Ejercicio 2.1

$x = X$ $\qquad f = (0,0,g)$

$y = Y$

$z = Z + \alpha(t)$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho 0 = \rho 0 \Rightarrow \underline{\nabla} \sigma_x = 0$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho 0 = \rho 0 \Rightarrow \underline{\nabla} \sigma_y = 0$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial z} + \frac{\partial \sigma_{zz}}{\partial z} + \rho g = \rho a_z = \rho \frac{\partial^2 z}{\partial t} = \rho \ddot{\alpha}$$

$$\Rightarrow \underline{\nabla} \sigma_z = \rho(\ddot{\alpha} - g)$$

Como σ es simétrico, pues no hay variación de momento, $\therefore \underline{\nabla} \sigma_x = \underline{\nabla}(\sigma_y + \underline{\nabla} \times \underline{B})$ podría solo moverse en una componente y seguiría valiendo.

$$\Rightarrow \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0 \Rightarrow \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0$$

$$\therefore \sigma_{xx} = \sigma_{xy} = \sigma_{yy} = \sigma_{yz} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0 \Rightarrow \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} = 0$$

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho(\ddot{\alpha} - g) \quad \therefore \frac{\partial \sigma_{zz}}{\partial z} = \rho(\ddot{\alpha} - g)$$

Sobre las caras dice que fuerza de contacto = 0

$\therefore \underline{\underline{\sigma}} \cdot \hat{x} = 0 = \sigma_{xx} = 0$

$\underline{\underline{\sigma}} \cdot \hat{y} = 0 = \sigma_{yy} = 0$

$\underline{\underline{\sigma}} \cdot \hat{z} \neq 0 \Rightarrow \sigma_{zz} \neq 0$

Ejercicio 2.2

Ejercicio 2.3 fluido en reposo con f de vol conservativas

ecuación indefinida $\rho \frac{\partial u_i}{\partial t} = \rho a_i \pm f_{vi} + \frac{\partial \sigma_{ij}}{\partial x_j} = 0$ $a_i = 0$ en reposo

ecuación de continuidad $\frac{\partial \rho}{\partial t} + \rho \underline{\nabla} \cdot \underline{u} = 0$ $\underline{F} = -\frac{\nabla \phi}{V}$ pot por unid de vol $\underline{f} = -\nabla \phi$ pot por unid de m

$\sigma_{ij} = -p\delta_{ij}$

a) Si el fluido es incompresible

$\Rightarrow \frac{\partial \rho}{\partial t} = 0 \therefore \underline{\nabla} \cdot \underline{u} = 0$ $\Rightarrow$ $f_{vi} = \frac{\partial}{\partial x_i}(\overbrace{-p\delta_{ij}}^{=-p})$ $\underline{f_v} = \rho(-\nabla\phi)$

$\Rightarrow (-\rho \nabla \phi)_i = \frac{\partial}{\partial x_i} p$ $\rho \neq \rho(x,y,z)$

$\Leftarrow$ $\underline{\nabla}(\rho\phi + p) = 0$ $\Rightarrow$ $\rho\phi + p = cte$

$\Rightarrow$

$$\boxed{\frac{p}{\rho} + \phi = cte}$$

b) si el fluido no intercambia calor

$H = U + pV$ entalpía $dH = \underbrace{\partial Q - PdV}_{dU} + PdV + dpV$

$\Rightarrow dH = \partial Q + Vdp = Vdp$ adiabático

$\frac{dH}{m} = dh = \frac{V}{m} dp = \frac{dp}{\rho} =$

$\Rightarrow h = \frac{p}{\rho} + c$ $\Rightarrow$ $h + \phi = \underbrace{\frac{p}{\rho} + \phi}_{cte} + c$

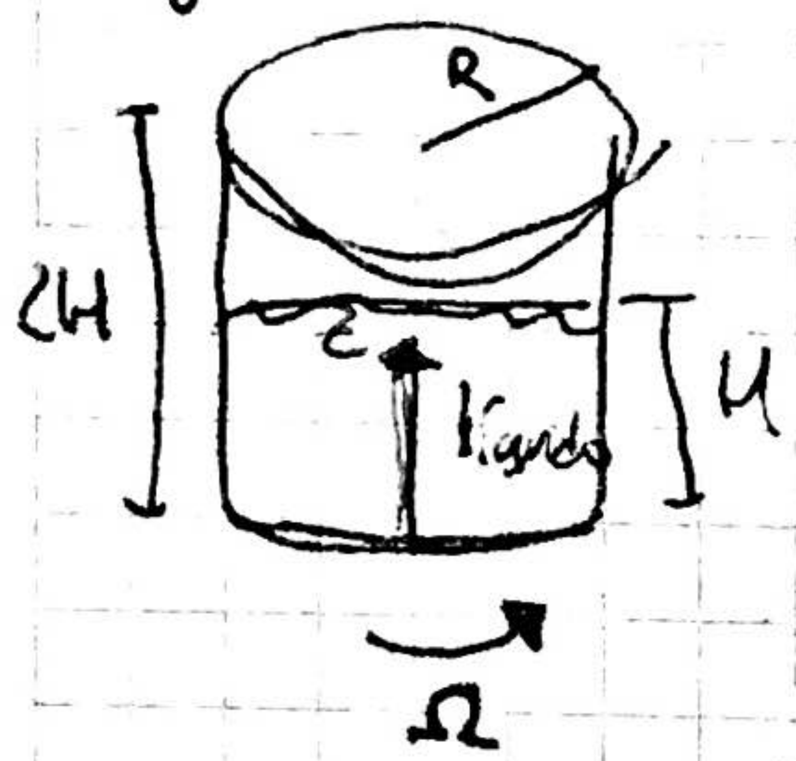
$$\Rightarrow \boxed{h + \phi = C}$$

c) $G = U - TS + PV$

$dG = \cancel{\partial Q} - \cancel{PdV} - \overbrace{\cancel{TdS}}^{\partial Q_{rev}} - \underset{\substack{=0\\ T\,cte}}{\partial T}S + \cancel{PdV} + Vdp$

$\Rightarrow dG = Vdp$ $\Rightarrow$ $dg = \frac{dp}{\rho}$ $\Rightarrow$ $\boxed{g + \phi = cte}$

Ejercicio 2.4 Tanquinche hidrostático



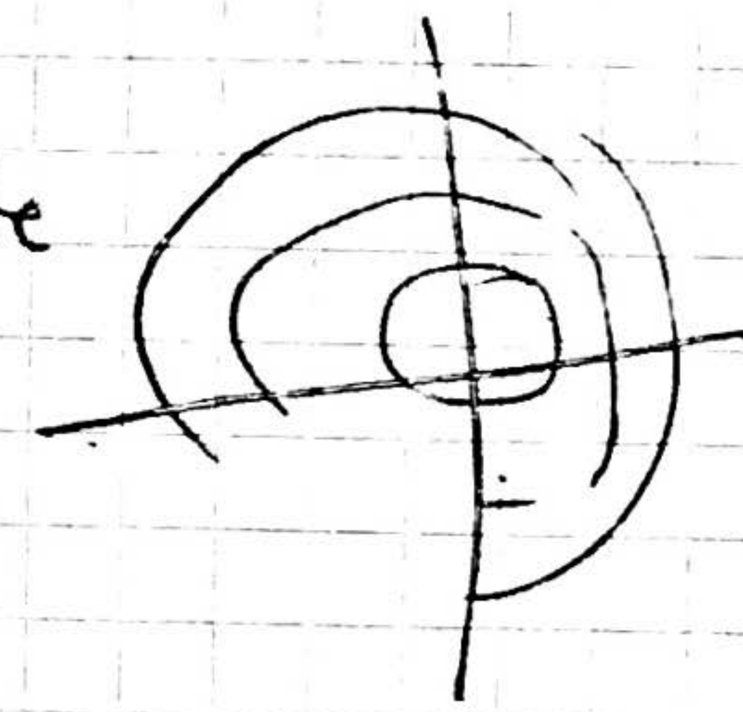
a) ¿Cuál es la forma de la superficie libre del líquido?

Sistemas de referencia posibles

FIJO (LABORATORIO)	NO INERCIAL (ROTANTE)
$\underline{v} \neq 0$	$\underline{v} = 0$
$\underline{v} = \underline{\Omega} \times \underline{r} = \Omega \hat{z} \times r\hat{r}$	$\underline{F} = \underline{F}_g + \underline{F}_{cor} + \underline{F}_{cp}$
$= \Omega r \hat{\varphi} = -\Omega y \hat{\imath} + \Omega x \hat{\jmath}$	$\underline{F}_{cor} = -2\underline{\Omega} \times \underline{v}$
	$\underline{F}_{cp} = -\underline{\Omega} \times (\underline{\Omega} \times \underline{r}) = -\Omega^2 r^2 \hat{r}$

lineas de corriente

$$\frac{dx}{v_x} = \frac{dy}{v_y} \qquad \frac{dx}{\Omega y} = \frac{dy}{\Omega x} \Leftrightarrow -\Omega x dx = \Omega y dy \Leftrightarrow \frac{x^2}{2} + \frac{y^2}{2} = cte$$



en el lab, desde el rotante?

b) $$\rho \underline{a} = \underline{f_v} - \underline{\nabla} \cdot p = \rho \frac{d\underline{v}}{dt} = \rho \Big(\underbrace{\frac{\partial \underline{v}}{\partial t}}_{=0} + (\underline{v} \cdot \underline{\nabla})\underline{v} \Big)$$

Euler

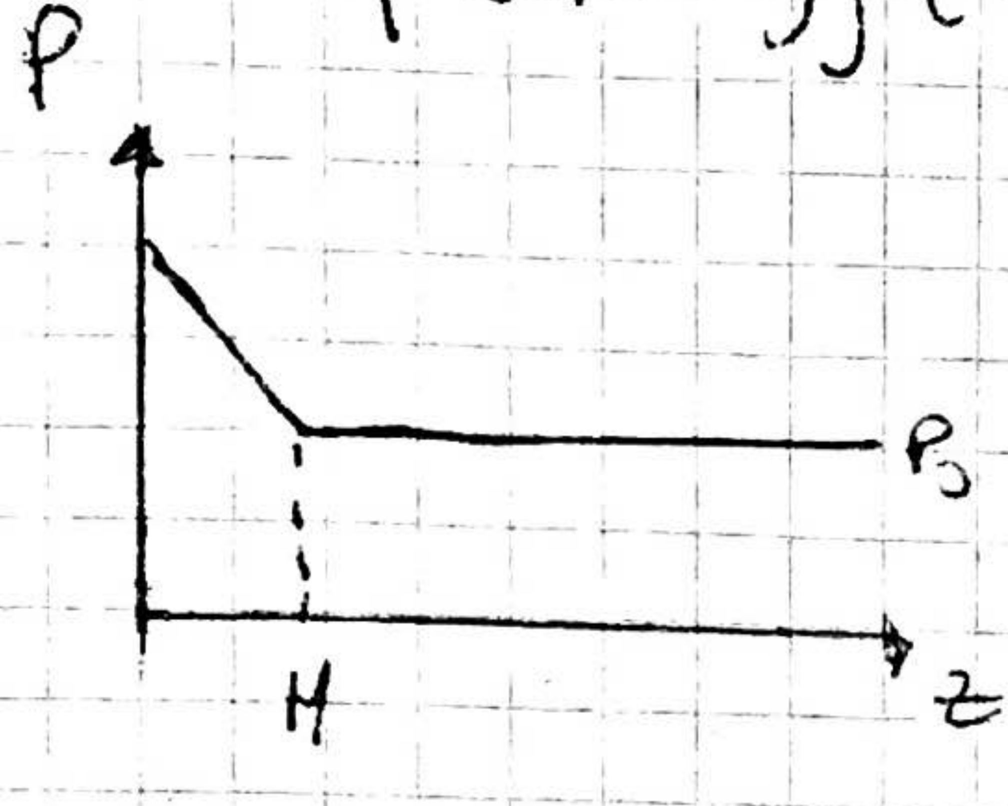
* Si $\Omega = 0$ (hidrostática)

$$\Rightarrow \underline{\nabla} p = \underline{f_v} = -g\rho \hat{z} \Rightarrow \int_{p_0(z_0)}^{p(z)} \partial p \, \hat{z} = -g\rho \int_{z_0}^{z} \partial z \, \hat{z}$$

$$\Rightarrow p - p_0 = -g\rho (z - z_0)$$

$z_0 = H \Rightarrow p(H) = p_0$

$$\Rightarrow p(z) = -g\rho(z - H) + p_0 \qquad 0 < z < H$$



* Si $\Omega \neq 0$

(si no lo hicieran así, con $(\underline{v}\cdot\underline{\nabla})\underline{v}$

$$=\left(\Omega r\hat{\varphi}\cdot\left(\partial_r\hat{r}+\frac{1}{r}\partial_\varphi\hat{\varphi}\right)\right)\Omega r\hat{\varphi}$$
$$=\Omega\partial\varphi\,\Omega r\hat{\varphi}=-\Omega^2 r\hat{r}$$

Me paro sobre un sistema no inercial

$$\frac{d\underline{v}}{dt}=0=\underline{f}_v-\underline{\nabla}p+\underline{F}_{\underline{v}cor}+\underline{F}_{\underline{v}cf}$$

$$\Leftrightarrow\ \underline{\nabla}p=-\rho g\hat{z}+\Omega^2 r\hat{r}\rho$$

$$\underline{\nabla}p=\partial_r p\,\hat{r}+\frac{1}{r}\partial_\varphi p\,\hat{\varphi}+\partial_z p\,\hat{z}$$

$$\Rightarrow\ \frac{\partial p}{\partial z}=-\rho g\ \Leftrightarrow\ p=-\rho g z+f(r,\varphi)\ \Rightarrow\ \frac{\partial p}{\partial\varphi}=\frac{\partial f(r,\varphi)}{\partial\varphi}=0\ \Rightarrow\ f(r,\varphi)=f(r)$$

$$\Rightarrow\ \frac{\partial p}{\partial r}=f(r)=\Omega^2 r\rho\ \Leftrightarrow\ p=-\rho g z+\rho\frac{\Omega^2 r^2}{2}+C$$

$$\Rightarrow\ z(p,r)=-\frac{p}{\rho g}+\frac{\Omega^2 r^2}{2g}+C$$

Sobre la superficie $p=p_0$ $\Rightarrow$ $z_{sup}(r)=\underbrace{-\frac{p_0}{\rho g}+C}_{K}+\frac{\Omega^2 r^2}{2g}$

$$\Rightarrow\ \boxed{z_{sup}(r)=\frac{\Omega^2 r^2}{2g}+K}$$

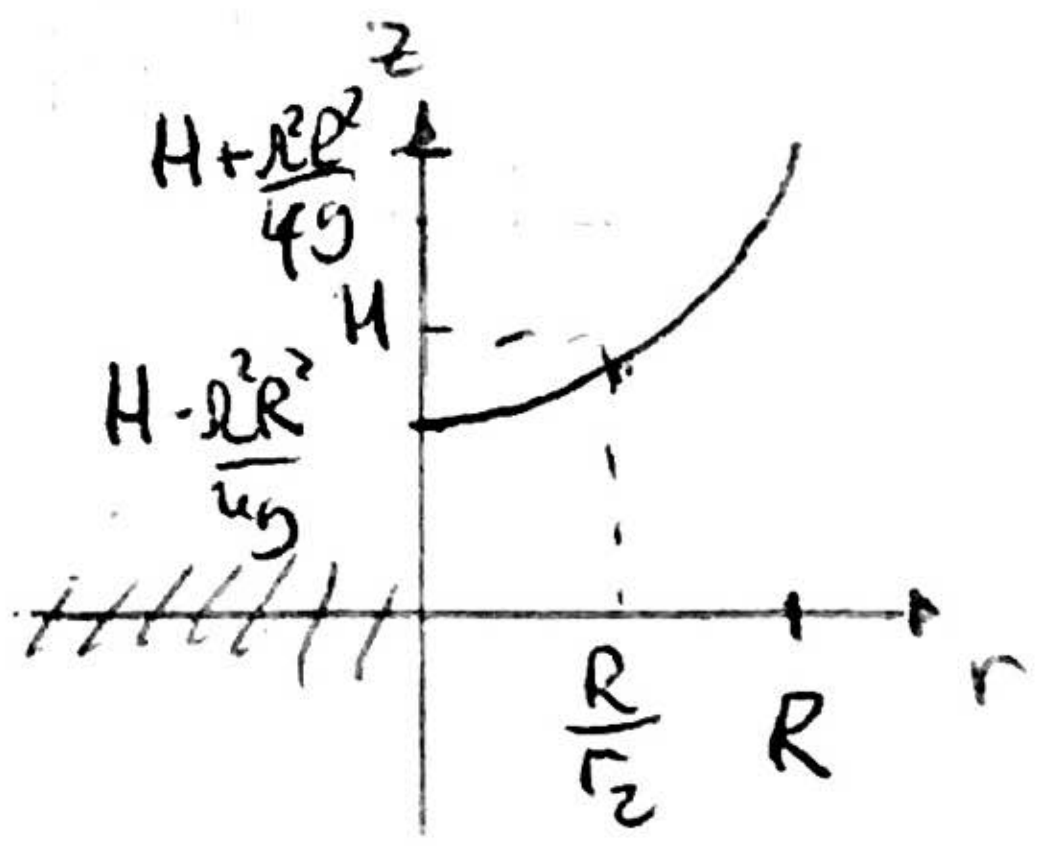
el volumen no cambia $\Rightarrow$ K sale de conservación de volumen

Si $\Omega=0$, $V=\pi R^2 H$

Si $\Omega\neq0$, $V=\int_0^R\int_0^{2\pi}z(r)\,r\,dr\,d\varphi=2\pi\left(\frac{\Omega^2R^4}{8g}+\frac{KR^2}{2}\right)$

$$\Leftrightarrow\ \pi R^2H=2\pi\left(\frac{\Omega^2R^4}{8g}+\frac{KR^2}{2}\right)\ \Leftrightarrow\ K=H-\frac{\Omega^2}{4g}R^2$$

$$\Rightarrow\ \boxed{z_{sup}(r)=\frac{\Omega^2r^2}{2g}+H-\frac{\Omega^2R^2}{4g}}$$



b) Para que Ω toca fondo? $z_{sup}(0)=0$ $\Leftrightarrow$ $\boxed{H=\frac{\Omega^2R^2}{4g}}$ $\Leftrightarrow$ $\Omega=\sqrt{\frac{H4g}{R^2}}$

c) ¿Para que Ω desborda? ~~$z_{sup}(2H)$~~ $z_{sup}(R)>2H$

$$\Leftrightarrow\ H+\frac{\Omega^2R^2}{4g}\geq 2H\ \Leftrightarrow\ \Omega^2>\frac{H4g}{R^2}$$

$\therefore$ cuando toca fondo, desborda

c) $\Omega^2 \geq \frac{7.5cm \cdot 4g}{(5cm)^2}$ límite $\Rightarrow \Omega^2 = \frac{7.5}{25} \cdot 4 \cdot 9.8 \left(\frac{100cm}{cm\, s^2}\right)$

$\Rightarrow |\Omega| \cong \frac{34,29}{s} = 34.29$ Hertz

d) $\underline{v} = 90 \frac{vueltas}{min} \cdot \hat{\varphi}$

Perímetro $= 2\pi R$ 1 vuelta $\Rightarrow 2\pi R$

$\Rightarrow \underline{v} = \frac{90\ 2\pi R}{60s} = \frac{90\ 2\pi \cdot 5cm}{60s} = 15\pi \frac{cm}{s} \hat{\varphi}$

P si $\Omega = 0$ ya lo hice

P si $\Omega \neq 0$

Sobre piso $p(0, r) = \frac{-\Omega^2 R^2}{2g} + p_0$

Sobre paredes $p(z, R) = -\rho g z + \frac{\Omega^2 R^2}{2g} + p_0$

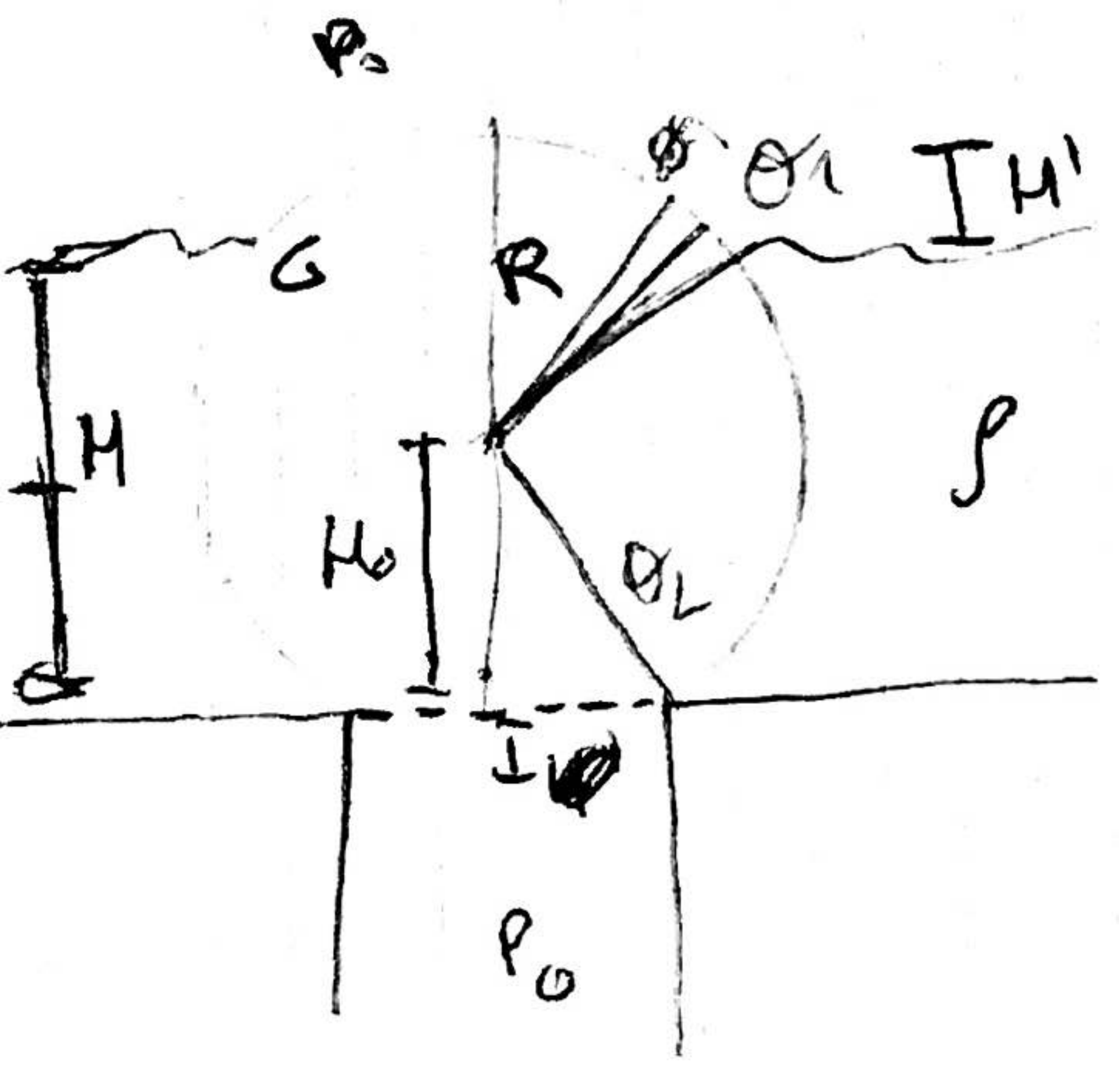
NO APORTA

e) Sobre la superficie se que $p = p(z_{sup}, R) \Rightarrow P_0 =$

$|\underline{v}| = |\Omega^2 r \hat{\varphi}|$ r puedo medirlo fácil, z también $\Rightarrow$ tengo $P(\Omega)$ o $\Omega(P)$

como sobre la sup $P = P_0 \Rightarrow$ Tengo Ω

Ejercicio 2.5



$\Rightarrow$ ~~$R\cos(\theta_1) = H'$ $H = R\cos(\theta_1) - R\cos(\theta_2)$~~

$R\cos(\theta_2) = H_0$ ~~$\Rightarrow H' =$~~

$H - H_0 = R\cos(\theta_1)$

a) Empuje (H) sobre la esfera

El empuje será igual al peso del líquido desalojado

$$\underline{E} = -\oint_S p\,\hat{n}\,dS = -\int_V \underline{\nabla} p\, dV$$

p continua

Hidrostática

$$\Rightarrow \frac{d\underline{v}}{dt} = 0 = \underline{f_v} - \underline{\nabla} p \Rightarrow \underline{\nabla} p = -\rho g \hat{z} \Rightarrow p = -\rho g(z - z_0) + p_0$$

fluido

$$p = \begin{cases} p_0 - \rho g (z - z_0) & -H_0 < z < H - H_0 \\ p_0 & \text{si el resto} \end{cases}$$

$$\Rightarrow \underline{E} = -\int_{S_1} p_0 \hat{n}\, dS + \int_{S_2} [p_0 - \rho g (z - z_0)]\,\hat{n}\, dS + \int_{S_3} p_0 \hat{n}\, dS$$

$$= \underbrace{\oint_S -p_0 \hat{n}\, dS}_{=0} + \int_{S_2} +\rho g (z - z_0)\,\hat{n}\, dS$$

$$\Rightarrow \underline{E} = \int_{S_2} \rho g (z - z_0)\,\hat{n}\, dS \qquad S_2 = (r\cos(\theta), r\sin(\theta), z) \qquad r = \sqrt{R^2 - z^2}$$

$$\Rightarrow \frac{\partial r}{\partial z} = -\frac{z}{r}$$

$$\partial_\theta S_2 = (-r\sin(\theta), r\cos(\theta), 0)$$

$$\partial_z S_2 = \left(-\frac{z}{r}\cos(\theta), -\frac{z}{r}\sin(\theta), 1\right)$$

$$\Rightarrow S_\theta \times S_r = (r\cos(\theta), r\sin(\theta), z) \qquad \|S_\theta \times S_r\| = \sqrt{(r^2 + z^2)} = R$$

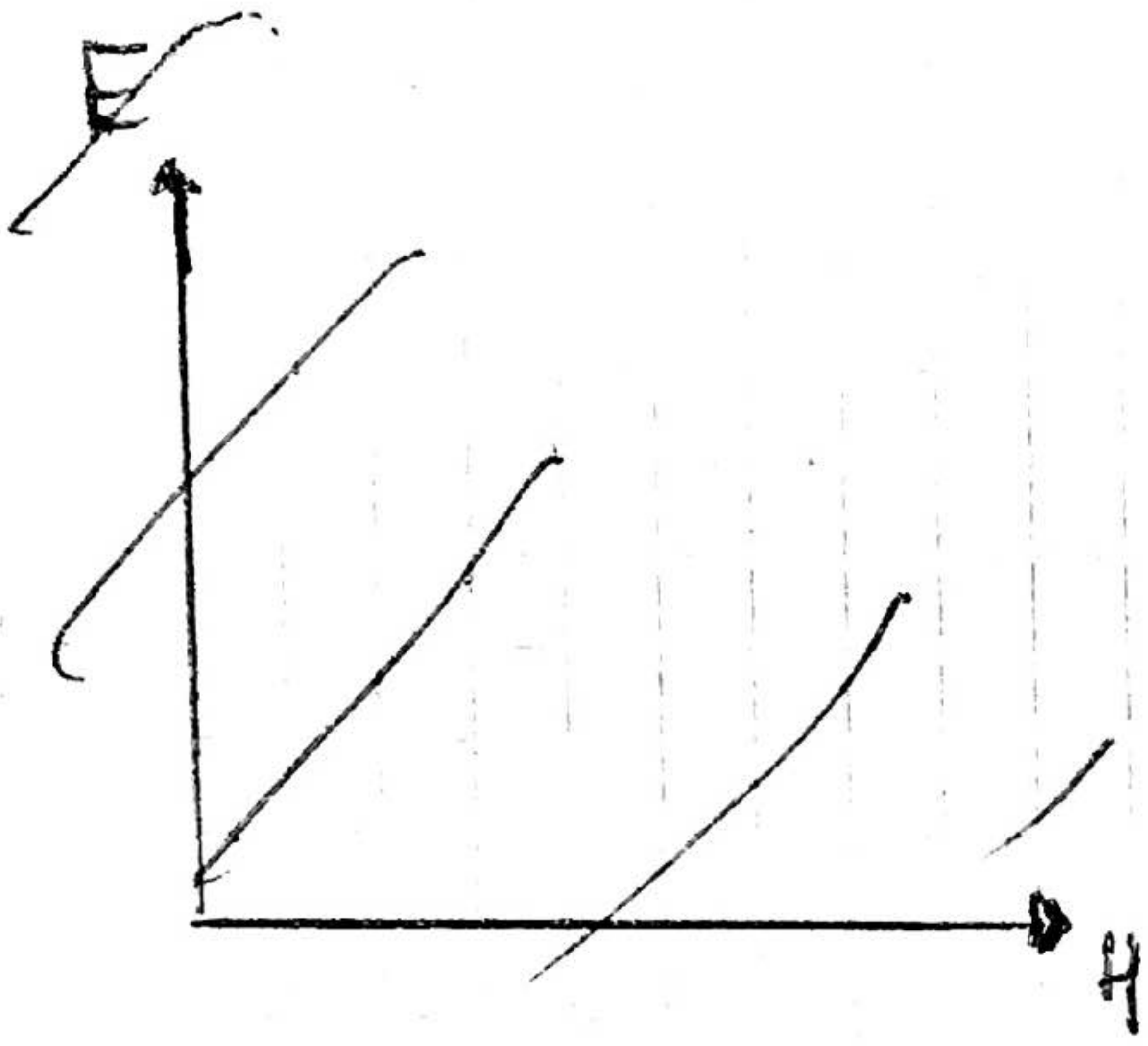
$$\Rightarrow \hat{n} = \left(\frac{r\cos(\theta)}{R}, \frac{r\sin(\theta)}{R}, \frac{z}{R}\right)$$

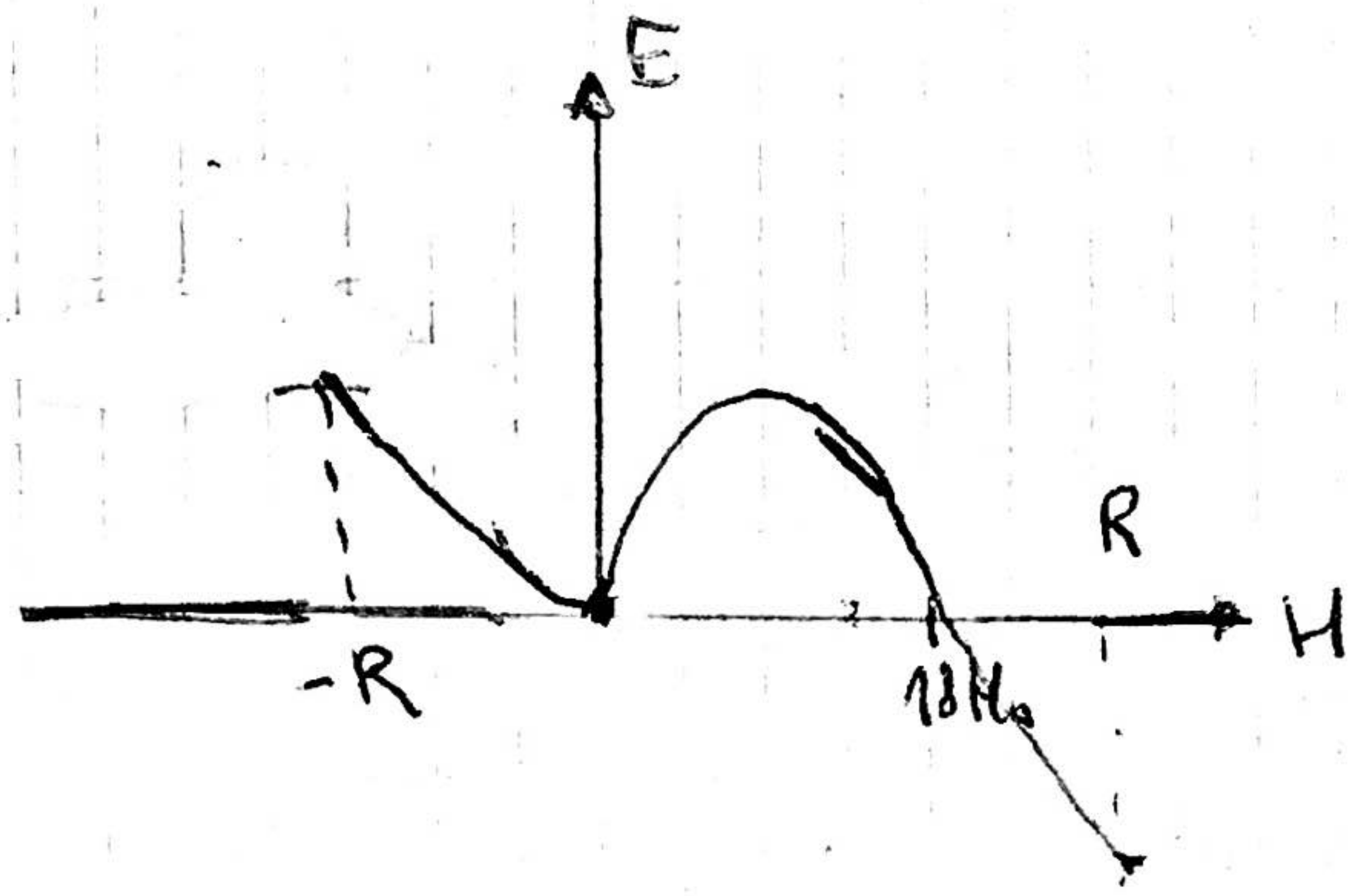
$$\Rightarrow \underline{E} = \int_0^{2\pi}\int_{-H_0}^{H-H_0} \rho g (z - R\cos(\theta_2)) \left(\frac{r\cos(\theta)}{R}, \frac{r\sin(\theta)}{R}, \frac{z}{R}\right) R\, d\theta\, dz$$

$$\Rightarrow \underline{E} = \int_{-H_0}^{H-H_0} 2\pi \rho g (z^2 - z z_0)\, dz\, \hat{z} = 2\pi\rho g \left(\frac{(H-H_0)^3}{3} + \frac{H_0^3}{3} - \frac{(H-H_0)^2}{2} + \frac{H_0^2}{2}\right)\hat{z}$$

Wolfram $= \boxed{2\pi\rho g\left(-\frac{H^3}{6}+3H_0H^2\right)\hat{z} = \underline{E}}$ $\quad = 2\pi\rho g H^2\left(-\frac{H}{6}+3H_0\right)=0$ $\quad H=0$

o $H = 18H_0$





~~[illegible]~~ $H = H^{máx} = 2R$

Obturación para todo H $\quad m\overset{\text{exten}}{\underline{A_{cm}}} = -gm + 2\pi\rho g\left(-\frac{H^3}{6}+3H_0H^2\right)$

Ejercicio 26   a)

$\sigma_{ij} = -p(\rho, T)\delta_{ij}$

$$\rho a_i = f v_i + \frac{\partial \sigma_{ij}}{\partial x_i} = f v_i - \frac{\partial}{\partial x_i} p(\rho, T)\delta_{ij}$$

cadena

$$\Rightarrow \rho a_i = f v_i - \frac{\partial p}{\partial \rho}\frac{\partial \rho}{\partial x_i} - \frac{\partial p}{\partial T}\frac{\partial T}{\partial x_i}$$

$$\boxed{\Rightarrow \rho \underline{a} = \underline{f v} - \frac{\partial p}{\partial \rho}\nabla\rho - \frac{\partial p}{\partial T}\nabla T}$$

b) El agua se modela con $p = K\frac{(\rho - \rho_0)}{\rho_0}$ K constante, ρ_0 = densidad en ausencia de presión

$\underline{f} = (0,0,g)$ $\quad z = 0,\ \rho = \rho_0$

$\Rightarrow$ ¿p, ρ? $\quad \rho \underline{a} = (0,0,+\rho g) - \frac{K}{\rho_0}\left(\frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z}\right)$

reposo $a = 0$

$\Rightarrow \hat{x}: \ 0 = -\frac{K}{\rho_0}\frac{\partial \rho}{\partial x} \qquad \hat{k}: \ 0 = +\rho g - \frac{K}{\rho_0}\frac{\partial \rho}{\partial z}$

$\hat{j}: \ 0 = -\frac{K}{\rho_0}\frac{\partial \rho}{\partial y}$

$\Rightarrow \frac{\partial \rho}{\partial z} - \frac{g}{K\rho_0}\rho = 0 \quad (0)$

$\Rightarrow \rho(x,y,z) = \rho(z)$

$(0) \Rightarrow \int_{\rho_0}^{\rho}\frac{d\rho}{\rho} = \int_{z_0=0}^{z}\frac{\cancel{\rho_0} g}{K\rho_0}dz \Rightarrow \ln\left(\frac{\rho}{\rho_0}\right) = \cancel{\rho_0}\frac{g}{K}z \Rightarrow e^{\ln(\rho)}e^{-\ln(\rho_0)} = e^{\cancel{\rho_0}\frac{g}{K}z}$

$\therefore \ \boxed{\rho = \rho_0 e^{\cancel{\rho_0}\frac{g}{K}z}}$ $\qquad \Rightarrow p = K\frac{\rho_0}{\rho_0}\left(e^{-\frac{g}{K}z} - 1\right)$

$\therefore \ \boxed{p = K\left(e^{\cancel{\rho_0}\frac{g}{K}z} - 1\right)}$

$\lim_{K\to\infty} K\left(e^{-\frac{g}{K}z} - 1\right) = \infty \cdot 0$

$\lim_{K\to\infty} \frac{\left(e^{-\frac{g}{K}z} - 1\right)}{\left(\frac{1}{K}\right)\to 0} = \frac{0}{0}$ L'Hôpital

c)

$\rho = \frac{p}{K} + cte \Rightarrow \rho \to \rho cte$

$\Rightarrow \rho \neq \rho(p,T) = \rho(x,y,z)$

$\Rightarrow 0 = (0,0,g) - \nabla p \Rightarrow p = p(z)$

$\frac{\partial p}{\partial z} = g\rho_0 \Leftrightarrow P = p_0 + g\rho_0 z$ despreciando p_0

$\Rightarrow P = K\left(e^{\frac{\rho_0 g z}{K}} - 1\right)$ si $z = 1000m$ $\rho_0 = 1000 kg/m^3$

$K = 2e^4 N/m^2$

$P \approx g\rho_0 z + p_0$

comp $P(0) = 0$ comp $P(1000m) \approx 9824049.264 \, N/m^2$

incomp $P(0) = P_0$ desprecio incomp $P(1000m) \approx 9800000 \, N/m^2$

$\frac{P_{comp}(1000)}{P_{incomp}(1000)} = 1.00245 4007$; $P_{comp} - P_{incomp} = 24049.264$

si $P_{com} = 100\%$ $\Rightarrow$ la dif es menor al 1%

$P_{comp} - P_{incomp} \approx 0.3\%$

d) $p = \frac{\rho R T}{m}$ $f = (0,0,-g)$

$\Rightarrow \rho = \rho(p,T)$

$\Rightarrow \hat{z} g\rho = -\frac{\partial \rho}{\partial p}\nabla p - \frac{\partial \rho}{\partial T}\nabla T = \left(-\underbrace{\frac{RT}{m}\frac{\partial \rho}{\partial z} - \frac{\rho R}{m}\frac{\partial T}{\partial z}}_{\frac{dp}{dz}}\right)\hat{z}$

$\Leftrightarrow \rho g + \frac{RT}{m}\frac{\partial \rho}{\partial z} + \frac{\rho R}{m}\frac{\partial T}{\partial z} = 0$

ó $\frac{g m p}{RT} + \frac{d}{dz}p = 0$

d) $p = \rho \frac{RT}{m}$ $\quad f = (0, 0, -g)$

$$-\hat{z} g \rho - \underline{\nabla} p = 0$$

reposo

$$\Leftrightarrow +\frac{g p m}{RT} + \frac{\partial}{\partial z} p = 0 \Leftrightarrow \int_{p_0}^{p} \frac{dp}{p} = \int_0^z -\frac{g m \, dz}{RT(z)}$$

$$\Leftrightarrow \ln\left(\frac{p}{p_0}\right) = \int_0^z -\frac{g m}{RT(z')} dz'$$

$$\Rightarrow \boxed{p = p_0 e^{-\int_0^z \frac{g m}{RT(z')} dz}}$$

e) $0 = 0 - \frac{\partial}{\partial x} p \Leftrightarrow p_0 e^{-\int_0^z \frac{g m}{RT(z')} dz} \cdot - \frac{mg}{R} \frac{\partial}{\partial x} \int_0^z \frac{dz'}{T(z')} = 0$

$$\Leftrightarrow \frac{\partial}{\partial x} \int_0^z \frac{dz'}{T(z')} = 0 \Leftrightarrow \int_0^z \frac{\partial}{\partial x}\left(\frac{1}{T(z')}\right) dz' = 0$$

$$\Leftrightarrow \int_0^z -\frac{1}{T^2(z')} \cdot \frac{\partial T(z')}{\partial x} dz' = 0 \qquad \Rightarrow = 0 \text{ si } \frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = 0$$

Analogo en y $\quad \int_0^z -\frac{1}{T^2(z')} \frac{\partial T}{\partial y}(z') dz' = 0$

$\therefore$ Sí depender de x, y no puede ser

$$\rho g + \frac{RT}{m} \frac{\partial \rho}{\partial z} + \frac{\rho R}{m} \frac{\partial T}{\partial z} = 0 \qquad \text{de } z \text{ no sé}$$

Ejercicio 2.7

$\rho\rho = de^{-\gamma}$ $\quad p = \rho\frac{RT}{m}$ $\quad \underline{f}_v = \rho g$

$\rho^{-\gamma} p = c$

$0 = (0, 0, -g) - \underline{\nabla} p$

$\rho^{-\gamma} p = c \Rightarrow \rho = c p^{-\gamma}$

$$0 = -\rho g - \frac{\partial}{\partial z} p \Leftrightarrow g c p^{-\gamma} + \frac{\partial p}{\partial z} = 0 \Leftrightarrow -\int_0^z g c\, dz = \int_{p_0}^{p} p^{\gamma} dp$$

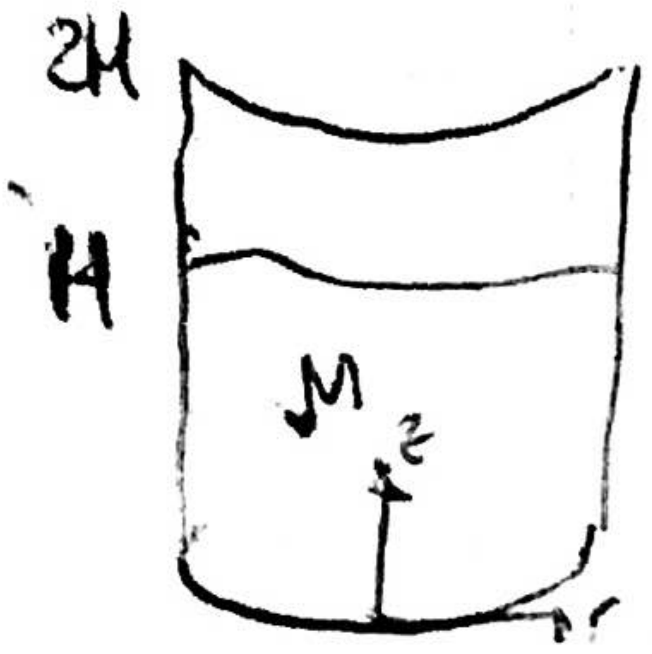
$$\Leftrightarrow -gcz = \left.\frac{p^{\gamma+1}}{\gamma+1}\right|_{p_0}^{p} \Leftrightarrow -gc(\gamma+1)z + p_0^{\gamma+1} = p^{\gamma+1}$$

$$\Rightarrow \rho = c\, p^{-\gamma} = c\left[\left(-gc(\gamma+1)z + p_0^{\gamma+1}\right)^{-(\gamma+1)}\right]^{-\gamma}$$

$$\Rightarrow \rho = c\left[-gc(\gamma+1)z + p_0^{\gamma+1}\right]^{\gamma(\gamma+1)}$$

$$T = \frac{mpR}{\rho} = \frac{mR\left(-gc(\gamma+1)z + p_0^{\gamma+1}\right)^{-(\gamma+1)}}{c\left(-gc(\gamma+1)z + p_0^{\gamma+1}\right)^{\gamma(\gamma+1)}} = \frac{mR}{c}\left(-gc(\gamma+1)z + p_0^{\gamma+1}\right)^{-(\gamma+1)(1+\gamma)}$$

Ejercicio 2.8



$T = cte$

$$p = \rho \frac{RT}{M}$$

$$0 = (0, 0, -\rho g) - \underline{\nabla} p$$

$$\Leftrightarrow \rho g + \frac{\partial p}{\partial z} = 0 \Leftrightarrow -\frac{pM}{RT} g = \frac{\partial p}{\partial z} \Leftrightarrow \int_H^z -\frac{Mg}{RT} dz = \int_{p_0}^{p} \frac{dp}{p}$$

$$-\frac{Mg}{RT}(z - H) = \frac{Mg}{RT}(H - z) = \ln\left(\frac{p}{p_0}\right)$$

$$\Rightarrow \boxed{p = p_0 e^{\frac{Mg}{RT}(H-z)}} \qquad \Rightarrow \rho = \frac{M}{RT} p_0 e^{\frac{Mg}{RT}(H-z)}$$

Ejercicio 2.9

$$\underline{E} = -\nabla \psi \quad / \quad \nabla^2 \psi = 4\pi G \rho$$

Usando la fórmula en pág 1

$$\frac{p}{\rho} + \psi = cte$$

a) $\rho = cte$

$$\Leftrightarrow \left(\frac{\underline{\nabla} p}{\rho} + \nabla \psi\right) = 0$$

$$\Leftrightarrow \underline{\nabla}\left(\frac{\underline{\nabla} p}{\rho}\right) = -\nabla^2 \psi = -4\pi G \rho$$

asumimos que solo importa la dirección r pues $\rho = cte$

$\Rightarrow$ por simetría no dependerá de direcciones angulares

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial p}{\partial r \, \rho}\right) = -4\pi G \rho \Leftrightarrow \frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial p}{\partial r}\right) = -4\pi G r^2 \rho \quad (1)$$

$$\Leftrightarrow \frac{2r}{\rho}\frac{\partial p}{\partial r} + \frac{r^2}{\rho}\frac{\partial^2 p}{\partial r^2} = -4\pi G r^2 \rho \Leftrightarrow \frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} + 4\pi G \rho^2 = 0$$

$$(1) \Leftrightarrow \frac{r^2}{\rho}\frac{\partial p}{\partial r} = -\frac{4\pi G r^3 \rho}{3} \Leftrightarrow \frac{\partial p}{\partial r} = -\frac{4\pi G \rho^2 r}{3} \Leftrightarrow p = +\frac{2\pi G}{3}\rho^2 (a^2 - r^2)$$

Si tuviera $r \to \infty$ tiene $p \to \infty$

b) $P = c\rho^{6/5}$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{\cancel{P}\rho^{6/5}}{\rho}\right)\right) = -4\pi G\rho \quad (2)$$

$$\Leftrightarrow 2r\frac{\partial}{\partial r}\left(c\rho^{1/5}\right) + r^2\frac{\partial^2}{\partial r^2}\left(c\rho^{1/5}\right) = -4\pi G\rho r^2$$

$$(2) \Leftrightarrow r^2\frac{\partial}{\partial r}\left(c\rho^{1/5}\right) = -\frac{4\pi G\rho r^3}{3} \Leftrightarrow c\rho^{1/5} = \frac{2\pi G\rho(a^2-r^2)}{3}$$

$$\Leftrightarrow \frac{3c}{2\pi G}\left(\frac{1}{a^2-r^2}\right) = \rho^{4/5}$$

$$\Leftrightarrow \rho = \left[\frac{3c}{2\pi G}\left(\frac{1}{a^2-r^2}\right)\right]^{5/4}$$

c) $\rho = \rho_c(1-\beta r^2)$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{P}{\rho_c(1-\beta r^2)}\right)\right) = -4\pi G r^2\rho_c(1-\beta r^2)$$

$$\Leftrightarrow r^2\frac{\partial}{\partial r}\left(\frac{P}{\rho_c(1-\beta r^2)}\right) = -4\pi G\rho_c\left(\frac{r^3}{3}-\frac{\beta r^5}{5}\right)$$

$$\Leftrightarrow \frac{P}{\rho_c(1-\beta r^2)} = -4\pi G\rho_c\left(\frac{r^2}{6}-\frac{\beta r^4}{20}\right)$$

$$\Rightarrow P = -4\pi G\rho_c^2(1-\beta r^2)\left(\frac{r^2}{6}-\frac{\beta r^4}{20}\right)$$

$$\Rightarrow P = -4\pi G\rho_c^2\left(\frac{\beta^2 r^6}{20}-\frac{\beta r^4}{6}-\frac{\beta r^4}{20}+\frac{r^2}{6}\right)$$

$$\therefore \boxed{P = -4\pi G\rho_c^2\left[\frac{\beta^2 r^6}{20}-\frac{13\beta r^4}{60}+\frac{r^2}{6}\right]}$$

Ejercicio 2.9

Estrella esférica autogravitante

$$\underline{E} = -\underline{\nabla}\psi \qquad \nabla^2\psi = 4\pi G\rho \qquad \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \text{cosas de fer}$$

En hidostática

$$\rho\underline{a} = 0 = \underline{f_v} - \underbrace{\underline{\nabla}p}_{\text{parcul}} \iff -\underline{\nabla}\psi = \frac{\underline{\nabla}p}{\rho}$$

$$\Rightarrow \underline{\nabla}\cdot\left(\frac{\underline{\nabla}p}{\rho}\right) = -4\pi G\rho$$

$$\iff \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{\partial p}{\partial r}\right) = -4\pi G r^2\rho$$

a) $\rho = \text{cte}$

$$\iff \frac{r^2}{\rho}\frac{dp}{\partial r} = -\frac{4\pi}{3}G r^3\rho + \cancel{\frac{4\pi}{3}Ga^3\rho}$$

$$\iff \int_{p_0=0}^{p} dp = \int_{a}^{r} -\frac{4\pi}{3} r G\rho^2 dr$$

$$\iff p = \frac{2\pi G}{3}\rho^2(a^2-r^2)+p_0$$

¿Condición de radio finito?

b) $p = C\rho^{6/5}$

$$-\frac{dp}{\rho^2} = \frac{4\pi}{3}G r\, dr \iff -\frac{dp}{p^{5/3}}C^{5/3} = \frac{4\pi}{3}G r\, dr$$

$$\iff \int_{p_0}^{p} -dp\, p^{-5/3} = \frac{3}{2}p^{-2/3} = \frac{4\pi}{6C^{5/3}}G(r^2-a^2)$$

$$\iff \frac{1}{p^{2/3}} = \frac{8\pi G}{18C^{5/3}}(r^2-a^2) \iff \frac{1}{p} = \frac{8}{27}\frac{\pi^{3/2}G^{3/2}}{C^{5/2}}(r^2-a^2)^{3/2}$$

$$\iff p = \frac{27}{8}\frac{C^{5/2}}{\pi^{3/2}G^{3/2}}\frac{1}{(r^2-a^2)^{3/2}} = \frac{27C^{5/2}}{(2\pi G)^{3/2}(r^2-a^2)^{3/2}}$$