrac{\bar{z}}{z \cdot \bar{z}} = \bar{z}}$

$W = \dfrac{1}{z} \cdot \dfrac{\bar{z}}{\bar{z}} = \dfrac{\bar{z}}{|z|^2}$ ✓

v) $Re(z) = \dfrac{z + \bar{z}}{2}$ $\quad Im(z) = \dfrac{z - \bar{z}}{2i}$

$Re(z) = x$
$Im(z) = y$

$\dfrac{z + \bar{z}}{2} = \dfrac{x + iy + x - iy}{2} = x$ ✓

b) Dados $z_1, z_2 \in \mathbb{C}$, probar que:

i) $|z_1||z_2| \geq \frac{1}{2}(z_1\overline{z_2} + \overline{z_1}z_2)$

$|z_1||z_2| = \sqrt{z_1\overline{z_1}} \cdot \sqrt{z_2\overline{z_2}} = \sqrt{z_1\overline{z_1}z_2\overline{z_2}}$

$= \cancel{x_1 x_2} \quad |z_1||z_2| = |x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2)| \leq$

$\cancel{(|z_1||z_2|)^2 = z_1\overline{z_1}\overline{z_2}z_2}$ $\qquad \cancel{|z_1| = |\overline{z_1}|}$

$\Rightarrow \cancel{\geq z_1\overline{z_2} \geq z_1\overline{z_2}}$ $\qquad z_1\overline{z_2} = \overline{z_1}z_2$

$|(x_1+iy_1)(x_2-iy_2)| = |(x_1-iy_1)(x_2+iy_2)|$

$|x_1x_2 + y_1y_2 + i(x_2y_1 - y_2x_1)| = |x_1x_2 + y_1y_2 + i(y_2x_1 - x_2y_1)|$

$\cancel{(x_1x_2+y_1y_2)^2} + (x_2y_1 - y_2y_1)^2 = \cancel{(x_1x_2+y_1y_2)^2} + (y_2x_1 - x_2y_1)^2$

$x_2^2y_1^2 - 2x_2y_1y_2x_1 + y_2^2y_1^2$

$\cancel{\Rightarrow (|z_1||z_2|)^2 >}$

$|z_1||z_2| = \sqrt{z_1\overline{z_1}z_2\overline{z_2}}$

$= \sqrt{z_1\overline{z_2}}\sqrt{\overline{z_1}z_2}$

$|z_1z_2| = \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2} \geq (x_1x_2 + y_1y_2)$

$\Leftrightarrow \cancel{x_1x_2^2 - 2x_1x_2y_1y_2 + y_1y_2^2 + (\ldots \geq x_1x_2^2 + 2x_1x_2y_1y_2}$

$= \sqrt{x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2}$

$= \quad x_1^2y_2^2 + y_1^2x_2^2 \geq 2x_1x_2y_1y_2$

$(x_1y_2 - y_1x_2)^2 \geq 0$

ii) $|z_1 \pm z_2| \leq |z_1| + |z_2|$

$|z_1 \pm z_2| = |x_1 \pm x_2 + i(y_1 \pm y_2)| = \sqrt{(x_1 \pm x_2)^2 + (y_1 \pm y_2)^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$

$\cancel{x_1^2} \pm 2x_1x_2 + \cancel{x_2^2} + \cancel{y_1^2} \pm 2y_1y_2 + \cancel{y_2^2} \leq \cancel{x_1^2} + \cancel{y_1^2} + \cancel{x_2^2} + \cancel{y_2^2} + 2\sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$

$\pm\cancel{2}(x_1x_2 + y_1y_2) \leq \cancel{2}\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$

$\cancel{x_1 x_2}$

Si es $-$ ~~basta~~ pues $\ominus \leq \oplus$

si es $+$ vale elevando al cuadrado otra vez

iii) $|z_1 - z_2| \geq ||z_1| - |z_2||$

$= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$ $\qquad \left|\sqrt{x_1^2+y_1^2} - \sqrt{x_2^2+y_2^2}\right|$

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \geq \sqrt{x_1^2+y_1^2 - 2\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)} + x_2^2+y_2^2}$$

$$\cancel{x_1^2} - 2x_1x_2 + \cancel{x_2^2} + \cancel{y_1^2} - 2y_1y_2 + \cancel{y_2^2} \geq \cancel{x_1^2} + \cancel{y_1^2} - 2\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)} + \cancel{x_2^2} + \cancel{y_2^2}$$

$$-\cancel{2}(x_1x_2 + y_1y_2) \geq -\cancel{2}\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}$$

$$(x_1x_2 + y_1y_2)^2 \leq (x_1^2+y_1^2)(x_2^2+y_2^2)$$

$$\cancel{x_1^2x_2^2} + 2x_1y_1y_2x_2 + \cancel{y_1^2y_2^2} \leq \cancel{x_1^2x_2^2} + \cancel{y_1^2y_2^2} + x_1^2y_2^2 + y_1^2x_2^2$$

$$0 \leq (x_1y_2 - y_1x_2)^2 \checkmark$$

③ Resolver las ecuaciones

a) $|z| - z = 1+2i$

$\sqrt{x^2+y^2} - x - iy = 1+2i$

$y = -2 \Rightarrow \sqrt{x^2+4} - x = 1$ $\qquad z = -\frac{2}{3} - 2i$

$\sqrt{x^2+4} = 1+x$

$\cancel{x^2} + 4 = 1 + 2x + \cancel{x^2}$

$2x = 3 \quad x = \frac{2}{3}$

b) $z\bar{z} - 2|z| + 1 = 0$

$|z|^2 - 2|z| + 1 = 0$ $\qquad |z| = 1$ $\qquad$ Rta: $\sqrt{x^2+y^2} = 1$

$\frac{2 \pm \sqrt{4-4}}{2} = 1$

c) $z^6 + 2 = 0$ $\qquad z^6 = -2$ $\qquad |z|^6 = 2 \quad |z| = \sqrt[6]{2}$

$z^n = |z|^n (\cos(\theta) + i\sin(\theta))^n = |z|^n(\cos(n\theta) + i\sin(n\theta))$

$\arg(-2) = \theta = \cancel{\pi} \Rightarrow \cancel{z^6 = \sqrt{2}}$

$2 = \cancel{z} \cos($

$z^6 = |z|^6 (\cos(6\theta) + i\sin(6\theta))$

$6\theta = \pi + 2k\pi$

$\theta = \pi/6 + k\pi/3 \quad k \in [0,5] \quad k \in \mathbb{Z}$

$\cancel{-\pi \leq \pi/6 + k\pi/3 \leq \pi}$ $\quad -6 \leq 1+2k \leq 6$

$-7 \leq 2k \leq 5$

$-3{,}5 = -7/2 \leq k \leq 5/2 = 2{,}5$

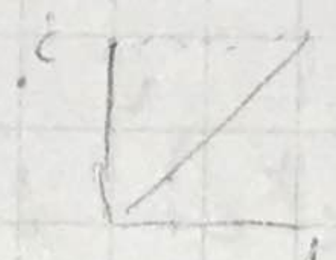
$-3, -2, -1, 0, 1, 2$

$z_k = \sqrt[6]{2}\, e^{i(\pi/6 + k\pi/3)} \quad k \in [-3, 2] \in \mathbb{Z}$

d) $z^4 - 1 - i = 0$

$z^4 = 1 + i \qquad |z|^4 = |z^4| = |1+i| = \sqrt{2}$

$|z| = \sqrt[8]{2}$



$\arg(z^4) = 4\arg(z) = \arg(1+i) =$

$4\arg(z) = \frac{\pi}{4}$

$\arg(z) = \frac{\pi}{16} + \frac{k\pi}{2} \qquad -\pi \leq \frac{\pi}{16} + \frac{k\pi}{2} \leq \pi$

$-\frac{17}{16} \leq \frac{k}{2} \leq \frac{15}{16}$

$-\frac{17}{8} \leq k \leq \frac{15}{8}$

$-2{,}1 \leq k \leq 1{,}9$

$-2, -1, 0, 1$

(4) Probar la resolución de las ecuaciones de segundo grado

$az^2 + bz + c = 0 \quad , \; a, b, c \in \mathbb{C}$ y $a \neq 0$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a(x^2 + 2xyi - y^2) + b(x + iy) + c = 0$

$i(2xya + by) = 0 \qquad y(2xa + b) = 0$ si $y = 0$

$a(x^2 - y^2) + bx + c \qquad ax^2 + bx + c$ estructura real

si $y \neq 0 \qquad x = \frac{-b}{2a} \qquad a\left(\frac{b^2}{4a^2} - y^2\right) + b\left(\frac{-b}{2a}\right) + c$

$$\frac{b^2}{4a} - y^2 a - \frac{b^2}{2a} + C = 0$$

$$-\frac{b^2}{4a^2} + = y^2$$

$$z^2 + \frac{b}{a} z + \frac{c}{a} = 0$$

$$z^2 + \frac{b}{a} z + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(z + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$w = z + \frac{b}{2a}$$

$$w^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$z = w - \frac{b}{2a}$ , W a determinar

b) $z^2 - (2i+4)z + 10i - 5$

$$w^2 = -10i + 5 + \frac{(-2i+4)^2}{4}$$

$$w^2 = -10i + 5 + \frac{4 + 16i + 16}{4}$$

$$w^2 = -6i + 8$$

$$a^2 - b^2 + 2abi = -6i + 8$$

$2ab = -6 \qquad a = -\frac{3}{b}$

$$a^2 - b^2 = 8$$

$$\frac{9}{b^2} - b^2 = 8$$

$$9 - b^4 = 8b^2$$

$$\frac{+8 \pm \sqrt{100}}{-2} \quad \frac{8+10}{-2} = -9 \quad \frac{8-10}{-2} = 1$$

$b^2 = 1 \quad b^2 = -9 \quad b = \pm 3i$

$\to a = \mp \frac{1}{i} = \pm i$

$\cancel{\to a = i \text{ es}}$

$\cancel{w_1 = \frac{1}{3} - 9i}$

$\cancel{w_2 = -3 + i}$

$$\boxed{\begin{array}{l} w_1 = -3 + i \\ w_2 = 3 - i \end{array}}$$

$$z = -3 + i + (2i + 4)$$

⑤ Probar que si $c \in \mathbb{R}_{>0}$, $|z-1| = c|z+1|$ es una circunf o una recta

$$\sqrt{(x-1)^2+y^2} = c\sqrt{(x+1)^2+y^2}$$

$$(x-1)^2+y^2 = c^2\left[(x+1)^2+y^2\right]$$

$$c^2\left[(x^2+2x+1)+y^2\right] = c^2\left[(x-1)^2+4x+y^2\right]$$

$$\left[y^2+(x-1)^2\right]\left[1-c^2\right] = c^2 4x$$

si $c=1$

$4x=0$ recta $x=0$

si no $y^2+x^2-$

Representar gráficamente $|z-3| = 2|z+3|$

$(x-3)^2 + y^2 = 4(x+3)^2 + 4y^2$

$x^2 - 6x + 9 + y^2 = 4x^2 + 24x + 36 + 4y^2$

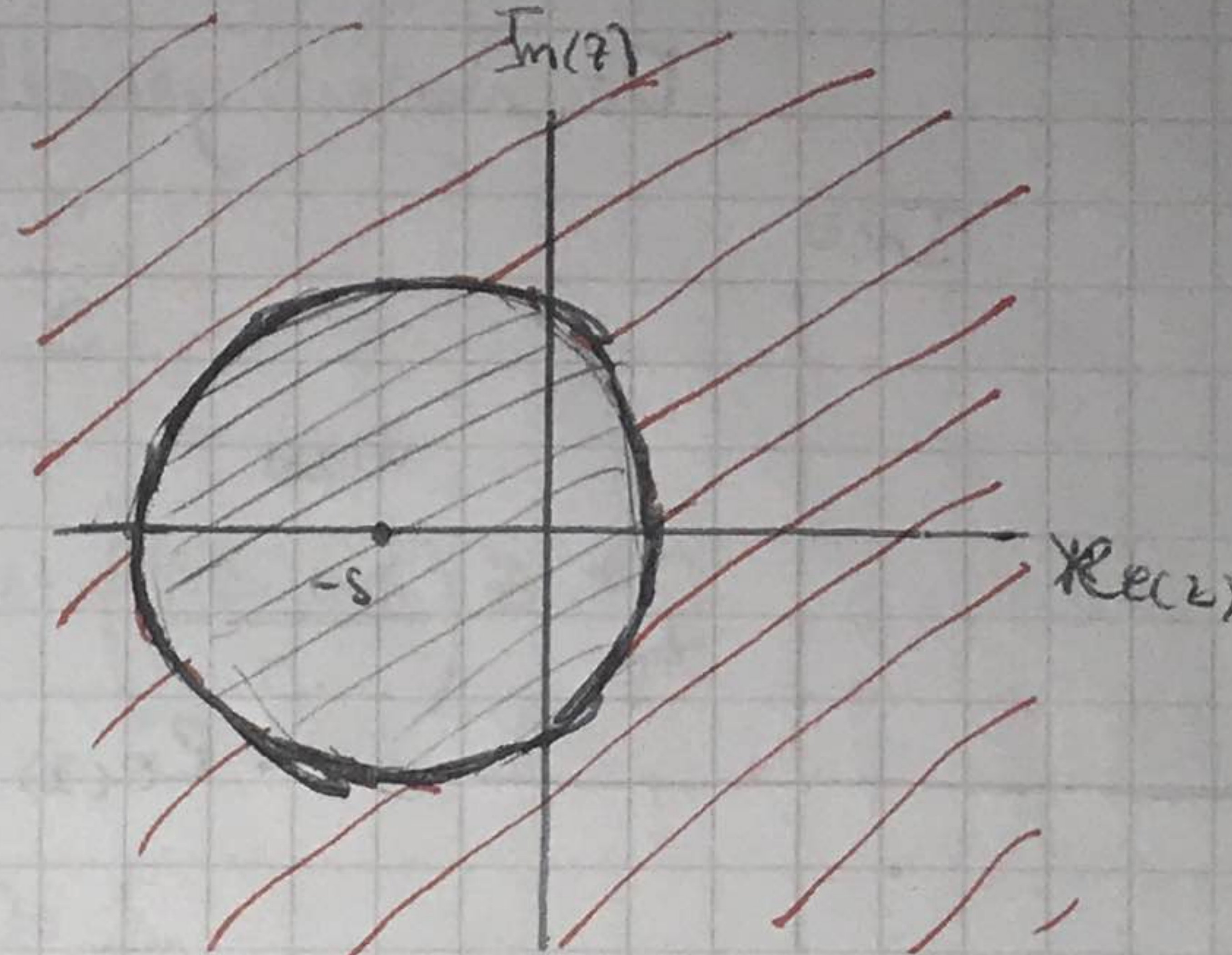
$-3x^2 - 3y^2 - 30x = 45$

$-3(x^2+y^2+10x) = 45$

$x^2+y^2+10x = -15$

$x^2 + 10x + 25 - 25 + y^2 = -15$

$(x+5)^2 + y^2 = 10$ circunferencia de radio $\sqrt{10}$ centrada en $(-5,0)$




en rojo $|z-3| < 2|z+3|$

⑥ Si $\alpha, \beta \in \mathbb{R}$, $c \in \mathbb{C}$, probar que $az\bar{z} + cz + \bar{c}\bar{z} + \beta = 0$ representa una circunf, una recta, un punto o $\{0\}$. Probar que toda recta o circunferencia puede escribirse de esta forma

$z\bar{z} = |z|^2 \Rightarrow$ si $c=0$ $\quad a|z|^2 = -\beta \Leftrightarrow a(x^2+y^2) = -\beta$

si $-\frac{\beta}{a} > 0 \Rightarrow$ circunf de radio $\sqrt{-\frac{\beta}{a}}$ $\quad x^2+y^2 = -\frac{\beta}{a}$

si $-\frac{\beta}{a} < 0 \Rightarrow \{0\}$ $\quad$ si $\beta/a = 0 \Rightarrow$ un punto $(0,0)$

si $a=0 \Rightarrow c = c_1 + ic_2$

$\Rightarrow (c_1+ic_2)(x+iy) + (c_1-ic_2)(x-iy) = c_1x + \cancel{ic_1y} + \cancel{ic_2x} - c_2y + c_1x - \cancel{ic_1y} - \cancel{ic_2x} - c$

$= 2(c_1x - c_2y)$

$\Rightarrow 2c_1x - 2c_2y + \beta = 0$

$c_1x + \frac{\beta}{2} = c_2y \qquad \overset{m}{\frac{c_1}{c_2}}x + \overset{b}{\frac{\beta}{2c_2}} = y \qquad mx+b=y$

circunferencia

$(x-x_0)^2 + (y-y_0)^2 = r^2$

$\Rightarrow x-x_0 = x' \quad y-y_0 = y' \qquad x'^2 + y'^2 = r^2$ $\qquad$ Sea $r^2 = -\frac{\beta}{a}$

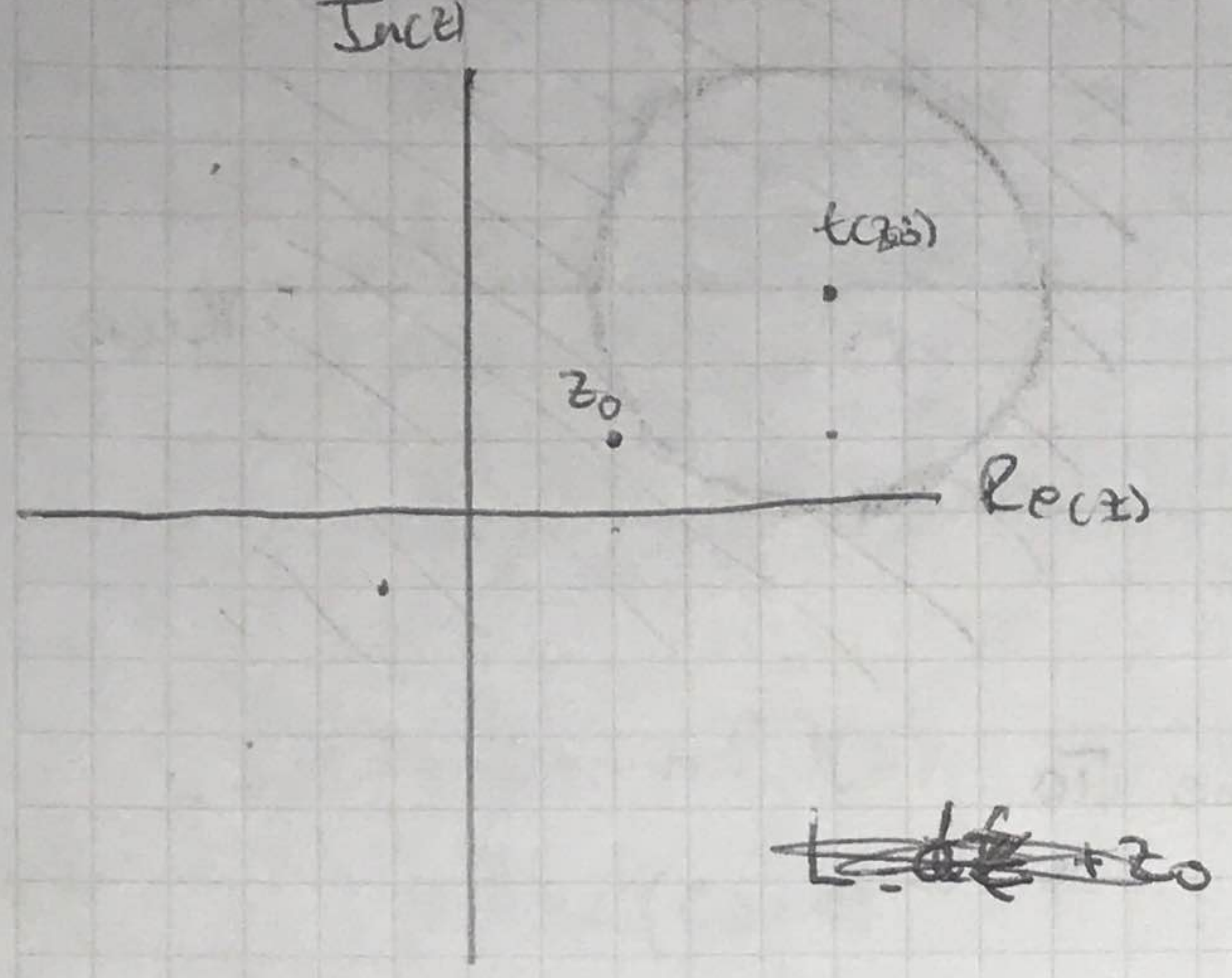
$\Rightarrow a(x'^2+y'^2) = -\beta$

7 a) Dadas $t(z) = z + c$, $c \in \mathbb{C}$ fijo (traslación)

$h(z) = a(z - z_0) + z_0$ con $a \in \mathbb{C}_{\neq 0}$, $z_0 \in \mathbb{C}$ (homotecia de centro $z_0$ y razón $a$)

$i(z) = \frac{1}{z}$, $z \neq 0$ (inversión)

Describirlas geométricamente. Imagen de cada una de una circunf y una recta




$L = \alpha z + z_0$ Circunf $z = x + iy$ / $x^2 + y^2 = r^2$

$t(L) = \alpha z + z_0 + c$ otra recta

$t(C) = \alpha(x + iy) + (x_0 + iy_0) + c$ circunf

$i(L) = \frac{1}{\alpha z + z_0} = \frac{1}{\alpha x + x_0 + i\alpha y + iy_0}$

$= \frac{(\alpha x + x_0) - i(\alpha y + y_0)}{(\alpha x + x_0)^2 + (\alpha y + y_0)^2}$ si $x_0 = y_0 = 0$ $\Rightarrow \frac{x}{\alpha(x^2+y^2)} - \frac{iy}{\alpha(x^2+y^2)}$

h manda circunf a circunf y lineal a lineal

$i(L) = \frac{Bu}{u^2+v^2} - \frac{Cv}{u^2+v^2} + D = 0$ $\quad D(u^2+v^2) + Bu - Cv = 0$

Si $D \neq 0 \Rightarrow$ circunf de centro $\left(-\frac{B}{2D}, \frac{C}{2D}\right)$ y radio $\frac{B^2+C^2}{4D^2}$ (recta que no pasa por el origen)

Si $D = 0 \Rightarrow$ a otra recta que pasa por el origen

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b) Probar que $f(z) = \dfrac{az+b}{cz+d}$ , $a,b,c,d \in \mathbb{C}$ / $ad-bc \neq 0$ (homografía)

Deducir Im(f) es una circunf ó una recta

si $c \neq 0 \Rightarrow \dfrac{bc-ad}{c}\left(\dfrac{1}{cz+d}\right) + \dfrac{a}{c}$

$\dfrac{bc}{c(cz+d)} - \dfrac{ad}{c(cz+d)} + \dfrac{a}{c}$

$\dfrac{\cancel{bc} - \cancel{ad} + a\cancel{c}z + \cancel{ad}}{\cancel{c}(cz+d)} = \dfrac{az+b}{cz+d}$

$h(z) = \left(\dfrac{bc-ad}{c}\right) z + \dfrac{a}{c}$

$i(z) = \dfrac{1}{z}$

$T(z) = cz+d$

$f(z) = (h \circ i \circ T)(z)$

$= h \circ i\,(cz+d)$

$= h\left(\dfrac{1}{cz+d}\right)$

$= \left(\dfrac{bc-ad}{c}\right)\dfrac{1}{cz+d} + \dfrac{a}{c}$ ✓

Si $z \in L$

$z = x+iy$ $\qquad h \circ i \circ T\,(z)$

lineal → lineal

circunf → circunf

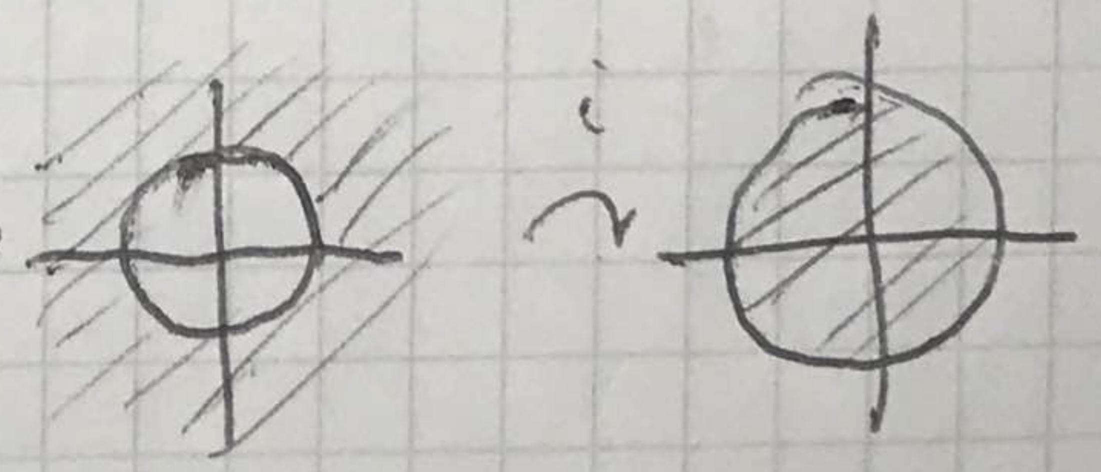
lineal → circunf o lineal

$i(C)$ $\qquad |z| = r$

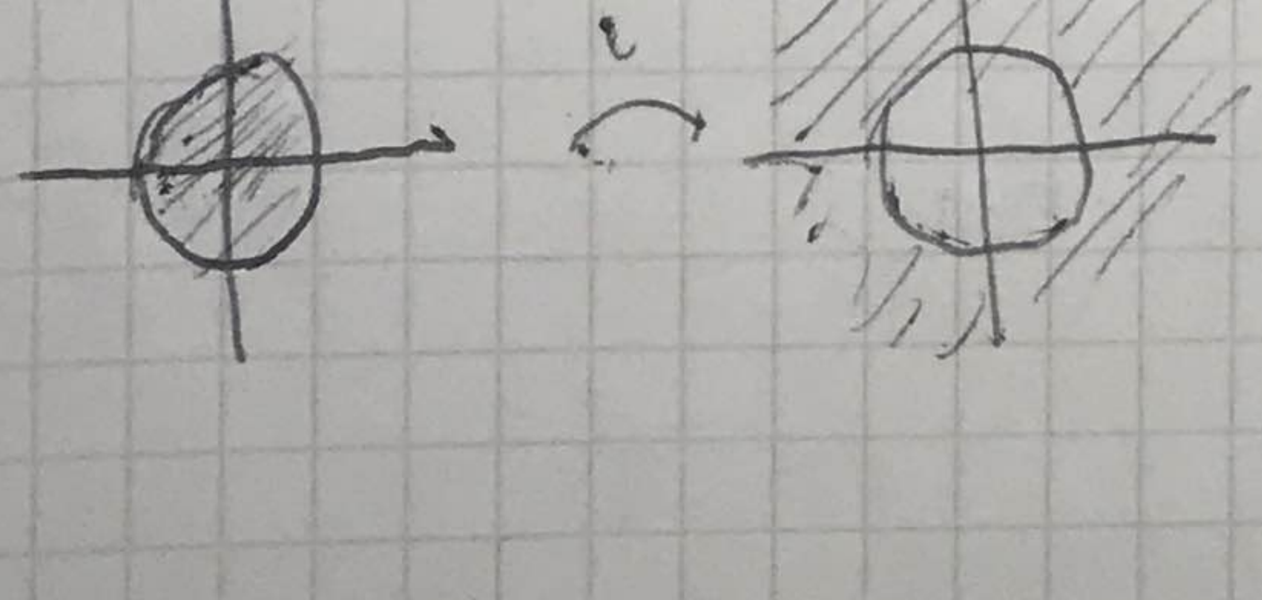
$i(z) = \dfrac{1}{z} = w$ $\qquad |z| = \left|\dfrac{1}{w}\right|$

circunf centrada en el origen

Si $|z| > 1 \Rightarrow |w| = \left|\dfrac{1}{z}\right| < 1 \Rightarrow$



si $|z| < 1 \Rightarrow |w| > 1$



NOTA

c) Verificar que $g(z) = \dfrac{dz-b}{-cz+a}$ es la homografía inversa de $\dfrac{az+b}{cz+d}$

$$g(z) = \frac{cz+d}{az+b}$$

$$i\left[\left(\frac{bc-ad}{c}\right) \cdot \frac{1}{cz+d} + \frac{a}{c}\right]$$

$$g(z) = \frac{cz^2+dz}{az^2+bz}$$

(8) Imagen

a) El cuadrante $\{z. Re(z)>0 \text{ e } Im(z)>0\}$ por $f(z) = \dfrac{z-i}{z+i}$

$$= \left(\frac{-i-i}{1}\right) \cdot \frac{1}{z+i} + 1$$

$$= \frac{-2i}{z+i} + 1$$

② Primeros términos y límites

a) $ni^n$

$= i, -2, -3i, 4, 5i, -6, -7i, 8$

no $\exists$ lim

b) $n\left(\frac{1+i}{2}\right)^n \nearrow n\left(\frac{1+i}{2}\right)\cdot\left|\frac{1+i}{2}\right|^{n-1} \to 0$ si n impar

$\searrow n\left|\frac{1+i}{2}\right|^n = n\left(\frac{1}{\sqrt{2}}\right)^n$ si n par

lim = 0

c) $\left(\frac{(-1)^n+i}{2}\right)^n \nearrow \left(\frac{(-1)^n+i}{2}\right)\cdot\frac{1}{\sqrt{2}^{\,n-1}} \to 0$ n impar

$\searrow \frac{1}{\sqrt{2}^{\,n}} \to 0$ si n par

d) $\cos(n\pi) + i\,\frac{\sin(\frac{n\pi}{2})}{n^2}$ $\quad -1+i, 1, -1-\frac{i}{9}$

oscila $\qquad \to 0$

$= (-1)^n + i\,0$

no $\exists$ lim

e) $\underbrace{\frac{n+1}{n}}_{\to 1} + i\underbrace{\left(\frac{n+1}{n}\right)^n}_{\to e}$ $\qquad \lim_{n\to\infty}\frac{n+1}{n} = \frac{1+\frac{1}{n}}{1} \to 1$

$\to 1+ei$

f) $\frac{e^{in\frac{\pi}{2}}}{n} = \frac{\cos(n\pi/2)+i\sin(n\pi/2)}{n} \to 0$

g) $\left(\frac{1+i}{\sqrt{2}}\right)^n \nearrow$ impar $\left|\frac{1+i}{\sqrt{2}}\right|$ no $\exists$ lim

$\searrow$ par $= 1$

h) $\left(\frac{1+i}{2}\right)^n \nearrow \left(\frac{1+i}{2}\right)\frac{1}{\sqrt{2}^{\,n-1}} \to 0$

$\searrow \left(\frac{1}{\sqrt{2}^{\,n}}\right) \to 0$

i) $z^{-n} = (re^{i\theta})^{-n} = (r)^{-n}(e^{i\theta})^{-n}$ $\qquad$ ~~si~~ si $0<r<1$ $z^{-n}\to\infty$

si $r=1$ $\quad z = e^{-in\theta} = \frac{1}{e^{in\theta}}$

si $r>1$ $\quad z^{-n}\to 0$

$= \frac{1}{\cos(n\theta)+i\sin(n\theta)}$

$\to 0$

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③ a) Probar $\lim_{z \to z_0} f(z) = L \Leftrightarrow \lim_{z \to z_0} \overline{f(z)} = \overline{L}$

$\lim_{z \to z_0} f(z) = L$

$\Rightarrow \lim_{z \to z_0} \overline{f(z)} = \overline{\left(\lim_{z \to z_0} f(z)\right)} = \overline{(L)} = \overline{L}$

$f: az + z_0$

$\Rightarrow f(z) = az + z_0 = R$

$\overline{f(z)} = \overline{a}\overline{z} + \overline{z}_0 = \overline{R}$

Análogo a la vuelta

b) $\lim_{z \to z_0} f(z) = L \Leftrightarrow \lim_{z \to z_0} Re(f(z)) = Re(L)$ y $\lim_{z \to z_0} Im(f(z)) = Im(L)$

$\Rightarrow$)

$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} Re(f(z)) + i Im(f(z)) = \lim_{z \to z_0} Re(f(z)) + \lim_{z \to z_0} i Im(f(z))$

$L = Re(L) + i Im(L) \qquad = L = Re(L) + i Im(L)$

$\Leftarrow$)

$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} Re(f(z)) + \lim_{z \to z_0} i Im(f(z)) = Re(L) + i Im(L) = L$

✓

④ Calcular

a) $\lim_{z \to -2i} \frac{z^2 + 2(1+i)z + 4i}{z + 2i} = \lim_{z \to -2i}$

si $a \neq 0 \Rightarrow z^2 + \frac{b}{a} z + \frac{c}{a} = 0 \Rightarrow$

$z^2 + \frac{b}{a} z + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad \left(z + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$

$\Rightarrow w^2 = -\frac{c}{a} + \frac{b^2}{4a^2} \qquad$ con $z = w - \frac{b}{2a}$

$w^2 = -\frac{4i}{1} + \frac{4(1+i)^2}{4}$

$(x+iy)^2 = x^2 + 2xiy - y^2$

$w^2 = -4i + \frac{\cancel{4}(1+2i-1)}{\cancel{4}} = -2i$

$w^2 = x^2 - y^2 + 2xyi = -2i$

$x^2 - y^2 = 0 \qquad x^2 = y^2 \qquad |x| = |y|$

$xy = -1 \qquad x = -\frac{1}{y} \Rightarrow \frac{1}{y^2} - y^2 = 0$

$y^2 = \frac{1}{y^2} \qquad y^4 = 1$

$y = \pm 1 \qquad \Rightarrow x = \mp 1$

$w_{1,2} = \pm 1 \mp i$

$z_1 = 1-i-(1+i) = -2i$

$z_2 = -1+\cancel{i}-1-\cancel{i} = -2$ $\qquad \Rightarrow (z+2i)(z+2)$

$$\lim_{z\to -2i} \frac{z^2+2(1+i)z+4i}{z+2i} = \lim_{z\to -2i} \frac{\cancel{(z+2i)}(z+2)}{\cancel{(z+2i)}} = \boxed{2-2i}$$

$\cancel{3+2i}\ \cancel{-8}\ \cancel{-2i}\ \cancel{+5}$

b) $$\lim_{z\to i} \frac{3z^4-2z^3+8z^2-2z+5}{z-i}$$

$$\begin{array}{l|l} 3z^4-2z^3+8z^2-2z+5 & z-i \\ -\ 3z^4-3iz^3 & 3z^3+(-2+3i)z^2+(5-2i)z+5i \\ 0-2z^3+3iz^3\ \ 8z^2 & \\ -\ (-2i+3i)z^3\ \ (2i+3)z^2 & \\ 0\ \ (5-2i)z^2-2z & \\ -\ (5-2i)z^2+(-5i-2)z & \\ 0\ \ \ 5iz+5 & \\ -\ \ 5iz+5 & \\ \ \ \ 0 & \end{array}$$

$(-2+3i)(-i) = 2i+3$

$(5-2i)(-i) = -5i-2$

$$= \lim_{z\to i} \frac{\cancel{(z-i)}(3z^3+(-2+3i)z^2+(5-2i)z+5i)}{\cancel{(z-i)}}$$

$= \boxed{4+4i}$

$-3i+(-2+3i)(-1)+(5-2i)i+5i$

$-3i+2-3i+5i+2+5i = 4+4i$

c) $\lim_{z\to -i} z.\bar{z} = 1$

$\frac{z}{|z|} = 1$

d) $$\lim_{z\to 0} \frac{\bar{z}^2}{z} = \lim_{z\to 0} \frac{\bar{z}^2}{z}.\frac{z}{z} = \frac{\bar{z}z.\bar{z}}{zz} = \lim_{z\to 0} \frac{\cancel{|z|^2}.\bar{z}}{\cancel{|z|^2}} = 0$$

$\lim_{z\to 0}$

$\frac{z}{\sqrt{z^2}} = \frac{z^2}{z^2} = 1$

$\bar{z}.z = (x+iy)(x-iy) = x^2+y^2$

$z.z = (x+iy)(x+iy) = x^2+2xiy-y^2$

$\bar{z}\bar{z} = x^2-2xiy-y^2 =$

$$\lim_{z\to 0} \frac{\bar{z}^2}{z}.\frac{\bar{z}}{\bar{z}} = \lim_{z\to 0} \frac{\bar{z}^2.z}{z^2} = \lim_{z\to 0} \frac{|z|^2.\bar{z}}{z^2}$$

$$= \lim_{z\to 0} \frac{\bar{z}}{\frac{z^2}{|z|^2}} = 0$$

$\frac{z^2}{|z|^2} = 1$

e) $\lim_{z\to i} f(z)$ con $f(z) = \begin{cases} z^2+2z, & z \neq i \\ 3+2i, & z = i \end{cases}$

$\lim_{z\to i^{\pm}} f(z) = \lim \lim_{z\to i^{\pm}} z^2+2z = -1+2i$ Pero $z = 3+2i$ $\therefore$ f no es continua

f) $\lim_{z \to \infty} \dfrac{z^3 - 3iz + 2+i}{z^4 + iz^2 - (3+4i)z + 6} = \lim_{z \to \infty} \dfrac{z^3\left(1 - \frac{3i}{z^2} + \frac{(2+i)}{z^3}\right)}{z^3\left(z + \frac{i}{z} - \frac{(3+4i)}{z^2} + \frac{6}{z^3}\right)} = 0$

⑤ Probar la continuidad de las siguientes funciones en el dominio indicado

a) $z, \bar{z}$, $Re(z)$ e $Im(z)$ en $\mathbb{C}$

$z = a + bi \qquad a, b \in \mathbb{R}$

$f$ continua a si $\lim_{z \to z_0^-} f = \lim_{z \to z_0^+} f = f(z_0)$

$\Rightarrow \lim_{z \to z_0^+} z = z_0^+$

$\lim_{z \to z_0^-} z = z_0^-$

$z(z_0) = z_0 \quad \therefore \quad z$ es continua

Analogo para $\bar{z}$, $Re(z)$ e $Im(z)$

b)

$\lim_{z \to z_0^\pm} \frac{1}{z} = \frac{1}{z_0} \qquad f(z_0) = \frac{1}{z_0}$

Si $z_0 = 0$

$\Rightarrow \lim_{z \to z_0^+} = \infty \quad \wedge \lim_{z \to z_0^-} = -\infty \quad \therefore$ no es continua en $z_0 = 0$

⑥ Hallar los puntos de discontinuidad de

a) $f(z) = \dfrac{z}{z^4 + 1}$ discontinua si $z^4 = -1$

$w = z^2$

$\Rightarrow z^4 = -1 \quad \Rightarrow w^2 = -1 \qquad w = \pm i$

$|z|^4 = 1 \Rightarrow |z| = 1$

$w_1 = i = z^2 \quad z = \pm\sqrt{i}$

$w_2 = -i = z^2$

$\sqrt{-i} = \sqrt{i^2 i} = i\sqrt{i} \quad z = \pm i\sqrt{i}$

$(i\sqrt{i})^2 = (-1)i \quad (\pm\sqrt{i}) = 1\sqrt{i}\,(i\sqrt{i}) = \pm 1$

$f: \mathbb{C} - \{\pm\sqrt{i}, \pm i\sqrt{i}\}$

b) $f(z) = \frac{1}{e^x(\cos(y) + i\sin(y)) + 1}$ $\quad \frac{1}{e^x . e^{iy} + 1}$

$\frac{1}{e^z + 1}$

$e^x i \sin(y) = 0$

$\sin(y) = \cancel{\pm 0}\ \pm k\pi$

$\Rightarrow e^x(\pm 1) + 1 = 0$

$e^x + 1 = 0$ absurdo

$-e^x + 1 = 0$

$e^x = |r|$

$x = \ln(1) = 0$ $\quad |r| + 1 = 0$ absurdo $\quad z = 1(\cos(k\pi) + i\sin(k\pi))$ $k = -\pi, -2\pi$

$-|r| + 1 = 0$

$|r| = 1$

$y = \pm k\pi$

$k = -1, 1, 3 \ldots$

~~$f: \mathbb{C} - \{0, z = -1\} \to \mathbb{C}$~~

$f: \mathbb{C} - \{ z = ik\pi \ / \ k = 2t - 1 \} \to \mathbb{C}$
$t \in \mathbb{Z}$

⑦ Sea $\varphi: \mathbb{C}_{\neq 0} \to (-\pi, \pi]$ definida por: $\varphi(z)$ es el único número de $(-\pi, \pi]$ / $z = |z| e^{i\varphi(z)}$. ¿Es continua en todo su dominio? ¿Dónde lo es?

$x + iy = \sqrt{x^2 + y^2}\, e^{i\varphi(z)}$ $\qquad z = |z| e^{i\,Arg(z)}$

$|z|$ continua en $\mathbb{C}$ $\qquad$ Arg continua en $\mathbb{C} - (-\infty, 0]$

⑧ Puntos de discontinuidad $z \neq 0$

a) $f(z) = arg(z)$ $\qquad \mathbb{C} - \{-\infty, 0]$

b) $f(z) = \underbrace{\log|z|}_{\text{continua excepto en } 0} + i\,Arg(z)$

HOJA N°

FECHA

9) ¿Cuales de las siguientes pueden definirse en $z=0$ / resulten continuas en $\mathbb{C}$?

a) $\frac{Re(z)}{|z|^2} = \frac{x}{x^2+y^2}$

$\lim_{\substack{x\to 0^\pm \\ y=0}} = \frac{x}{x^2} = \lim_{\substack{x\to 0^\pm \\ y=0}} \frac{1}{x} = \pm\infty$ ∴ no

0 si $x=0$ $\Rightarrow$ $=0$

b) $\frac{Re(z)}{|z|} = \frac{x}{\sqrt{x^2+y^2}}$ $\lim_{\substack{x\to 0^\pm \\ y=0}} = \frac{x}{|x|}$ si $x\to 0^+ = 1$ si $x\to 0^- = -1$

si $x=0$ $\frac{0}{|0|}$ ∴ no puede

c) $\frac{(Re(z))^2}{|z|} = \frac{x^2}{\sqrt{x^2+y^2}}$ $\lim_{x\to 0^\pm} \frac{x^2}{|x|} =$ $x\to 0^+ = \frac{x^2}{x} = x \to 0$ ; $x\to 0^- = \frac{x^2}{-x} = x \to 0$

si $y=0$ $=0$ a principio pareceria ser reescribible

d) $\frac{Re(z^2)}{|z|^2}$ $\left|\frac{x^2}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{|z|^2}{|z|}\right|$

$x^2 - y^2$ $\frac{x^2-y^2}{x^2+y^2} = \lim_{\substack{x\to 0 \\ y=0}} = 1$ $\lim_{\substack{x=0 \\ y\to 0}} = -\frac{y^2}{y^2} = -1$ no es reescribible

Práctica 3

① a) $f(z) = z$ $\quad \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \frac{z+h-z}{h} = 1$

$f'(z) = 1$

b) $f(z) = z^2 = \lim_{h\to 0} \frac{(z+h)^2 - z^2}{h} = \frac{\cancel{z^2}+2hz-\cancel{z^2}+h^2}{h} = 2z+h = 2z$

$f'(z) = 2z$

c) $f(z) = \frac{1}{z}$ $\quad \lim_{h\to 0} \frac{\frac{1}{(z+h)} - \frac{1}{z}}{h} = \frac{\cancel{z}-\cancel{z}-h}{hz(z+h)} = -\frac{1}{z(z+h)} = -\frac{1}{z^2}$

$z \neq 0$

$f'(z) = -\frac{1}{z^2}$

② $f: \mathbb{C} \to \mathbb{C}$ derivable en $z_0 \in \mathbb{C}$, $f(z) = u(x,y) + iv(x,y)$

a) Calcular:

i) $\lim_{\substack{h\to 0 \\ h\in\mathbb{R}}} \frac{f(z_0+h)-f(z_0)}{h}$ ; $\lim_{\substack{h\to 0 \\ h\in\mathbb{R}}} \frac{f(z_0+ih)-f(z_0)}{ih}$ ii)

en términos de $u$ y $v$.

¿Qué se deduce?

i) $= \lim_{h\to 0} \frac{u(x_0+h, y_0+h) + iv(x_0+h, y_0+h) - u(x_0,y_0) - iv(x_0,y_0)}{h}$

$=$

ii) $\lim_{h\to 0} \frac{u(x_0+ih, y_0+ih) + iv(x_0+ih, y_0+ih) - u(x_0,y_0) - iv(x_0,y_0)}{ih}$

¿

$f(x,y) = (u(x,y), v(x,y))$ $\quad f[(x_0,y_0)+(h_1,h_2)] = f(x_0,y_0) + Df(x_0,y_0)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \|(h_1,h_2)\|$

i) $= \frac{u(x_0+h, y_0+h) - u(x_0,y_0)}{h}$

b) Suponiendo que $u$ y $v$ son $C^2$, calcular $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$ en $z_0$

$\partial \quad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

⑤ Calcular

a) $\lim\limits_{z \to i} \frac{z^{10}+1}{z^6+1} = \frac{0}{0}$ ? $\quad i^{10} = i\,i\,i\,i\,i\,i\,i\,i\,i\,i$ $\quad \lim\limits_{h\to 0} \frac{(z+h)^{10} - z^{10}}{h} =$

L'Hopital $\lim\limits_{z\to i} \frac{(z^{10}+1)'}{(z^6+1)'} = \lim\limits_{z\to i} \frac{10z^9}{6z^5} = \lim\limits_{z\to i} \frac{5}{3} z^4 = \frac{5}{3}$

b) $\lim\limits_{z\to 2i} \frac{z^2+4}{2z^2+(3-4i)z-6i}$ = L'Hospital $\lim\limits_{z\to 2i} \frac{2z}{4z+(3-4i)-6} = \frac{4i}{8i+3-4i-6} = \frac{4i}{4i-3}$

c) $\lim\limits_{z\to e^{\frac{\pi i}{3}}} \frac{z-e^{\frac{\pi i}{3}}}{z^3+1} = \lim\limits_{z\to e^{\frac{\pi i}{3}}} \frac{1}{3z^2} = \frac{1}{3e^{\frac{2\pi i}{3}}}$

d) $\lim\limits_{z\to i} \frac{z^2-2iz+1}{z^4+2z^2+1} = \lim\limits_{z\to i} \frac{2z-2i}{4z^3+4z} = \lim\limits_{z\to i} \frac{2}{12z^2+4} = \frac{2}{-8}$

⑥ Sea $\gamma: \mathbb{R} \to \mathbb{C}$. Condiciones necesarias y suficientes sobre su parte real e imaginaria / sea derivable en $a \in \mathbb{R}$ y calcular $\gamma'(a)$. Calcular $\gamma'(t)$ para $\gamma(t) = \cos(t) + i\sin(t)$

$\gamma(t) = u + iv \quad \to \gamma'(t) = \frac{\partial u}{\partial t} + i\frac{\partial v}{\partial t}$

⑦ Sean $f, g: \mathbb{C} \to \mathbb{C}$ dadas por

$f(x,y) = \sqrt{|xy|}$

$f, g$ continuas en $0$ y cumplen C-R pero no derivables

$\lim\limits_{(x,y)\to 0} = 0 = f(0,0) = 0 \Rightarrow f$ es continua

$\frac{\partial \sqrt{x^2}}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2}} = \frac{x}{|x|}$

$$\frac{\partial f}{\partial x} = \sqrt{\sqrt{x^2 y^2}}' = (xy)^{2/4}{}' = (xy)^{1/2}{}' = \frac{|y|^{1/2}}{2|xy|^{1/2}} \cdot \frac{xy}{|xy|}$$

$$= \frac{xy|^{1/2}}{2|xy|^{3/2}} \qquad = \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial x} \sqrt{|y|}\sqrt{|x|} = \frac{\sqrt{|y|}}{2} \frac{}{\sqrt{|x|}} \cdot \frac{x}{|x|} = \frac{x|y|^{1/2}}{2|x|^{3/2}}$$

$$\frac{\partial f}{\partial y} = \frac{y|x|^{1/2}}{2|y|^{3/2}}$$

$$\nabla f = \left( \frac{x|y|^{1/2}}{2|x|^{3/2}}, \frac{y|x|^{1/2}}{2|y|^{3/2}} \right)$$

(9) C-R polar

$x = r\cos(\varphi)$
$y = r\sin(\varphi)$

$u_x = v_y \quad \Rightarrow \quad u_r = u_x \cos(\varphi) + u_y \sin(\varphi)$

$u_y = -v_x \quad u_\theta = u_x(-r\sin(\theta)) + u_y r\cos(\varphi) = r(u_x(-\sin\theta) + u_y \cos$

$\cancel{u_\theta = r u_{v\theta}}$

$v_r = V_x \cos(\varphi) + V_y \sin(\varphi)$

$V_\theta = r(V_x(-\sin\varphi) + V_y \cos(\varphi))$

$V_x = u_y \qquad V_\theta = r(u_y \sin(\varphi) + u_x \cos(\varphi)) = r u_r$

$V_\theta = r u_r \qquad V_r = -u_y \cos(\varphi) + u_x \sin(\varphi)$

$\boxed{\frac{\partial v}{\partial \theta} \cdot \frac{1}{r} = \frac{\partial u}{\partial r}}$ ✓ $\qquad V_r = -\frac{u_\theta}{r}$

$\boxed{\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}\,\frac{1}{r}}$ ✓

b) $f' = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{e^{-i\theta}}{r}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right)$

~~$f' = u_x + i v_x = \frac{u_r - u_y \sin(\theta)}{\cos(\theta)} + i u_y = \frac{u_r}{\cos(\theta)}$~~

⑩ Donde f es derivable y donde es holomorfa

a) $f(z) = \begin{cases} \frac{x+iy}{x^2+y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$ $\quad \frac{z}{|z|}$

derivable $\forall z - \{\cancel{0}\}$ $\quad x^2 = -y^2$ $\quad \cancel{x = iy}$ $\quad z - \{0\}$

~~$z - \{z \in \mathbb{C} / z = x + ix$~~

$\frac{\partial u}{\partial x} = \frac{x^2 - 2x^2}{(x^2+y^2)^2} = \frac{-x^2}{x^2+y^2}$

$\frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$ $\quad -2xiy$

$Df = \begin{pmatrix} \frac{-x^2}{(x^2+y^2)^2} & \frac{-2xy}{(x^2+y^2)^2} \\ -\frac{2xy}{(x^2+y^2)^2} & \frac{-iy^2}{(x^2+y^2)^2} \end{pmatrix}$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\quad -x^2 = -iy^2$ $\quad y = x = 0$

$\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x}$ $\quad -2i = -2$ $\quad$ f no es holomorfa y no es derivable en ningún punto

b) $f(z) = \bar{z}$

$\lim_{h \to 0} \frac{\overline{z+h} - \bar{z}}{h} = \frac{\bar{h}}{h}$ derivable en ningún punto

c) $f(z) = x^2 + iy^2$ $\quad \frac{\partial u}{\partial x} = 2x$ $\quad \frac{\partial v}{\partial y} = 2y$ $\to$ derivables en $z = 0$

$\frac{\partial u}{\partial y} = 0$ $\quad \frac{\partial v}{\partial x} = 0$

d) $f(z) = x^2 - y^2 - 2xy + i(x^2 - y^2 + 2xy)$

$\frac{\partial u}{\partial x} = 2x - 2y$ $\quad \frac{\partial u}{\partial y} = \cancel{-2y - 2y - 4y} - 2y - 2x$ $\quad$ ~~$2x - 2y = -2y + 2x$~~

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ✓

$\frac{\partial v}{\partial x} = 2x + 2y$ $\quad \frac{\partial v}{\partial y} = -2y + 2x$ $\quad 2x + 2y = -2y - 2x$ ✓

f es $\mathbb{C}$ derivable $\forall z \in \mathbb{C}$

(11) $f(z) = \dfrac{3+2z}{i+2z}$ en : $D\ |z|<1$

$z = x+iy$

$$\lim_{h\to 0} \frac{\dfrac{3+2(z+h)}{i+2(z+h)} - \dfrac{3+2z}{i+2z}}{h} =$$

$$f(z) = \frac{3+2x+2iy}{i+2x+2iy} \qquad z = h(i(T(z)))$$

$$\frac{az+b}{cz+d} = \frac{3+2z}{i+2z} = \frac{a}{c} - \frac{ad-bc}{c}\frac{1}{cz+d} = 1 - \frac{2i-6}{2}\cdot\frac{1}{i+2z} = 1 - \left[\frac{i-3}{i+2z}\right]$$

$$\lim_{h\to 0} \frac{\cancel{1} - \dfrac{i-3}{i+2(z+h)} - \cancel{1} + \dfrac{i-3}{i+2z}}{h} = \frac{i-3}{h}\left(\frac{-1}{i+2(z+h)} + \frac{1}{i+2z}\right)$$

$$= \frac{i-3}{\cancel{h}}\cdot\left[\frac{\cancel{-i}\cancel{-2z}\cancel{+i}\cancel{+2z}+2\cancel{h}}{(i+2(z+h))(i+2z)}\right] = \frac{i-3}{(i+2(z+h))(i+2z)} = \text{algo}$$

$\therefore$ es $\mathbb{C}$ derivable en $\mathbb{C} \in D \Rightarrow f$ es holomorfa en $D$

b) $f(z) = \cos(x)$ $\qquad |z|<1$

$$\frac{\partial u}{\partial x} = -\sin(x) \qquad \frac{\partial u}{\partial y} = 0 \qquad -\sin(x) = 0$$

$$x = n\pi$$

$z = |r|(\cos(\theta) + i\sin(\theta))$ $\qquad r\cos(\theta)$

$f$ es $\mathbb{C}$ deriv en $0$ $\therefore$ no es holomorfa en ninguna parte

c) $f(z) = e^{-y}(\cos(x) + i\sin(x))$ en $\mathbb{C}$

$$\frac{\partial u}{\partial x} = e^{-y}(-\sin(x)) \qquad \frac{\partial u}{\partial y} = -e^{-y}\cos(x)$$

$$\frac{\partial v}{\partial x} = e^{-y}\cos(x) \qquad \frac{\partial v}{\partial y} = -e^{-y}\sin(x) \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \forall x y \ ✓$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \ ✓$$

$\Rightarrow$ es holomorfa

(12) a) $f(z) = z^3 - 2z =$

$$\lim_{h\to 0} \frac{(z+h)^3 - 2(z+h) - z^3 + 2z}{h} \quad \frac{\left[(z^2+2zh+h^2)(z+h) - 2\right](z+h) - z^3 + 2z}{h}$$

$$= \frac{\cancel{z^3} + 2z^2 h + h^2 z - \cancel{2z} + hz^2 + 2zh^2 + h^3 - 2h - \cancel{z^3} + \cancel{2z}}{\cancel{h}}$$

$\mathbb{C}$ deriv en todo $\mathbb{C}$

Por composición de funciones holomorfas $\Rightarrow$ es holomorfa en $\mathbb{C}$

$f'(z) = 3z^2 - 2$

b) $f(z) = \dfrac{z+1}{1-z} = \dfrac{z+1}{-z+1} = \dfrac{a}{c} - \dfrac{ad-bc}{c} \cdot \dfrac{1}{cz+d}$ $\quad z \neq 1$

$$= -1 - \left(\frac{1+1}{-1}\right) \cdot \left(\frac{1}{-z+1}\right) = -1 + \frac{2}{-z+1}$$

$$\lim_{h\to 0} \frac{\cancel{-1} + \frac{2}{-z-h+1} \cancel{+1} - \frac{2}{-z+1}}{h} = \frac{1}{h}\left(\frac{2(-z+1) - 2(-z-h+1)}{(-z-h+1)(-z+1)}\right)$$

$= \frac{1}{\cancel{h}} \cdot \left(\cancel{-2z + 2 + 2z + 2h - 2}\right)$ $\quad \therefore \mathbb{C}$ deriv si $z \neq 1$

$$= \frac{2}{(-z-h+1)(-z+1)} = \frac{2}{(-z+1)^2} \quad \text{holomorfa en } \mathbb{C} - \{1\}$$

c) $f(z) = z^2 \cdot \bar{z} = (x^2 - y^2 + 2xyi)(x - iy) = x^3 - iyx^2 - xy^2 + iy^3 + 2x^2yi + 2xy^2$

$= x^3 + xy^2 + i(y^3 + x^2 y)$

$$\frac{\partial u}{\partial x} = 3x^2 + y^2 \qquad \frac{\partial v}{\partial y} = 3y^2 + x^2$$

$$\frac{\partial u}{\partial y} = 2xy \qquad \frac{\partial v}{\partial x} = 2xy$$

$\Rightarrow 3x^2 + y^2 = 3y^2 + x^2$

$\quad$ si $x = 0 \Rightarrow y^2 = 3y^2 \quad y = 0$

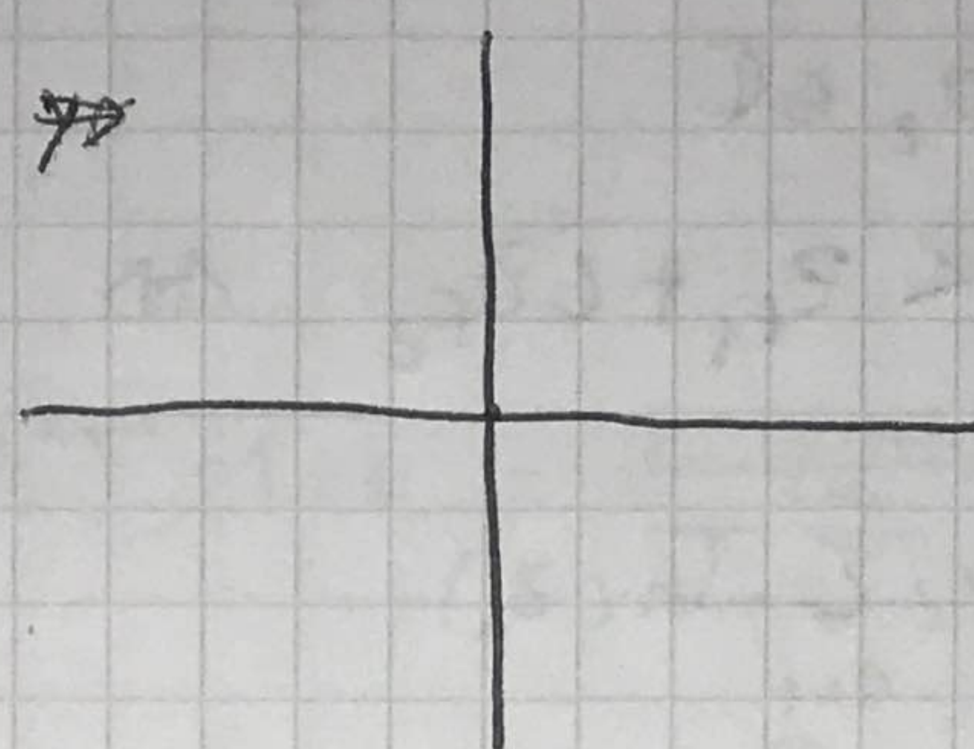
$2xy = -2xy \Rightarrow x = 0$ o $y = 0$ $\Rightarrow$ no es holomorfa, $\mathbb{C}$ deriv en $x, y = 0$

$f'(z) = 3x^2 + y^2 + 2xyi \quad x, y = 0 \quad = 0$

d) $f(z) = x^2 + iy^3$

$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = 3y^2$

$2x = 3y^2 \qquad x = \frac{3}{2}y^2 \Rightarrow$



holomorfa sobre la parabola $x = \frac{3}{2}y^2$

$f'(z) = 2x$

e) $f(z) = e^x(\cos(y) + i\sin(y))$

$\frac{\partial u}{\partial x} = e^x\cos(y) \qquad \frac{\partial v}{\partial x} = e^x\sin(y) \qquad \frac{\partial u}{\partial y} = -e^x\sin(y)$

$\frac{\partial v}{\partial y} = e^x\cos(y)$

holomorfa en $\mathbb{C}$

(13) Sea $\Omega \subset \mathbb{C}$ abierto y conexo. Probar:

a) Si $f$ y $f'$ son holomorfas en $\Omega \Rightarrow f$ es constante

$\Leftarrow$) Si $f = $ cte $\Rightarrow f' = 0 \therefore$ son holomorfas

$\Rightarrow$)

Práctica 4

① Probar que si $\overbrace{\sum_{n=1}^{\infty} z_n}^{a}$ converge || $\Leftrightarrow$ $\overbrace{\sum_{n=1}^{\infty} Re(z_n)}^{b}$, $\overbrace{\sum_{n=1}^{\infty} Im(z_n)}^{c}$ convergen ||

$\Rightarrow$ Si a converge $\Rightarrow \exists\ z_F \in \mathbb{C}$ /

$a \leqslant z_F$

pero $a = a_1 + ia_2 \in \mathbb{C}$

$\Rightarrow a_1 + ia_2 < z_{F_1} + iz_{F_2} \Leftrightarrow a_1 < z_{F_1}$

$a_2 < z_{F_2}$

$a_1 = \sum_{n=1}^{\infty} Re(z_n) \qquad a_2 \sum_{n=1}^{\infty} Im(z_n)$

b) Si $\sum_{n=1}^{\infty} z_n$ converge ||

$\Rightarrow \left|\sum_{n=1}^{\infty} z_n\right| \leq \sum_{n=1}^{\infty} |z_n|$

$\Rightarrow \sum_{n=1}^{\infty} |z_n|$ converge $\therefore \exists c$ / $\sum_{n=1}^{\infty} |z_n| \leq C$

$y = |z_1| + |z_2| + \dots + |z_n| \leq C$

$|a+b+\dots+c_n| \leq |a| + |b| + \dots + |c_n|$

$\Rightarrow$ ✓

②a) Sea $a \in \mathbb{C}$, $|\alpha| < 1$, ¿Cuánto vale $\lim_{n\to\infty} a^n$? Demo

$\lim_{n\to\infty} a^n = a.a.a\dots \qquad a = e^{i\theta}$

$r < 1$

$\lim_{n\to\infty} r^n e^{ni\theta} = \lim_{n\to\infty} \overset{\to 0}{r^n}(\cos(n\theta) + i\sin(n\theta))$ ~~si $\theta = 2p\pi$ $p\in\mathbb{Z}$ $\Rightarrow$ converge~~

$n \to h$

converge $1, 0, i$ $\left(\frac{\pi}{2} + 2k\pi\right)$ ~~no converge~~

b) $|a| > 1$ diverge

$|a| = 1 \Rightarrow \theta = 2p\pi$ $p\in\mathbb{Z} \Rightarrow$ converge

c) Probar que $\sum_{n=1}^{\infty} z^n$ converge si $|z| < 1$ y diverge si $|z| \geqslant 1$

~~$z^n \cdot z = z^{n+1}$~~

$= 1 + \cancel{z + z^2 + z^3 + \dots + z^n} = \cancel{(1-z)(1+z+z^2+\dots+z^{n-1})} = 1 - r^{n+1}$

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$S_n = 1 + z + z^2 + z^3 + \dots + z^n$

$(1-z)S_n = (1-z)(1+z+z^2+\dots+z^n) = 1 - \cancel{z} + \cancel{z} - \cancel{z^2} + \dots + \cancel{z^n} - z^{n+1}$

$\Rightarrow S_n = \dfrac{1-z^{n+1}}{1-z}$ converge si $|z|<1$ y diverge si $|z|>1$

$$\sum_{n=1}^{\infty} z^n = \sum_{n=0}^{\infty} z^n - 1$$

$$= \frac{1-z^{n+1}}{1-z} - 1 = \frac{\cancel{1}-z^{n+1}-\cancel{1}+z}{\cancel{1-z}} = \frac{z - z^{n+1}}{1-z} = \frac{z(1-z^n)}{(1-z)}$$

$$S_n = z + z^2 + z^3 + \dots + z^n = z(1+z+\dots+z^{n-1}) = z\sum_{n=0}^{\infty} z_n^{n+1}$$

$$= \frac{z(1-z^n)}{(1-z)} \Rightarrow \quad \text{si } |z|<1 \Rightarrow z^n \to 0$$

$$= \frac{z}{1-z}$$

b) $\Rightarrow z = re^{i\theta} \quad 0<r<1$

$\Rightarrow$ ~~$\sum_{n=1}^{\infty} r^n \cos(n\theta) = r\cos(n\theta)(1-$~~

$$\Rightarrow \sum_{n=1}^{\infty} z^n = \frac{re^{i\theta}(1-r^n e^{ni\theta})}{1-re^{i\theta}}$$

$$= \frac{r[\cos(\theta)+i\sin(\theta)]}{1-r[\cos(\theta)+i\sin(\theta)]} = \frac{re^{i\theta}(1+re^{i\theta})}{1-r^2e^{2i\theta}} =$$

$$\frac{re^{i\theta}+r^2e^{2i\theta}}{1-r^2e^{2i\theta}} = \frac{r[\cos(\theta)+i\sin(\theta)]^2+r^2}{1-r^2[\cos(\theta)+i\sin(\theta)]^2} \qquad \cos(2\theta) =$$

NOTA

4) a) $\frac{2i}{3^n}$ $\quad 2i\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ converge

b) $\left(\frac{-3}{2}\right)^n$ no converge

c) $\frac{2^n}{5^n-n} =$ $\qquad 5^n - n < 5^n$

$= \left(\frac{2}{5}\right)^n \overbrace{\left(\frac{1}{1-n/5^n}\right)}^{a_n}$ $\qquad \lim_{n\to\infty} \frac{\cancel{\left(\frac{2}{5}\right)^n}\left(\frac{1}{1-n/5^n}\right)}{\cancel{\left(\frac{2}{5}\right)^n}} = 1$

y $v_n$ converge, pues es serie geométrica de razón $\frac{2}{5} < 1$

d) $\frac{1}{n!} =$ $\qquad n! > n^2$ $\qquad\qquad n! > 2^n$

$\cancel{1 > 1}$

$\cancel{2 > 4}$ $\qquad \Rightarrow n! > n^2 \ \forall n \geq 4$

$6 > 9$

$24 > 16$

$120 > 25$

| | | |
|---|---|---|
| $n=1$ | 1 | 2 |
| $n=2$ | 2 | 4 |
| $n=3$ | 6 | 8 |
| $n=4$ | 24 | 16 |

$\Rightarrow \frac{1}{n^2} =$ $\qquad \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} \leq \int_0^{l\to\infty} \frac{1}{x^2}\,dx = -\frac{1}{x} \to 0$

$\frac{1}{n!} < \frac{1}{2^n}$ $\quad \forall n > 4 \Rightarrow$ como $\left(\frac{1}{2}\right)^n$ converge n! también

e) $\frac{n}{n^n} = \frac{1}{n^{n-1}}$ $\qquad \frac{n}{n^n}$ Cauchy

$\lim_{n\to\infty} \frac{\sqrt[n]{n}}{\sqrt[n]{n^n}} = \frac{\sqrt[n]{n}}{n} \to 0$

$\overset{\to 0}{\sqrt[n]{n}} \cdot \overset{\to 0}{\frac{1}{n}} = 0$

$\Rightarrow$ converge

f) $\frac{n}{n^2-n} = \frac{1}{(n-1)}$ $\qquad n > n-1$ $\qquad \frac{1}{n-1} > \frac{1}{n}$

$\Rightarrow$ y $\frac{1}{n}$ diverge $\Rightarrow \frac{1}{n-1}$ diverge

g) $\sin\left(\frac{1}{n^2}\right)$

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# Práctica 6

Ejercicio 1 : Sea $f(z) = \frac{1}{z(z-1)(z-2)}$. Hallar el desarrollo en serie de Laurent de $f$ en los siguientes anillos

a) $0<|z|<1$

Primero desarrollo la función

$$\frac{A}{z} + \frac{B}{(z-1)} + \frac{C}{(z-2)} = \frac{1}{z(z-1)(z-2)} = \frac{A(z-1)(z-2)+Bz(z-2)+Cz(z-1)}{z(z-1)(z-2)}$$

Si $z=3$ $A(2)+B(3)+C(6)=1$

Si $z=4$ $6A+8B+12C=1$

Si $z=-1$ $6A+3B+2C=1$

$$\left(\begin{array}{ccc|c} 2 & 3 & 6 & 1 \\ 6 & 8 & 12 & 1 \\ 6 & 3 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 3 & 6 & 1 \\ 0 & -1 & -6 & -2 \\ 0 & \underset{1}{\cancel{-8}} & \underset{2}{\cancel{-10}} & 0 \end{array}\right)$$

$B=-2C \Rightarrow 2C-6C=-2 \Leftrightarrow C=\frac{1}{2}$

$B=-1$ $A=1-\cancel{3}+\cancel{3}$

$A=\frac{1}{2}$

$$\Rightarrow f(z) = \frac{1}{2}\frac{1}{z} - \frac{1}{(z-1)} + \frac{1}{2}\frac{1}{(z-2)}$$

El desarrollo de Laurent para este anillo $\left|\frac{1}{z}\right|>1$

$0<|z|<1$

En $-\frac{1}{(z-1)} = \frac{1}{(1-z)} = \sum_{n=0}^{\infty} z^n$

$|z|<1$

No se trabaja con el término en el que se esta centra

$$\boxed{f(z) = \frac{1}{2}\frac{1}{z} + \sum_{n=0}^{\infty} z^n + \frac{1}{2}\frac{1}{(z-2)}}$$

b) $1<|z|<2$

$\left|\frac{1}{z}\right|<1$

En $\frac{-1}{(z-1)} = \frac{-1}{z(1-\frac{1}{z})} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$

$$\frac{1}{2}\frac{1}{(z-2)} = \frac{1}{4}\frac{1}{(\frac{z}{2}-1)} = -\frac{1}{4}\frac{1}{(1-\frac{z}{2})} = -\frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{2^n} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}}$$

$\left|\frac{z}{2}\right|<1$

$$\boxed{f(z) = \frac{1}{2}\frac{1}{z} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{z^n}{2^{n+2}}}$$

NOTA

c) $|z|>2 \Rightarrow \left|\frac{1}{z}\right|<2$ además $\left|\frac{1}{z}\right|<1$ y $\left|\frac{2}{z}\right|<1$

$$\frac{-1}{(z-1)} = \frac{1}{(1-z)} = \frac{1}{z}\frac{-1}{\left(1-\frac{1}{z}\right)} \underset{\left|\frac{1}{z}\right|<1}{=} -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

$$\frac{1}{2}\cdot\frac{1}{(z-2)} = \frac{1}{2z}\cdot\frac{1}{\left(1-\frac{2}{z}\right)} = \frac{1}{2z}\sum_{n=0}^{\infty}\frac{2^n}{z^n} = \sum_{n=0}^{\infty}\frac{2^{n-1}}{z^{n+1}}$$

$$\boxed{f(z) = \frac{1}{2}\frac{1}{z} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{2^{n-1}}{z^{n+1}}}$$

e) $0<|z-1|<1 \Rightarrow$ Busco un desarrollo de potencias de $z-1$

$$\frac{1}{2}\cdot\frac{1}{z} = \frac{1}{2}\cdot\frac{1}{1+(z-1)} = \frac{1}{2}\cdot\frac{1}{1-\underbrace{(1-z)}_{<1}} = \frac{1}{2}\sum_{n=0}^{\infty}(1-z)^n = \sum_{n=0}^{\infty}\frac{(1-z)^n}{2}$$

$|z-1| = |1-z| < 1$

$$\frac{1}{2}\frac{1}{(z-2)} = \frac{1}{-1+(z-1)} = \frac{1}{2}\frac{-1}{1-\underbrace{(z-1)}_{<1}} = -\sum_{n=0}^{\infty}\frac{(z-1)^n}{2}$$

$$f(z) = \sum_{n=0}^{\infty}\frac{(1-z)^n}{2} - \frac{1}{(z-1)} - \sum_{n=0}^{\infty}\frac{(z-1)^n}{2} = \left[\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{2} - \frac{(z-1)^n}{2}\right] - \frac{1}{(z-1)}$$

$$\boxed{f(z) = \sum_{n=0}^{\infty}\frac{(z-1)^{2n+1}}{2} - \frac{1}{(z-1)}}$$

d) $1<|z-2|<2$

$\left|\frac{1}{z-2}\right|<1$

$1 < 2\left|\frac{z}{2}-1\right| < 2$

$\frac{1}{2} < \left|\frac{z}{2}-1\right| < 1$

$$\frac{1}{2}\frac{1}{z} = \frac{1}{2}\cdot\frac{1}{2+(z-2)} = \frac{1}{2}\cdot\frac{1}{2}\left(\frac{1}{1+\left(\frac{z}{2}-1\right)}\right)$$

$$= \frac{1}{4}\cdot\frac{1}{\left(1-\left(1-\frac{z}{2}\right)\right)} \underset{\left|\frac{z}{2}-1\right|<1}{=} \frac{1}{4}\sum_{n=0}^{\infty}\left(1-\frac{z}{2}\right)^n = \frac{1}{4}\sum_{n=0}^{\infty}\left(1-\frac{z}{2}\right)^n$$

$\frac{1}{2}$

Como el anillo atrapa dos singularidades no puede reescribirse más

$$\boxed{f(z) = \frac{1}{4}\sum_{n=0}^{\infty}\left(1-\frac{z}{2}\right)^n - \frac{1}{(z-1)} + \frac{1}{2}\cdot\frac{1}{(z-2)}}$$

NOTA

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Ejercicio 2

a) $\frac{1}{z(z-1)^2}$ en $0<|z-1|<1$ y en $|z-1|>1$

$$\frac{A}{z}+\frac{B}{(z-1)}+\frac{C}{(z-1)}=\frac{A(z-1)^2+B(z)(z-1)+C(z)(z-1)}{z(z-1)^2}$$

Si $z=2$

$$A+2B+2C=1$$

Si $z=3$ $\Rightarrow A=-2$

$$4A+6\overbrace{(B+C)}^{D}=1 \qquad -2+2D=1 \qquad \Rightarrow f(z)=-\frac{2}{z}+\frac{3}{2}\cdot\frac{1}{(z-1)^2}$$

$$D=3/2$$

Si $0<|z-1|<1$ $\qquad |1-z|=|z+1|<1$

$$\Rightarrow -\frac{2}{z}=\frac{-2}{1+(z-1)} \qquad \frac{-2}{1-\underbrace{(1-z)}_{<1}}=-2\sum_{n=0}^{\infty}(1-z)^n=-2\sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

$$\boxed{f(z)=2\sum_{n=0}^{\infty}(-1)^{n+1}(z-1)^n+\frac{3}{2}\frac{1}{(z-1)^2}}$$

$|z-1|>1$ $\qquad \frac{1}{|z-1|}<1$ $\qquad |1-z|>1 \Rightarrow \frac{1}{|1-z|}<1$

$$\frac{-2}{z}=\frac{-2}{1+(z-1)}=\frac{-2}{(z-1)}\cdot\frac{1}{\frac{1}{(z-1)}+1}=\frac{-2}{(z-1)}\cdot\frac{1}{1-\left(\frac{-1}{z-1}\right)}=\frac{-2}{(z-1)}\cdot\frac{1}{\left(1-\underbrace{\frac{1}{(1-z)}}_{<1}\right)}$$

$$=\frac{-2}{(z-1)}\sum_{n=0}^{\infty}\frac{1}{(1-z)^n}=\frac{-2}{(z-1)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^n}=2\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(z-1)^{n+1}}$$

$$\boxed{f(z)=\frac{3}{2}\frac{1}{(z-1)^2}+2\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(z-1)^{n+1}}}$$

NOTA

b) $\dfrac{z^2-1}{(z+2)(z+3)^2}$ en $0<|z+3|<1$ y $2<|z|<3$

$= (z+1)(z-1)\left[\dfrac{A}{(z+2)} + \dfrac{B}{(z+3)^2}\right]$ $\qquad \dfrac{A(z+3)^2 + B(z+2)}{(z+2)(z+3)^2} = \dfrac{1}{(z+2)(z+3)^2}$

Si $z=0$ $\quad 9A+2B=1$

Si $z=-1$ $\quad 4A+B=1$

$A=-1 \quad B=5$

$\Rightarrow f(z) = (z+1)(z-1)\cdot\left[\dfrac{-1}{(z+2)} + \dfrac{5}{(z+3)^2}\right]$

Si $0<|z+3|<1$

$$\frac{-1}{(z+2)} = \frac{-1}{-1+(z+3)} = \frac{1}{1-\underbrace{(z+3)}_{<1}} = \sum_{n=0}^{\infty} (z+3)^n$$

$$f(z) = (z^2-1)\cdot\left[\sum_{n=0}^{\infty} (z+3)^n + \frac{5}{(z+3)^2}\right]$$

Si $2<|z|<3$ $\qquad 2<|z|<3$ $\quad \dfrac{1}{|z|}<1$ y $\dfrac{2}{|z|}<1$

$$\frac{-1}{(z+2)} = \frac{-1}{z\left(1+\frac{2}{z}\right)} = \frac{-1}{z\left(1-\left(-\frac{2}{z}\right)\right)} = \frac{-1}{z}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^n} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^n}{z^{n+1}}$$

$\left|\frac{z}{3}\right|<1$

$$\frac{5}{(z+3)^2} = \frac{5}{(z+3)(z+3)} = \frac{5}{15\left(\frac{z}{3}+1\right)\left(\frac{z}{3}+1\right)} = \frac{1}{3}\,\frac{1}{\left(1-\left(-\frac{z}{3}\right)\right)}\cdot\frac{1}{\left(1-\left(-\frac{z}{3}\right)\right)}$$

$$= \frac{1}{3}\left(\sum_{n=0}^{\infty}(-1)^n\frac{(z)^n}{3^n}\right)^2$$

$$f(z) = \sum_{n=0}^{\infty}(-1)^{n+1}\cdot\frac{2^n}{z^{n+1}} + \frac{1}{3}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n\right]^2$$

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Ejercicio 6

$f(z) = e^{-\frac{1}{z^2}}$. Mostrar que $f$ tiene una singularidad esencial en $z=0$ y explicar por que este hecho muestra que $F: \mathbb{R} \to \mathbb{R}$ dada por:

$$F(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0 & x = 0 \end{cases}$$ es derivable en $\mathbb{R}$, no coincide con su Taylor en torno al cero

$f(z) = e^{-\frac{1}{z^2}}$ $\quad z = a+bi \quad z^2 = a^2 + 2abi - b^2$

$\Rightarrow f(a+bi) = e^{-\frac{1}{a^2+2abi-b^2}}$

$z=0$ es una singularidad esencial $\Leftrightarrow \lim_{z \to 0} f(z)$ no $\exists$

$\Rightarrow$ Sea $b=0$

$\lim_{a \to 0^+} e^{-\frac{1}{a^2}} = e^{-\infty} \to 0$

$\therefore$ no $\exists$ lim

Sea $a=0$

$\lim_{b \to 0} e^{-\frac{1}{-b^2}} = e^{\infty} \to +\infty$

NOTA

Ejercicio 7. Ver si son singularidades evitables y redefinirla o si es polo, la parte singular

a) $f(z) = \dfrac{z^2+1}{z(z-1)}$

La parte singular es $\dfrac{1}{z}$

b) $f(z) = \dfrac{\cos(z)-1}{z}$

$$\lim_{z\to 0} \frac{\cos(z)-1}{z} = \frac{0}{0} \Rightarrow \text{L'Hôpital} \lim_{z\to 0} \frac{-\sin(z)}{1} = 0$$

∴ la singularidad es evitable

$$\boxed{f(z) = \begin{cases} \dfrac{\cos(z)-1}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}}$$

c) $f(z) = e^{1/z} = e^{\frac{1}{a+bi}}$ si $b=0$ $\lim_{a\to 0^+} e^{1/a} = e^{\infty} = \infty$

$\lim_{a\to 0^-} e^{1/a} = \dfrac{1}{e^{\infty}} = 0$

∴ no ∃ lim. ∴ es un polo

La parte singular es el $\dfrac{1}{z}$

d) $f(z) = \dfrac{\log(z+1)}{z} \Rightarrow$ $f(re^{i\varphi}) =$

Si $z = r$ $(\varphi = 0)$ $\Rightarrow \dfrac{\log(r+1)}{r} = \dfrac{\ln(r+1)}{r}$

$$\lim_{r\to 0} \frac{\ln(r+1)}{r} = \frac{0}{0}$$

L'Hôpital $\lim_{r\to 0} \dfrac{1}{r.1} = \infty$

∴ no ∃ lim ∴ la singularidad es un polo, y nuevamente vierde $\dfrac{1}{z}$

$\operatorname{Log}(z) = \ln|z| + i\arg(z)$

$e^{\operatorname{Log}(z)} = e^{\ln(z) + i\arg(z)} = e^{\ln(z)} \cdot e^{i\arg(z)}$

$= |z| e^{i\arg(z)} = z$

$e^{\operatorname{Log}(z)} = z \Rightarrow (e^{\operatorname{Log}(z)})' = z'$

$(\operatorname{Log}(z))' e^{\operatorname{Log}(z)} = 1$

$\operatorname{Log}(z)' = \dfrac{1}{z}$

e) $f(z) = \dfrac{1}{z}\cos\left(\dfrac{1}{z}\right) = \dfrac{1}{z} \cdot \dfrac{e^{\frac{i}{z}} + e^{-\frac{i}{z}}}{2}$ si $z =$ Real puro $= a$

$\Rightarrow \dfrac{1}{a} \cdot \dfrac{e^{\frac{i}{a}} + e^{-\frac{i}{a}}}{2}$ $\lim_{a\to 0} \underbrace{\frac{1}{a}}_{\to\infty} \overbrace{\frac{e^{i/a} + e^{-i/a}}{2}}^{\text{acotado}} = \infty$ ∴ no ∃ lim

la singularidad es $\dfrac{1}{z}$

f) $f(z)=\dfrac{1}{1-e^z}$ $\quad \lim\limits_{z\to 0}\dfrac{1}{1-e^z}=\dfrac{1}{0}=\infty$

$\therefore$ polo. La parte singular es $1-e^z=0 \iff z=0$

g) $\sin(z)\sin\left(\frac{1}{z}\right)$ $\quad \lim\limits_{z\to 0}\underbrace{\sin(z)}_{\to 0}\cdot\sin\left(\frac{1}{z}\right)=0$

$\therefore$ la singularidad es evitable

$$\boxed{f(z)=\begin{cases}\sin(z)\sin\left(\frac{1}{z}\right) & z\neq 0\\ 0 & z=0\end{cases}}$$

Ejercicio 8. Hallar y clasificar las singularidades de $\mathbb{C}$ en:

a) $f(z)=\dfrac{1+z^2}{z(z-1)^2}$ $\quad$ Singularidades $=\{0,1,\infty\}$

$z=0$
$z=1$

$z=0$ $\quad \lim\limits_{z\to 0^+}\dfrac{1+z^2}{z(z-1)^2}=\infty$

$\lim\limits_{z\to 0^-}\dfrac{1+z^2}{z(z-1)^2}=-\infty$ $\quad$ lo que es análogo a $\lim\limits_{z\to 0}\dfrac{|1+z^2|}{|z||z-1|^2}=+\infty$

$\therefore$ es un polo

y de orden 1 pues

$\lim\limits_{z\to 0}\dfrac{\cancel{z}(1+z^2)}{\cancel{z}(z-1)^2}=1\neq 0$

$z=1$ $\quad \lim\limits_{z\to 1}\dfrac{\overbrace{1+z^2}^{\to 2}}{z(z-1)^2}$

$\lim\limits_{z\to 1}\dfrac{|1+z^2|}{\underbrace{|z|}_{\to 1}\underbrace{|z-1|^2}_{\to 0}}=\infty$

$z=1$ es un polo de orden 2 $\quad \dfrac{\cancel{(z-1)^2}(1+z^2)}{z\cancel{(z-1)^2}}=L\neq 0=2$ $\therefore$ polo de orden 2

$z=\infty$ o sea $z=0$ de $f\left(\frac{1}{z}\right)$

$f\left(\frac{1}{z}\right)=z\left(\dfrac{1+\frac{1}{z^2}}{\left(\frac{1}{z}-1\right)^2}\right)$ $\quad \lim\limits_{z\to 0} z\,\dfrac{\overbrace{1+\frac{1}{z^2}}^{\to\infty}}{\underbrace{\left(\frac{1}{z}-1\right)^2}_{\to\infty}}=\lim\limits_{z\to 0}\dfrac{z\cdot\frac{1}{z^2}(z^2+1)}{\frac{1}{z^2}-\frac{2}{z}+1}=\lim\limits_{z\to 0}\dfrac{\cancel{\frac{1}{z}}\overbrace{(z^2+1)}^{\to 1}}{\cancel{\frac{1}{z}}\underbrace{\left(\frac{1}{z}-2+z\right)}_{\to\infty}}=0$

Por lo tanto $z=\infty$ es una singularidad evitable

$\boxed{\text{Rta: } 0 \text{ polo } n=1;\ 1 \text{ polo } n=2;\ \infty \text{ evitable}}$

b) $f(z)=\dfrac{1}{z^3}\sin(z)$ $\quad$ Singularidades $=\{0,\infty\}$

$z=0$ $\quad \lim\limits_{z\to 0}\underbrace{\dfrac{1}{z^2}}_{\to\infty}\cdot\underbrace{\dfrac{\sin(z)}{z}}_{\to 1}=\infty$ $\quad \therefore z=0$ es polo

$z=0$ es polo de orden 2 $\iff \lim\limits_{z\to 0}\dfrac{z^2\sin(z)}{z^3}=1\neq 0$ ✓

$z=\infty \Rightarrow f\left(\frac{1}{z}\right)=z^3 \cdot \sin\left(\frac{1}{z}\right)$

$\lim\limits_{z\to 0} \underbrace{z^3}_{\to 0} \cdot \underbrace{\sin\left(\frac{1}{z}\right)}_{\text{acot}} = 0 \quad \therefore$ infinito es una singularidad evitable

$$\boxed{\begin{array}{ll} \text{Rta: } z=0 & \text{polo orden 2} \\ z=\infty & \text{evitable} \end{array}}$$

e) $f(z)=\dfrac{\cos(z)}{z+1}$ Singularidades $\{-1,\infty\}$

$z=-1 \quad \lim\limits_{z\to -1} \dfrac{\cos(z)}{z+1} = \infty \quad \therefore$ polo de orden 1 ($\cos(z)\to\cos(-1)$, $z+1\to 0$)

Pues $\lim\limits_{z\to -1} \dfrac{\cancel{(z+1)}\cos(z)}{\cancel{(z+1)}} = \cos(-1) \neq 0$

$z=\infty \quad \lim\limits_{z\to 0} \dfrac{\cos(1/z)}{\frac{1}{z}+1} = 0$ $z=\infty$ es una singularidad evitable ($\cos(1/z)$ acot., $\frac{1}{z}+1 \to \infty$)

$$\boxed{\begin{array}{ll} \text{Rta: } z=-1 & \text{polo orden 1} \\ z=\infty & \text{evitable} \end{array}}$$

F) $f(z)=e^{\frac{1}{z^2}}$ singularidades $=\{0,\infty\}$

$z=0 \quad z=a+bi$

$\Rightarrow f(a+bi)=e^{\frac{1}{a^2-b^2+2abi}}$

$\lim\limits_{a=0,\, b\to 0} f(a+bi) = e^{\left(\frac{1}{-b^2}\right)} = 0$ ($\to -\infty$)

$\lim\limits_{b=0,\, a\to 0} f(a+bi) = e^{\left(\frac{1}{a^2}\right)} = \infty$ ($\to\infty$) $\quad \therefore$ no $\exists$ lim. sing esencial

$z=\infty \quad \lim\limits_{z\to 0} e^{z^2}$ $\quad z=a+bi$

Analizo $\lim\limits_{(a,b)\to 0} f(a+bi) = e^{a^2-b^2+2abi}$ $\quad \lim\limits_{a=0,\, b\to 0} e^{a^2-b^2+2abi} = e^{-\infty} = 0$

$\lim\limits_{b=0,\, a\to 0} e^{a^2-b^2+2abi} = e^{\infty} = \infty$ $\quad \therefore$ no $\exists$ lim, sing esencial

$$\boxed{\text{Rta: } \left.\begin{array}{l} z=\infty \\ z=0 \end{array}\right\} \text{singularidades esenciales}}$$

g) $f(z)=\cos\left(\dfrac{\pi}{z-\pi}\right)$ Singularidades $=\{z=\pi,\ z=\infty\}$

$z=\pi \quad \lim\limits_{z\to\pi} \cos\left(\dfrac{\pi}{z-\pi}\right) =$ no $\exists$ lim singularidad esencial ($\frac{\pi}{z-\pi}\to\infty$, $z-\pi\to 0$)

$$\boxed{\begin{array}{ll} \text{Rta: } z=\pi & \text{esencial} \\ z=\infty & \text{removible} \end{array}}$$

$z=\infty \quad \lim\limits_{z\to 0} \cos\left(\dfrac{\pi}{\frac{1}{z}-\pi}\right) = \cos(0)$ $\quad \therefore$ singularidad removible ($\frac{1}{z}\to\infty$, $\to 0$)

h) $f(z) = -\dfrac{1}{\sin\left(\frac{1}{z^2+1}\right)}$  Singularidades $\left\{ i, -i, \pm\sqrt{\frac{1}{k\pi} - 1} \right\} + \{\infty\}$ $k \in \mathbb{Z}_{-0}$

$$\frac{1}{z^2+1} = k\pi \iff \underset{k \neq 0}{\frac{1}{k\pi} - 1} = z^2$$

$i$ y $-i$ $\lim\limits_{z \to \pm i} -\dfrac{1}{\underbrace{\sin\left(\underbrace{\frac{1}{z^2+1}}_{\to\infty}\right)}_{\to\text{ no tiene lim}}} = \infty \ \nexists \lim$

$\therefore$ singularidades esenciales

$\pm\sqrt{\frac{1}{k\pi} - 1}$

$$\lim_{z \to \pm\sqrt{\frac{1}{k\pi}-1}} \left| \frac{1}{\underbrace{\sin\left(\underbrace{\frac{1}{z^2+1}}_{\to k\pi}\right)}_{\to 0}} \right| = +\infty \quad \therefore \text{ polos} \iff \text{analizo } u = \underbrace{\frac{1}{z^2+1}}_{\to\infty}$$

$$\lim_{u \to k\pi} \frac{(u - k\pi)}{\underbrace{\sin(u)}_{\to 0}} = \text{L'Hôpital} = \lim_{u \to k\pi} \frac{1}{\cos(u)} = 1 \text{ o } -1$$

son polos

$\infty$ $\lim\limits_{z \to 0} \left| \dfrac{1}{\sin\left(\frac{1}{\frac{1}{z^2}+1}\right)} \right| = \lim\limits_{z \to 0} \left| \dfrac{1}{\sin\left(\frac{z^2}{1+z^2}\right)} \right| = +\infty$ $\therefore$ es un polo

$$\lim_{z \to 0} \frac{\overbrace{z}^{\to 0}}{\underbrace{\sin\left(\frac{z^2}{1+z^2}\right)}_{\to 0}} = \text{L'Hôpital} = \lim_{z \to 0} \frac{1}{\cos\left(\frac{z^2}{1+z^2}\right) \cdot \left(\frac{2z(1+z^2) - z^2 \cdot 2z}{(1+z^2)^2}\right)}$$

$$= \lim_{z \to 0} = \infty$$

¿y si fuera polo de orden 2

$$\lim_{z \to 0} \frac{z^2}{\sin\left(\frac{z^2}{1+z^2}\right)} = \text{L'Hôpital} \lim_{z \to 0} \frac{2\cancel{z}}{\cos\left(\frac{z^2}{1+z^2}\right) \cdot \frac{2\cancel{z}}{(1+z^2)^2}} = 1 \quad \therefore \text{ polo de orden 2}$$

$\boxed{\text{Rta: } z = \infty \text{ polo orden 2} \quad |z| = \sqrt{\frac{1}{k\pi} - 1} \text{ polo de orden 1.} \quad z = i,\ z = -i \text{ esenciales}}$

i) $f(z) = \dfrac{e^{\frac{1}{z}}}{(z+1)z}$  Singularidades $\{ z = 0,\ z = -1,\ z = \infty \}$

$z = -1$ $\lim\limits_{z \to -1} \left| \dfrac{\overbrace{e^{1/z}}^{\to 1/e}}{\underbrace{(z+1)z}_{\to 0 \cdot -1}} \right| = +\infty$ $\therefore$ $z=0$ es un polo

De orden uno $\iff \lim\limits_{z \to -1} \left| \dfrac{\cancel{(z+1)} e^{1/z}}{\cancel{(z+1)} z} \right| \neq 0 \cdot = \dfrac{1}{+e}$

$z=0$ $\lim_{z\to 0} \left| \frac{e^{1/z}}{(z+1)z} \right|$ $z=a+bi$ si $b=0$

$\lim_{a\to 0^+} \left| \frac{e^{1/a}}{(a+1)a} \right| = +\infty$ no $\exists$ lim $\therefore$ es esencial

$\lim_{a\to 0^-} \left| \frac{e^{1/a}}{(a+1)a} \right| = \frac{0}{0} = ?$ L'Hopital $\lim_{a\to 0^-} \frac{e^{1/a} \cdot -\frac{1}{a^2}}{1}$

$z=\infty$ $\lim_{z\to 0} \left| \frac{e^{z^{-1}}}{\left(\frac{1}{z}+1\right)} \cdot z \right| = \lim_{z\to 0} \left| \frac{e^{z} z}{1+z} \right| = 0$ $\therefore$ es una singularidad removible

$\boxed{\text{Rta: } z=-1 = \text{polo orden } 1 \quad z=\infty \text{ removible} \quad z=0 \text{ esencial}}$

## Ejercicio 9

a) Mostrar que $f(z)=tg(z)$ es meromorfa en $\mathbb{C}$

$f(z)=tg(z)=\frac{\sin(z)}{\cos(z)}$ Singularidades $\left\{\left(\frac{\pi}{2}+k\pi\right) k\in\mathbb{Z}, \infty\right\}$

$\lim_{z\to \frac{\pi}{2}+k\pi} \left| \frac{\sin(z)}{\cos(z)} \right| = +\infty$ $\therefore$ es un polo

Polo de orden $1 \Leftrightarrow \lim_{z\to \frac{\pi}{2}+k\pi} \left| \frac{\left(z-\left(\frac{\pi}{2}+k\pi\right)\right) \sin(z)}{\cos(z)} \right| = \frac{0}{0} \overset{?}{=} 1$ L'Hopital

$\lim_{z\to \frac{\pi}{2}+k\pi} \left| \frac{1 \cdot \sin(z)}{-\sin(z)} \right| = 1$ $\therefore$ polos de orden 1 $\therefore$ $tg(z)$ es meromorfa pues todas sus singularidades son polos

$\lim_{z\to 0} \frac{\sin\left(\frac{1}{z}\right)}{\cos\left(\frac{1}{z}\right)}$ no $\exists$ lim

c) $\operatorname{Res}\left(f, \frac{\pi}{2}+k\pi\right) = \frac{1}{0!} \lim_{z\to \frac{\pi}{2}+k\pi} \frac{\left(z-\frac{\pi}{2}+k\pi\right)\sin(z)}{\cos(z)}$ L'Hopital $\lim_{z\to \frac{\pi}{2}+k\pi} \frac{\cancel{\sin(z)}}{-\cancel{\sin(z)}} = -1$

parte singular $=-1$

## Ejercicio 10

Sea $f(z)=\frac{a_m z^m + \ldots + a_1 z + a_0}{b_n z^n + \ldots + b_1 z + b_0}$. Probar que $\infty$ es a lo sumo una singularidad aislada de $f$

$$\lim_{z\to 0} \frac{a_m \frac{1}{z^m} + \dots + a_1 \frac{1}{z} + a_0}{b_n \frac{1}{z^n} + \dots + b_1\frac{1}{z} + b_0} = \frac{\frac{1}{z^m}\left(a_m + \dots + a_1 z^{m-1} + a_0 z^m\right)}{\frac{1}{z^m}\left(b_n z^{m-n} + \dots + b_1 z^{m-1} + b_0 z^m\right)}$$

Si $n \geq m \Rightarrow$ $f(\infty) =$

Si $m>n$ hay un polo de orden $m-n$

a)

## Ejercicio 11. Clasificar las singularidad de las siguientes funciones en $\mathbb{C}_\infty$:

a) $\dfrac{e^z-1-z}{z^2}$ Singularidades $\{0, \infty\}$

$z=0$: $\displaystyle\lim_{z\to 0}\frac{e^z-1-z}{z^2} = \frac{0}{0}$? L'Hopital $= \displaystyle\lim_{z\to 0}\frac{e^z-1}{2z} = \frac{0}{0}$? = L'Hopital

$\displaystyle\lim_{z\to 0}\frac{e^z}{2} = \frac{1}{2}$ ∴ singularidad removible

$z=\infty$: $\displaystyle\lim_{z\to 0}\left(e^{1/z}-1-\frac{1}{z}\right)\cdot z^2 = \lim_{z\to 0} z^2 e^{1/z} - z^2 - z$

Sea $z=a+bi$ con $b=0$, Analizo el caso real

$\displaystyle\lim_{a\to 0^+} a^2\left(e^{1/a}-1-\frac{1}{a}\right) = \lim_{a\to 0^+} a^2 e^{1/a} - a^2 - a = 0\cdot\infty = ? = \lim_{a\to 0^+}\frac{e^{1/a}}{\frac{1}{a^2}}$ L'Hopital

$= \displaystyle\lim_{a\to 0^+}\frac{+\frac{1}{a^2}e^{1/a}}{+\frac{1}{a^3}} = \lim_{a\to 0^+}\frac{e^{1/a}}{\frac{1}{a}} = \frac{\infty}{\infty}$ L'Hopital $\displaystyle\lim_{a\to 0^+}\frac{-\frac{1}{a^2}\cdot e^{1/a}}{-\frac{1}{a^2}} = \infty$

$\displaystyle\lim_{a\to 0^-} a^2 e^{1/a} - a^2 - a = 0$ ∴ no $\exists$ lim $\Rightarrow$ singularidad esencial

$$\boxed{\begin{array}{l}\text{Rta: } 0 = \text{removible} \\ \infty = \text{esencial}\end{array}}$$

b) $\dfrac{\cos(z)-1}{z^6+z^4} = \dfrac{\cos(z)-1}{z^4(z^2+1)}$ Singularidades $\{0, i, -i, \infty\}$

$z=0$ $\lim\limits_{z\to 0} \dfrac{\cos(z)-1}{z^4(z^2+1)} = \dfrac{0}{0}$ ? L'Hôpital $\lim\limits_{z\to 0} \dfrac{1}{(z^2+1)} \cdot \dfrac{-\sin(z)}{4z^3} = \lim\limits_{z\to 0} \dfrac{1}{(z^2+1)} \cdot \dfrac{-\cos(z)}{12z^2} = \infty$

$\Rightarrow$ polo de orden 2

$z=i$ $\cos(i) = \dfrac{e^{i^2}+e^{-i^2}}{2} = \dfrac{e^{-1}+e}{2} = \dfrac{1+e^2}{2e} = \cos(-i)$

$\lim\limits_{z\to \pm i} f(z) =$ infinito $\Rightarrow$ polos de orden 1

$\lim\limits_{z\to 0} \dfrac{\overset{\text{acotado}}{\cos\left(\frac{1}{z}\right)-1}}{\underbrace{\left(\frac{1}{z^2}+1\right)}_{\to\infty}} \cdot \overset{\to 0}{z^4} = 0$ $\therefore$ evitable

$$\boxed{\begin{array}{ll} \text{Rtas} & z=0 \text{ polo orden } 2 \\ & z=\pm i \text{ polo orden } 1 \\ & z=\infty \text{ evitable} \end{array}}$$

c) $\dfrac{\cos(z)-1}{(z-2\pi)^2} + \dfrac{z-3}{(z+1)^2(z-2)}$ Singularidades $\left[\{z=2\pi, z=-1, z=2\}\ z=\infty\right]$

$\lim\limits_{z\to 2\pi} \underbrace{\dfrac{\cos(z)-1}{(z-2\pi)^2}}_{\frac{0}{0}} + \overset{\text{acot}}{\dfrac{(z-3)}{(z+1)^2(z-2)}}$ L'Hôpital $\lim\limits_{z\to 2\pi} \dfrac{-\overset{\to 0}{\sin(z)}}{2\underbrace{(z-2\pi)}_{\to 0}} = \dfrac{0}{0}$ L'Hôpital

$\lim\limits_{z\to 2\pi} \dfrac{-\cos(z)}{2} = -\dfrac{1}{2} \Rightarrow z=2\pi$ evitable

$\lim\limits_{z\to -1} \overset{\text{acot}}{\dfrac{\cos(z)-1}{(z-2\pi)^2}} + \dfrac{z-3}{(z+1)^2(z-2)} = \infty$ polo de orden 2

$\lim\limits_{z\to 2}$ ... polo de orden 1

$\lim\limits_{z\to 0} \dfrac{\cos\left(\frac{1}{z}\right)-1}{\left(\frac{1}{z}-2\pi\right)^2} + \dfrac{\frac{1}{z}-3}{\left(\frac{1}{z}+1\right)^2\left(\frac{1}{z}-2\right)} = 0$ evitable

(first term $\to 0$, second term $\to 0$)

$$\boxed{\begin{array}{ll} \text{Rta:} & 2\pi = \text{evitable} \\ & \infty = \text{evitable} \\ & -1 = \text{polo de orden } 2 \\ & 2 = \text{polo orden } 1 \end{array}}$$

d) $e^{\frac{z}{1-z}}$ Singularidades $\{1, \infty\}$

$\lim\limits_{z\to 1} e^{\frac{z}{1-z}} =$ Sea $z=a+bi$ $b=0$

$\Rightarrow \lim\limits_{a\to 1^+} e^{\frac{a}{1-a}} = 0$ $\lim\limits_{a\to 1^-} e^{\frac{a}{1-a}} = \infty$ $\therefore$ no $\exists$ lim

$$\lim_{z\to 0} e^{\frac{\frac{1}{z}}{1-\frac{1}{z}}} = \lim_{z\to 0} e^{\frac{1}{z-1}} = \frac{1}{e} \quad \text{evitable}$$

$$\boxed{\begin{array}{l}\text{Rta: } 1 \text{ esencial} \\ \infty \text{ evitable}\end{array}}$$

e) $\dfrac{e^{z^2+\frac{1}{z^2}-1}}{z^2-1}$ Singularidades $\{0, 1, -1, \infty\}$

$$\lim_{z\to 0} \frac{e^{z^2+\frac{1}{z^2}-1}}{z^2-1} \qquad z^2 = a^2-b^2+2abi$$

Sea $a=0$

$$\lim_{b\to 0^+} \frac{e^{-b^2-1+\frac{1}{b^2}}}{-b^2-1} = 0$$

$\therefore$ no $\exists$ lim

Sea $b=0$

$$\lim_{a\to 0} \frac{e^{a^2-1+\frac{1}{a^2}}}{a^2-1} = -\infty$$

$$\lim_{z\to\pm 1} \frac{e^{z^2+\frac{1}{z^2}-1}}{z^2-1} = \lim_{z\to\pm 1} \frac{e^{z^2+\frac{1}{z^2}-1}}{(z+1)(z-1)} = \lim_{z\to\pm 1} \frac{e^{\frac{1}{z^2}+(z+1)(z-1)}}{(z+1)(z-1)} = \infty \Rightarrow \text{polo de orden } 1$$

$$\lim_{z\to 0} \frac{e^{\frac{1}{z^2}+z^2-1}}{\frac{1}{z^2}-1} = \lim_{(a,b)\to 0} \frac{e^{\frac{1}{a^2-b^2+2abi}+a^2-b^2+2abi-1}}{\frac{1}{a^2-b^2+2abi}-1}$$

$$= \lim_{z\to 0} \frac{e^{\frac{1}{z^2}} \cdot e^{z^2-1}}{\frac{1}{z^2}-1}$$

Sea $b=0 \Rightarrow \lim_{a\to 0} \dfrac{e^{\frac{1}{a^2}} \cdot e^{a^2} \cdot e^{-1}}{\frac{1}{a^2}-1} = \dfrac{\infty}{\infty}$ ? L'Hopital $\lim_{a\to 0} \dfrac{-\frac{1}{a^3} \cdot e^{1/a^2}}{-\frac{1}{a^3}} = \infty$

Sea $a=0$ $\lim_{b\to 0} \dfrac{e^{\frac{1}{-b^2}} \cdot e^{-b^2} e^{-1}}{-\frac{1}{b^2}-1} = 0 \quad \therefore$ sing esencial

$$\boxed{\begin{array}{l}\text{Rta: } z=0, \infty \text{ esenciales} \\ z=1,-1 \text{ polos de orden } 1\end{array}}$$

## Ejercicio 12

a) Hallar $f$ no holomorfa en $0$ / $\text{Res}(f,0)=0$

$\text{Res}(f,0) = \left((z-z_0)f(z)\right) = 0$

$\lim_{z\to z_0} f(z) = \infty$ $\qquad f(z) = \frac{1}{z^2}$ no es holomorfa en cero (trivial)

$$\text{Res}(f,0) = \frac{1}{1!}\lim_{z\to 0}\frac{\partial}{\partial z}\left(z^2\frac{1}{z^2}\right) = 0$$

b) Mostrar que una función puede ser holomorfa en $\infty$ y tener residuo nulo allí

c) $f$ tiene una singularidad esencial en $z=a$

$\Rightarrow \lim_{z\to a} f$ no $\exists$

## Ejercicio 1

a) $\displaystyle\int_0^{\pi} \frac{\cos(2\varphi)}{1-2a\cos(\varphi)+a^2}\,d\varphi \,,\; a^2<1$ SOL = cambiamos la integral por una de contorno complejo

$\cos(2\varphi) = \dfrac{e^{2i\varphi}+e^{-2i\varphi}}{2}$  sea $z = e^{i\varphi}$  Sea $f(\varphi) = \dfrac{\cos(2\varphi)}{1-2a\cos(\varphi)+a^2}$

$\cos(\varphi) = \dfrac{e^{i\varphi}+e^{-i\varphi}}{2}$  $dz = ie^{i\varphi}d\varphi$  tanto $\cos(2\varphi)$ como $\cos(\varphi)$

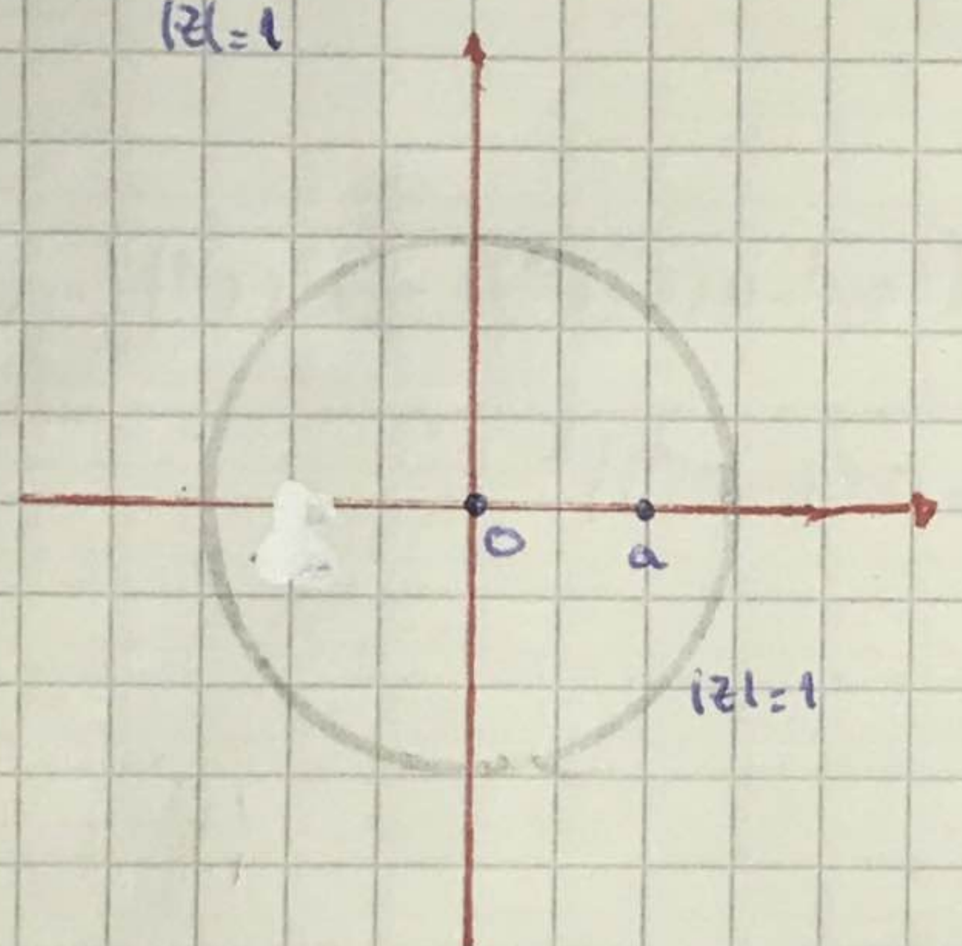
$\dfrac{dz}{iz} = d\varphi$  son par en $[-\pi,\pi]$, o sea

$\Rightarrow \cos(2\varphi) = \dfrac{z^2+z^{-2}}{2}$  $f(\varphi) = f(-\varphi)\ \forall \varphi \in [-\pi,\pi]$

$\cos(\varphi) = \dfrac{z+z^{-1}}{2}$  $\Rightarrow$ con $|z|=1$

Por paridad $\displaystyle\int_0^{\pi} f(\varphi)d\varphi = \frac{1}{2}\oint_{-\pi}^{\pi} f(\varphi)d\varphi = \frac{1}{2}\oint_{|z|=1} \frac{1}{2}\frac{z^2+z^{-2}}{(1-a(z+z^{-1})+a^2)}\cdot\frac{1}{zi}dz$

$\displaystyle = \frac{1}{4i}\oint_{|z|=1}\frac{z+z^{-3}}{(1-a(z+z^{-1})+a^2)}dz = \frac{1}{{}_2\cancel{4}i}\cdot 2\pi i\ \Sigma$ residuos de singularidades encerradas



$1-a(z+z^{-1})+a^2 = 0$

$\dfrac{1+a^2}{a} = z+z^{-1} = \dfrac{z^2+1}{z}$

$\Leftrightarrow z = a < 1$

polo de orden 1

$\displaystyle\Rightarrow \operatorname{Res}(f,a) = \frac{1}{0!}\lim_{z\to a}\frac{(z-a)(z+z^{-3})}{(1-a(z+z^{-1})+a^2)} = \frac{a}{-a^2+1}\cdot\left(a+\frac{1}{a^3}\right)$

L'Hôpital $\displaystyle\lim_{z\to a}\frac{1}{-a+\frac{a}{z^2}} = \frac{a}{-a^2+1}$

$\displaystyle\lim_{z\to 0}\frac{(z+z^{-3})}{(1-a(z+z^{-1})+a^2)} = \frac{\infty}{\infty}!$

L'Hôpital

$\displaystyle\lim_{z\to 0}\frac{1-3z^{-4}}{-a(1-z^{-2})} = \frac{\infty}{\infty}$

L'Hôpital $\displaystyle\lim_{z\to 0}\frac{12z^{-5}}{-2az^{-3}} = -\frac{6}{a}z^{-2} = \infty$ polo

$\displaystyle\lim_{z\to 0}\frac{z^2(z+z^{-3})}{(1-a(z+z^{-1})+a^2} = \lim_{z\to 0}\frac{z^3}{(1-a(z+z^{-1})+a^2)} + \lim_{z\to 0}\frac{\cancel{z^{-1}}}{\cancel{z^{-1}}(z-az^2-a+a^2z)} = -\frac{1}{a}$

$\therefore$ polo de orden 2

$$\frac{z+z^{-3}}{(1-a(z+z^{-1})+a^2)} = \frac{z^{-3}(z^4+1)}{z^{-1}\left[(1+a^2)z-az^2-a\right]}$$

$$= z^{-2}\cdot\varphi(z) \qquad Res(f,0) = \frac{\varphi^{(n-1)}(0)}{(n-1)!} = \frac{(4z^3)\left[(1+a^2)z-az^2-a\right]-(z^4+1)\left[(1+a^2)-2az\right]}{\left[(1+a^2)z-az^2-a\right]^2}$$

$$= \frac{-1(1+a^2)}{a^2} = -\frac{(1+a^2)}{a^2}$$

Finalmente $\boxed{\int_0^\pi \frac{\cos(2\varphi)}{1-2a\cos(\varphi)+a^2}\,d\varphi,\ a^2<1 = \frac{\pi}{2}\left[-\frac{(1+a^2)}{a^2} + \frac{a}{-a^2+1}\cdot\left(a+\frac{1}{a^3}\right)\right]}$

b) $\int_0^\pi \frac{1}{(a+\cos(\varphi))^2}\,d\varphi,\ a>1 \qquad f(\varphi) = \frac{1}{(a+\cos(\varphi))^2}$ es par en $[-\pi,\pi]$

$\cos(\varphi) = \frac{z+z^{-1}}{2}$ con $z = e^{i\varphi}$

$\frac{dz}{zi} = d\varphi$

$$\Rightarrow \int_0^\pi \frac{1}{(a+\cos(\varphi))^2}\,d\varphi = \frac{1}{2}\oint_{-\pi}^{\pi} \frac{1}{(a+\cos(\varphi))^2}\,d\varphi$$

$$= \frac{1}{2}\oint_{|z|=1} \frac{1}{\left(a+\frac{z+z^{-1}}{2}\right)^2\cdot z\,i}\,dz = \frac{1}{\cancel{2i}}2\pi\cancel{i}\sum \text{residuos encerrados por la curva}$$

$$\frac{1}{\left(a+\frac{z^2+1}{2z}\right)^2}\cdot\frac{1}{z} = \frac{1}{\left(\frac{1}{2z}\left(a2z+z^2+1\right)\right)^2}\cdot\frac{1}{z} = \frac{4z^2}{(a2z+z^2+1)^2}\cdot\frac{1}{z}$$

$$= \frac{4z}{(z^2+2az+1)^2} \qquad z^2+2az+1=0 \Leftrightarrow z = \frac{-2a\pm\sqrt{4a^2-4}}{2} \begin{cases} -a+\sqrt{a^2-1} \\ -a-\sqrt{a^2-1}\end{cases}$$

$$\Rightarrow \int_0^\pi \frac{1}{(a+\cos(\varphi))^2} = \pi\left[Res(f(z),-a+\sqrt{a^2-1}) + Res(f(z),-a-\sqrt{a^2-1})\right]$$

puede ser $-a+\sqrt{a^2-1} = 1$?

$a^2-1 = (1+a)^2$

$\cancel{a^2}-1 = 1+2a+\cancel{a^2}$

$-2 = 2a$ no

y $-a-\sqrt{a^2-1} = 1$?

$a^2-1 = (1+a^2)$ no

$-a+\sqrt{a^2-1}<1$

$-a-\sqrt{a^2-1}<1$ ✓

$\cancel{a^2}-1 < 1+2a+\cancel{a^2}$

$-2<2a$ ✓

$$\lim_{z\to(-a+\sqrt{a^2-1})} \frac{4z}{(z^2+2az+1)^2}$$ polo de orden 2

$$f(z)=\frac{4z}{\left((z+a-\sqrt{a^2-1})(z+a+\sqrt{a^2-1})\right)^2}$$

Polo de orden 2

$\Rightarrow \text{Res}(f,-a+\sqrt{a^2-1})$ $\ast$ Sea $\varphi(z)=\dfrac{4z}{(z+a+\sqrt{a^2-1})^2}$

$$\text{Res}(f,-a+\sqrt{a^2-1})=\frac{\partial\varphi(-a+\sqrt{a^2-1})}{\partial z \; 1!}=\frac{4(z+a+\sqrt{a^2-1})^2-4z.2(z+a+\sqrt{a^2-1})}{(z+a+\sqrt{a^2-1})^4}$$

$$=\frac{4(2\sqrt{a^2-1})^2-4(-a+\sqrt{a^2-1}).2(2\sqrt{a^2-1})}{(2\sqrt{a^2-1})^4}$$

$= 4.4.(a^2-1) -$ ... terminar de resolver no aporta conocimiento al cálculo.

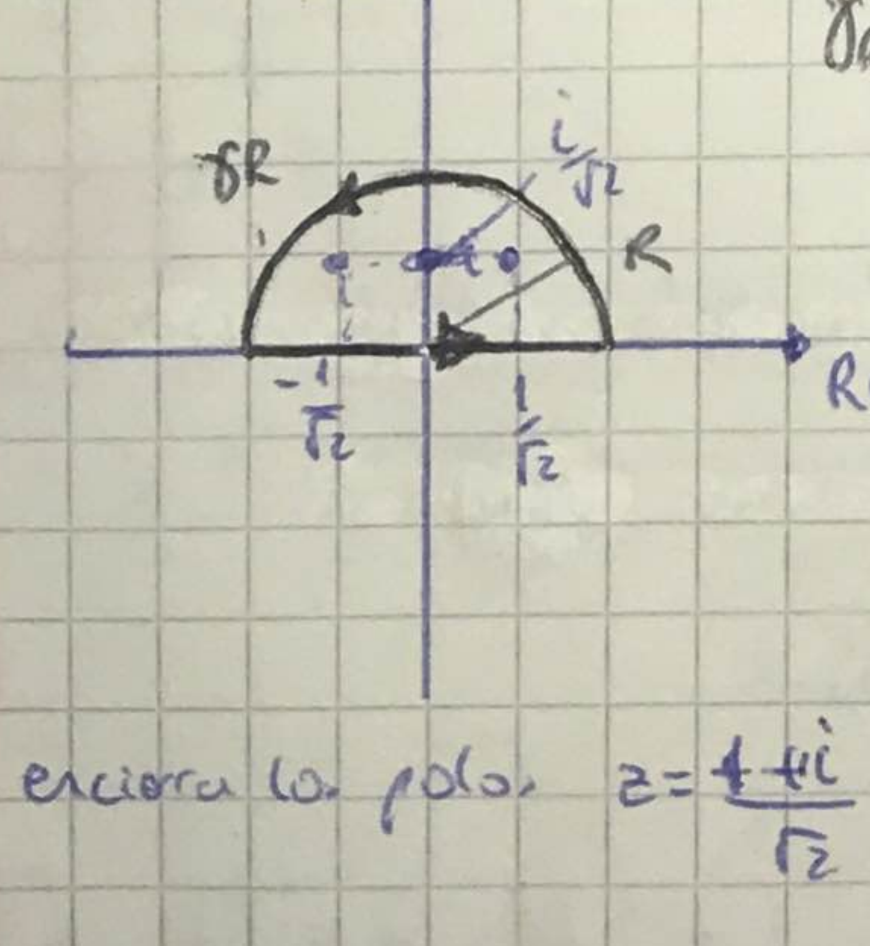
## Ejercicio 2

Probar que $\displaystyle\int_{-\infty}^{+\infty}\frac{x^2}{1+x^4}\,dx=\frac{\pi}{\sqrt{2}}$

$$\lim_{x\to\pm\infty}\frac{\frac{x^2}{1+x^4}}{\frac{x^2}{x^4}}=\lim_{x\to\pm\infty}\frac{1}{\frac{1}{x^4}+1}=1$$

$\Rightarrow$ como $\dfrac{x^2}{x^4}$ converge, esta también y converge a su V.P

Consideramos $f(z)=\dfrac{z^2}{1+z^4}$, que es una función holomorfa y la vamos a integrar en el siguiente intervalo




$\gamma_R=C_R\cup[-R,R]$ $1+z^4=0 \Leftrightarrow z^4=-1$

sea $w=z^2 \Rightarrow w^2=-1 \Leftrightarrow w=\pm i$

$w=i \Rightarrow i=z^2$

$a^2-b^2+2abi=i$

$2ab=1$ $\quad a=\frac{1}{2b}$

$a^2-b^2=0$

$\frac{1}{4b^2}-b^2=0$ $\quad b^4=\frac{1}{4}$

$z=\pm\frac{1+i}{\sqrt{2}}$ $\quad z=\pm\frac{1-i}{\sqrt{2}}$

encierra los polos $z=\frac{1+i}{\sqrt{2}}$, $z=\frac{-1+i}{\sqrt{2}}$

$$\Rightarrow \oint_{\gamma_R}\frac{z^2}{1+z^4}dz=2\pi i\left[\text{Res}\left(f,\frac{1+i}{\sqrt{2}}\right)+\text{Res}\left(f,\frac{-1+i}{\sqrt{2}}\right)\right]$$

$z=Re^{i\varphi}$

$dz=Rie^{i\varphi}d\varphi$

$$= \int_0^{\pi} \frac{R^3 e^{3i\varphi}\, i\, d\varphi}{1+R^4 e^{4i\varphi}} + \int_{-R}^{R} \frac{x^2}{1+x^4}\,dx = 2\pi i \left( Res(f, P_1) + Res(f, P_2) \right)$$

$P_1$ y $P_2$ son polos de orden 1 $\Rightarrow$

L'Hôpital: $$\lim_{z \to \frac{1+i}{\sqrt{2}}} \frac{1}{4z^3} = \frac{1}{4i\left(\frac{1+i}{\sqrt{2}}\right)} = \frac{1}{2\sqrt{2}(i-1)}$$

Finalmente $$Res\left(f, \frac{1+i}{\sqrt{2}}\right) = \frac{i}{2\sqrt{2}(1+i)} = \frac{i\ (1-i)}{\sqrt{2}^3 (1+i)(1-i)} = \frac{-i+1}{\sqrt{2}^3 (2)}$$

$$= \frac{-i+1}{4\sqrt{2}}$$

$$\lim_{z \to \frac{-1+i}{\sqrt{2}}} \frac{z^2}{1+z^4}\left(z - \left(\frac{-1+i}{\sqrt{2}}\right)\right) = \frac{0}{0} \quad \to -i$$

L'Hôpital: $$\lim_{z \to \frac{-1+i}{\sqrt{2}}} \frac{1}{4z^3} = \frac{1}{-4i\left(\frac{-1+i}{\sqrt{2}}\right)} = \frac{+1}{2\sqrt{2}(+i+1)}$$

$Res(f, -1$

$\Rightarrow 2\pi i$

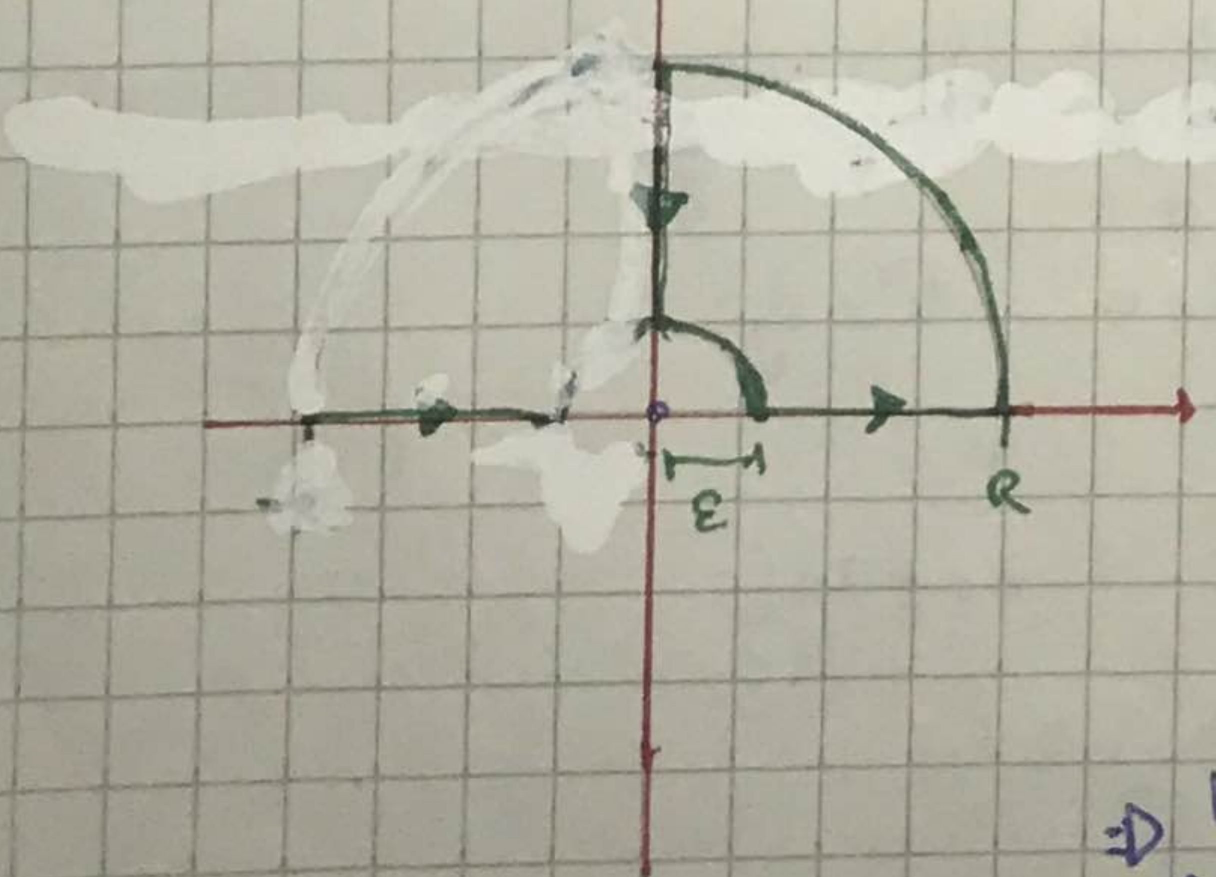
$$\lim_{R\to+\infty} \int_{-R}^{R} \frac{x^2}{1+x^4}\,dx = \lim_{R\to+\infty} \int_0^{\pi} \frac{R^3 e^{3i\varphi}\, i\, d\varphi}{1+R^4 e^{4i\varphi}} + \frac{\pi}{\sqrt{2}}$$

$$\int_0^{\pi} \frac{R^3 e^{3i\varphi}\, i\, d\varphi}{1+R^4 e^{4i\varphi}} \leq \left| \int_0^{\pi} \frac{R^3 e^{3i\varphi}\, i\, d\varphi}{1+R^4 e^{4i\varphi}} \right| \leq \int_0^{\pi} \frac{R^3\, d\varphi}{|1+R^4 e^{4i\varphi}|} \qquad |1+R^4 e^{4i\varphi}| \geq |R^4 e^{4i\varphi}| - |1|$$

$$\leq \int_0^{\pi} \frac{R^3}{R^4-1}\,d\varphi = \frac{R^3}{R^4-1}\cdot \pi \qquad \lim_{R\to+\infty} \frac{R^3}{R^4-1} = 0$$

$$\boxed{\text{Finalmente} \quad \int_{-\infty}^{+\infty} \frac{x^2}{1+x^4}\,dx = \frac{\pi}{\sqrt{2}}}$$

b) $$\int_0^{\infty} \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$$ Sea $f(z) = \frac{e^{iz}}{z}$ (Bien el resultado, mal resuelto, conviene agarrar entre 0 a $\pi$)




Integrando sobre este contorno

$$\oint_{|z|=R} \frac{e^{iz}}{z}\,dz + \cancel{\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx} + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx + \oint_{|z|=\varepsilon} \frac{e^{iz}}{z}\,dz = 0$$

eso no

holomorfa

$$\Rightarrow \lim_{R\to+\infty} \int_{\varepsilon}^{R} \frac{e^{ix}}{x} = -\int_{|z|=R} \frac{e^{iz}}{z}\,dz - \int_{|z|=\varepsilon} \frac{e^{iz}}{z}\,dz$$

$R \to +\infty$ $\qquad \to \varepsilon \to 0$

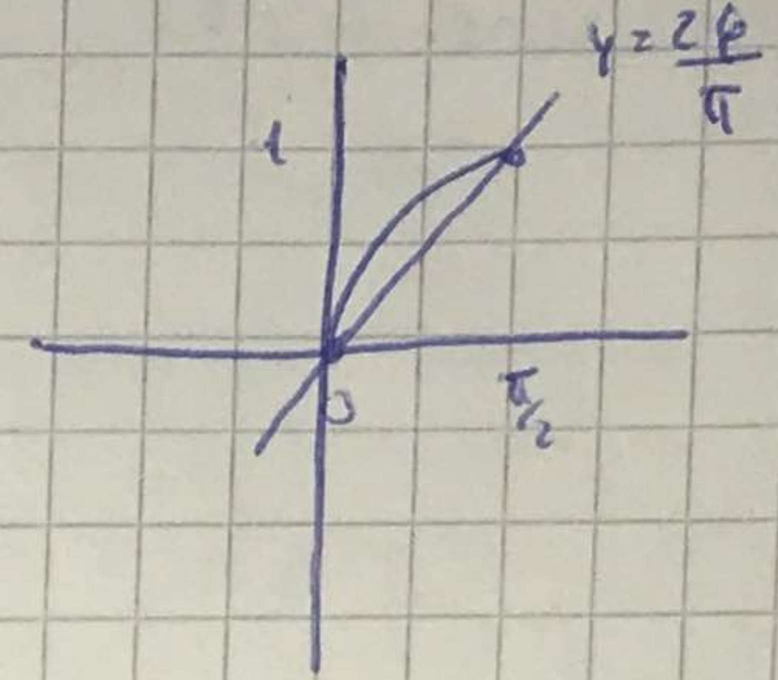
Parametrizando $z = Re^{i\varphi}$

$$dz = Rie^{i\varphi}d\varphi \qquad \varphi \in [0, \tfrac{\pi}{2}]$$

$$\Rightarrow \int_0^{\pi/2} \frac{e^{iRe^{i\varphi}}}{\cancel{Re^{i\varphi}}} \cancel{Rie^{i\varphi}} d\varphi \leq \left| \int_0^{\pi/2} e^{iRe^{i\varphi}} i d\varphi \right| \leq \int_0^{\pi} \left| e^{iR(\cos(\varphi) + i\sin(\varphi))} \right| d\varphi$$

$$= \int_0^{\pi/2} \left| e^{-R\sin(\varphi)} \right| d\varphi = \int_0^{\pi/2} e^{-R\sin(\varphi)} d\varphi$$

entre $0$ y $\pi/2$ $\int$




$$\sin(\varphi) \geq \frac{2\varphi}{\pi}$$

$$\Leftrightarrow -\sin(\varphi) \leq -\frac{2\varphi}{\pi}$$

$$\Rightarrow \int_0^{\pi/2} e^{-R\sin(\varphi)} d\varphi \leq \int_0^{\pi/2} e^{-\frac{R2\varphi}{\pi}} d\varphi = -\frac{\pi e^{-\frac{R2\varphi}{\pi}}}{2R} \Big|_0^{\pi/2} = -\frac{\pi}{2R}\left(\frac{1}{e^R} - 1\right) \xrightarrow[R \to +\infty]{} 0$$

$$\Rightarrow \underbrace{\lim_{\substack{R\to\infty \\ \varepsilon \to 0}}}_{\text{no formal}} \left( \int_\varepsilon^R \frac{e^{ix}}{x} dx = \int_{|z|=\varepsilon} \frac{e^{iz}}{z} dz \right)$$

Lema: Si $g$ tiene un polo simple en $z_0$ entonces $\int_{C_{\varepsilon,\theta}} g \xrightarrow{\varepsilon \to 0} i\theta \, Res(g, z_0)$

$$\Rightarrow \int_{|z|=\varepsilon} \frac{e^{iz}}{z} dz = i\frac{\pi}{2} \underbrace{Res(f,0)}_{=1 \text{ sing evitable con valor } 1} = i\frac{\pi}{2}$$

$$\Rightarrow \boxed{\int_0^{+\infty} \frac{\cos(x)}{x} dx + i\int_0^{+\infty} \frac{\sin(x)}{x} dx = i\frac{\pi}{2}}$$

## Ejercicio 3

a) $\int_0^{\infty} \frac{x^2}{x^4 + x^2 + 1} dx$ cambiando a complejo

$z = e^{i\varphi}R$

$dz = iRe^{i\varphi}d\varphi$

$z^4 + z^2 + 1 = 0$

$\omega = z^2 \quad \omega^2 + \omega + 1 = 0$

$\frac{-1 \pm \sqrt{1-4}}{2} \to \frac{-1+i\sqrt{3}}{2}, \ \frac{-1-i\sqrt{3}}{2}$

$z^2 = a^2 - b^2 + 2abi$

$z^2 = \frac{-1+i\sqrt{3}}{2} = e^{i120°}$

$z^2 = \frac{-1-i\sqrt{3}}{2} = e^{i240°}$

$a^2 - b^2 = -\frac{1}{2}$

$2ab = \frac{\sqrt{3}}{2} \qquad b = \frac{\sqrt{3}}{4a}$

$\Rightarrow a^2 - \frac{3}{16a^2} = -\frac{1}{2}$

$a^4 - \frac{3}{16} + \frac{a^2}{2} = 0$

$\frac{3}{16} \pm \sqrt{\frac{9}{16^2} - 4 \cdot \frac{1}{2}}$

$|z| = e^{i60°}$

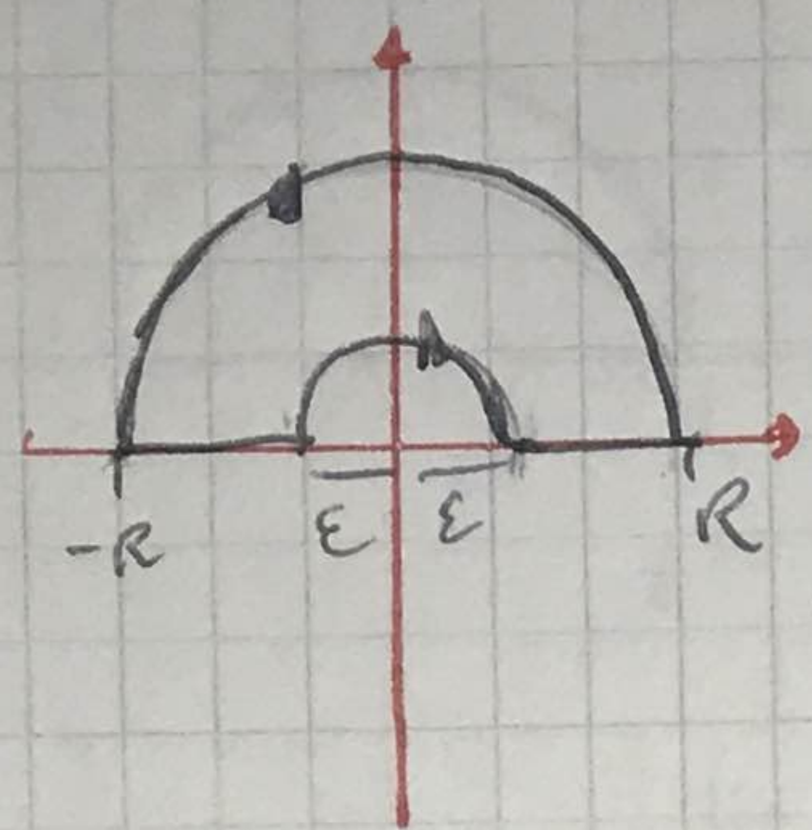
$|z| = e^{i120°}$

HOJA N°

FECHA

## Ejercicio 5

Calcular $\int_0^{\infty} \frac{\sin^2(x)}{x^2}\,dx$ considerando $f(z)=\frac{1-e^{2iz}}{z^2}$ y usando teo de res en un recinto apropiado




$\cos(2x) = \cos^2(x) - \sin^2(x)$

$\sin^2(x) = \cos^2(x) - \cos(2x)$

$\sin^2(x) = 1-\sin^2(x)-\cos(2x)$

$2\sin^2(x) = 1-\cos(2x)$

$\sin^2(x) = \frac{1-\cos(2x)}{2}$

$\sin^2(x) = \frac{1}{2} - \frac{e^{2ix}+e^{-2ix}}{4}$

$e^{2ix} = \cos(2x)+i\sin(2x)$

$= 2\sin^2(x)-1+i\sin(2x)$

$$\oint_{\gamma_R} \frac{1-e^{2iz}}{z^2} = ⊛$$

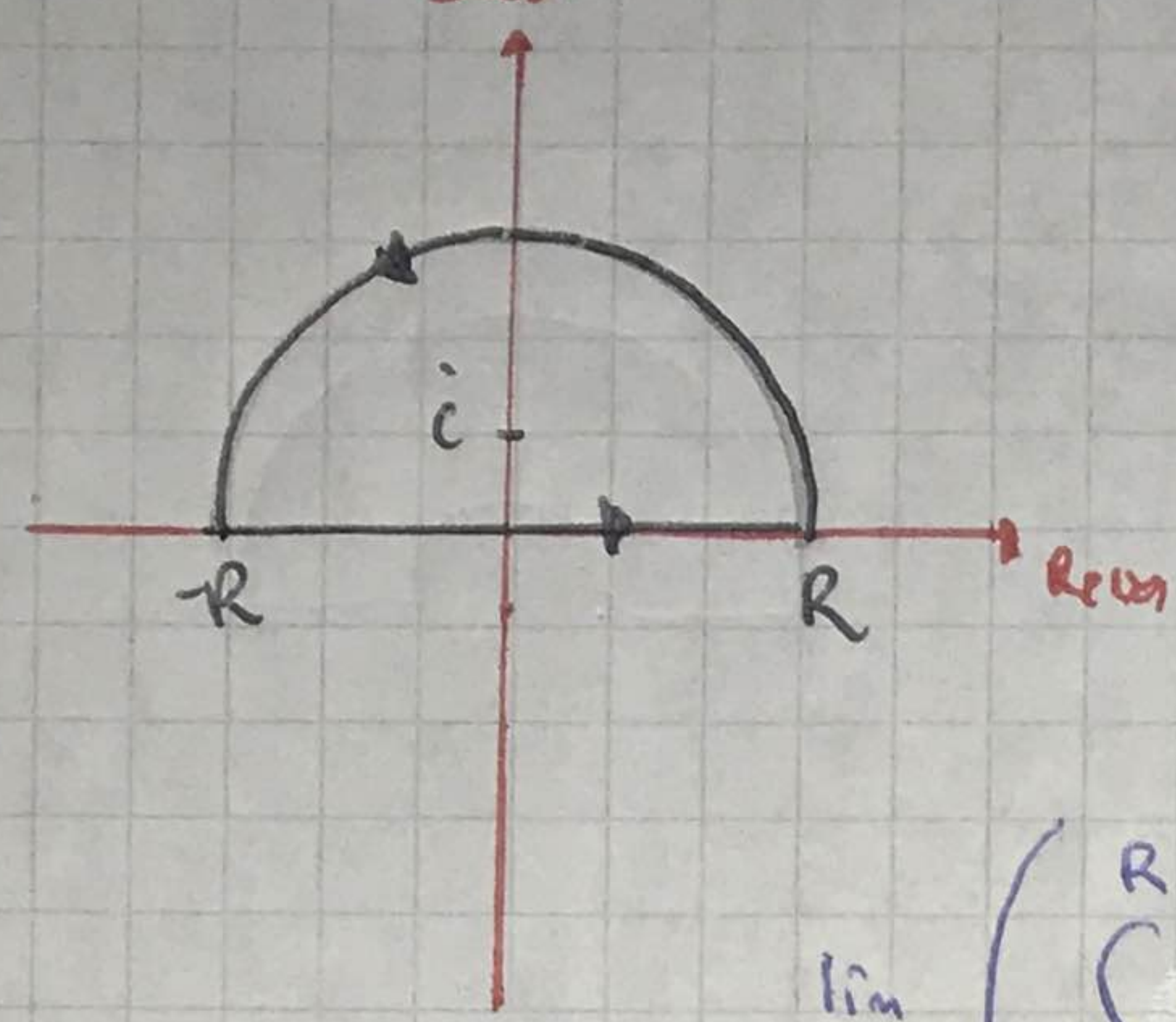
$$⊛ \quad \int_{\substack{0\\|z|=R}}^{\pi} \frac{1-e^{2iz}}{z^2}\,dz + \int_{-R}^{-\varepsilon}\frac{1-e^{2ix}}{x^2}\,dx + \int_{\varepsilon}^{R} \frac{1-e^{2ix}}{x^2}\,dx + \int_{|z|=\varepsilon} \frac{1-e^{2iz}}{z^2}\,dz = 0$$

NOTA

Ejercicio 6 Probar que:

b) $$\int_{-\infty}^{+\infty} \frac{\cos(\omega x)}{1+x^2}\,dx = \pi e^{-|\omega|}$$

$\cos(|\omega|x) = \cos(\omega x)$




tomo $f(z) = \dfrac{e^{i|\omega|z}}{1+z^2}$

$1+z^2=0 \iff z=i$
$z=-i$

Por residuos

$$\lim_{R\to+\infty}\left(\int_{-R}^{R} \frac{e^{i|\omega|x}}{1+x^2}\,dx + \int_{|z|=R} \frac{e^{i|\omega|z}}{1+z^2}\,dz\right) = 2\pi i \operatorname{Res}(f,i)$$

$$\operatorname{Res}(f,i) = \frac{e^{i|\omega|i}}{2i} \Rightarrow 2\pi i \operatorname{Res}(f,i) = \pi e^{-|\omega|}$$

$$\int_0^{\pi} \frac{e^{i|\omega|Re^{i\varphi}}}{1+R^2e^{2i\varphi}}\, iRe^{i\varphi}\,d\varphi \leq \left|\int_0^{\pi} \frac{e^{i|\omega|Re^{i\varphi}}}{1+R^2e^{2i\varphi}}\, iRe^{i\varphi}\,d\varphi\right|$$

$$\leq \int_0^{\pi} \frac{|e^{i|\omega|R(\cos(\varphi)+i\sin(\varphi))}|}{R^2-1}\, R\,d\varphi = \frac{R}{R^2-1}\int_0^{\pi} |e^{-|\omega|R\sin(\varphi)}|\,d\varphi$$

$$\leq \frac{R}{R^2-1}\int_0^{\pi} 1\,d\varphi = \frac{\pi R}{R^2-1} \xrightarrow{R\to+\infty} 0$$

$e^{-|\omega|R\sin(\varphi)} \leq 1$

$$\Rightarrow \boxed{\int_{-\infty}^{+\infty} \frac{\cos(|\omega|x) + i\sin(|\omega|x)}{1+x^2}\,dx = \pi e^{-|\omega|}}$$