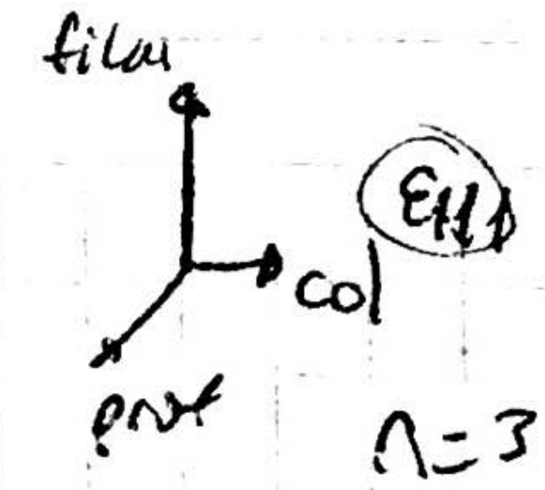

Ejercicio 1

i) La densidad tensorial de Levi-Civita es de orden n $\quad n=3$ ⇒

$$\begin{matrix} \varepsilon_{111} & \varepsilon_{121} & \varepsilon_{131} \\ \varepsilon_{112} & \varepsilon_{122} & \varepsilon_{132} \\ \varepsilon_{113} & \varepsilon_{123} & \varepsilon_{133} \end{matrix} \quad \text{luego} \quad \begin{matrix} \varepsilon_{211} & \varepsilon_{221} & \varepsilon_{231} \\ \varepsilon_{212} & \varepsilon_{222} & \varepsilon_{232} \\ \varepsilon_{213} & \varepsilon_{223} & \varepsilon_{233} \end{matrix} \quad \text{luego} \quad \begin{matrix} \varepsilon_{311} & \varepsilon_{321} & \varepsilon_{331} \\ \varepsilon_{312} & \varepsilon_{322} & \varepsilon_{332} \\ \varepsilon_{313} & \varepsilon_{323} & \varepsilon_{333} \end{matrix}$$

ii) $\varepsilon_{ijk}\varepsilon_{pqr}$: Para que no sea cero ijk todos $\neq$ y pqr también.

∴ ó $i=p$, $i=q$ ó $i=r$

Si $i=p$

$\Rightarrow \ldots = \delta_{ip}(\delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq})$ (*)

Porque $\varepsilon_{ijk}\varepsilon_{iqr} = \begin{cases} 1 & \text{si } jk\ qr:\ 23\ 23,\ 32\ 32 \\ -1 & \text{si } jk \neq qr:\ 23\ 32,\ 32\ 23 \end{cases}$

fácil ver reemplazando j lo que p.

De igual forma

Si $i=r$ ⇒ $\delta_{ir}(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp})$ ← porque reescribimos $\varepsilon_{ijk}\varepsilon_{rpq}$ con permutación cíclica

Si $i=q$ hay que tener cuidado porque la permutación no es cíclica $\quad \varepsilon_{ijk} = \varepsilon_{qpr} \Rightarrow (-1)$

$\Rightarrow \delta_{iq}(-1)(\delta_{jp}\delta_{kr} - \delta_{jr}\delta_{kp})$

El mismo razonamiento vale si tomamos $j=p$, $j=q$, $j=r$ y $k=q$, $k=p$, $k=r$

⇒ Pasando a un determinante

$$\boxed{\varepsilon_{ijk}\varepsilon_{pqr} = \begin{vmatrix} \delta_{ip} & \delta_{iq} & \delta_{ir} \\ \delta_{jp} & \delta_{jq} & \delta_{jr} \\ \delta_{kp} & \delta_{kq} & \delta_{kr} \end{vmatrix}}$$

~~qed~~ QED

iii) a) Se deduce instantaneamente del ejercicio anterior (*). Si $\varepsilon_{ijk}\varepsilon_{iqr}$

$\Rightarrow \varepsilon_{ijk}\varepsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{qk})$ ✓

b) $\varepsilon_{ijk}\varepsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$ (trivial, es solo permutar $\varepsilon_{ijk}\varepsilon_{pqk}$
$= \varepsilon_{kij}\varepsilon_{kpq}$)

c) $$\varepsilon_{ijk}\varepsilon_{ijl} = \begin{vmatrix} \delta_{ii} & \delta_{ij} & \delta_{il} \\ \delta_{ji} & \delta_{jj} & \delta_{jl} \\ \delta_{ki} & \delta_{kj} & \delta_{kl} \end{vmatrix} = \begin{vmatrix} 1 & \delta_{ij} & \delta_{il} \\ \delta_{ij} & 1 & \delta_{jl} \\ \delta_{ki} & \delta_{kj} & \delta_{kl} \end{vmatrix}$$

$$\varepsilon_{ijk}\varepsilon_{ijl} = \delta_{ii}(\delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj}) = \sum_{i=1}^{3}\sum_{j=1}^{3} \delta_{kl} - \delta_{jl}\delta_{kj}$$

$\delta_{jl}\delta_{kj} = 1$ si $l=j$, $k=j$ $\therefore k=l=j$ Pero si $k=l=j \Rightarrow 1-1=0$

$$= \delta_{kl} - \underbrace{\delta_{1l}}_{=0}\underbrace{\delta_{1k}}_{=0} + \delta_{kl} - \overset{=0}{\delta_{2l}}\,\overset{=0}{\delta_{2k}} \quad \text{si } k,l=3$$

$$+ \underbrace{\delta_{kl}}_{=1} - \underbrace{\delta_{3l}\delta_{3k}}_{=1} = 2\delta_{kl}$$

d) $\varepsilon_{ijk}\varepsilon_{ijk} = 6$

$$= (\delta_{jj}\delta_{kk} - \delta_{jk}\delta_{jk})\delta_{ii} : \ldots + 2\delta_{12}\delta_{12} - \ldots$$

$$= \delta_{11}\delta_{11} - \delta_{11}\delta_{11} + \delta_{11}\delta_{22} - \delta_{12}\delta_{12} + \delta_{11}\delta_{33} - (\delta_{13})^2 + \ldots$$

$$= \delta_{11}\delta_{22} + \delta_{11}\delta_{33} + \delta_{22}\delta_{11} + \delta_{33}\delta_{11} + \delta_{22}\delta_{33} + \delta_{33}\delta_{22} = 6$$

e) $$\delta_{mn}\delta_{mn} = 3 = \delta_{11}\delta_{11} + \delta_{12}\delta_{12} + \delta_{13}\delta_{13} + \delta_{21}\delta_{21} + \delta_{22}\delta_{22} + \delta_{23}\delta_{23} + \delta_{31}\delta_{31} + \delta_{32}\delta_{32} + \delta_{33}\delta_{33}$$
$$= 3$$

iv)
a) $\underline{A} \times \underline{B} = (A_i \hat{\imath} + A_j \hat{\jmath} + A_k \hat{k}) \times (B_i \hat{\imath} + B_j \hat{\jmath} + B_k \hat{k}) = A_i \times B_i$

$\Rightarrow \quad = \underset{\sim}{A} \times \underset{\sim}{B} = [A_j B_k - A_k B_j] \hat{e_i}$

$= \varepsilon_{ijk} A_j B_k \hat{e_i} = \varepsilon_{123} A_2 B_3 \hat{e}_1 + \varepsilon_{132} A_3 B_2 \hat{e}_1 + \ldots + \varepsilon_{312} A_1 B_2 \hat{e}_3 + \varepsilon_{321} \ldots$

$= (A_2 B_3 - A_3 B_2)\hat{e}_1 + (-A_1 B_3 + A_3 B_1)\hat{e}_2 + (A_1 B_2 - A_2 B_1)\hat{e}_3$ ✓

b) $\underline{\nabla} \times \underline{C} = \varepsilon_{ijk} \dfrac{\partial C_k}{\partial x_j} \hat{e_i}$

Es lo mismo pero ahora A pasa a ser ∇ y B, C

$\Rightarrow \underline{\nabla} \times \underline{C} = (\partial_2 C_3 - \partial_3 C_2)\hat{e}_1 + (-\partial_1 C_3 + \partial_3 C_1)\hat{e}_2 + (\partial_1 C_2 - \partial_2 C_1)\hat{e}_3$

$= \varepsilon_{ijk} \dfrac{\partial C_k}{\partial x_j} \hat{e_i}$ ✓

<u>Ejercicio 2</u>

$G_{ij} = G_{ij}^{(I)} + G_{ij}^{(S)} + G_{ij}^{(A)} \quad 1 \leq i,j \leq n$

$G_{ij}^{(I)} = \lambda \delta_{ij}$ tensor isótropo

$G_{ij}^{(S)} =$ tensor simétrico de traza nula

$G_{ij}^{(A)} =$ tensor antisimétrico

Pista: $G_{ij} = \frac{1}{2}\underbrace{(G_{ij} + G_{ji})}_{a_{ij}} + \frac{1}{2}\underbrace{(G_{ij} - G_{ji})}_{b_{ij}}$

~~G_{ij} simétrico~~

~~G_{ji} antisimétrico~~

$\Rightarrow G_{ij} = \frac{1}{2} a_{ij} + \frac{1}{2} b_{ij}$

a_{ij} = es simétrico pue $a_{ij} = a_{ji}$ $(G_{ij} + G_{ji} = G_{ji} + G \ldots$

b_{ij} es antisimétrico pues $b_{ij} = -b_{ji}$

$(G_{ij} - G_{ji}) = -(G_{ji} - G_{ij})$

$tr(a_{ij}) = a_{11} + a_{22} + \ldots + a_{nn}$

$= \lambda_{ij} \delta_{ij}$

$\lambda_{ij} = a_{ij}$

~~Pero~~ además $a_{ii} = G_{ii} + G_{ii} = 2G_{ii}$

$tr(b_{ij}) = G_{ii} - G_{ii} = 0$

$b_{ii} = 0$

$tr(a_{ij}) = a_{11} + a_{22} + \dots + a_{NN} = C$ sí la quiero hacer 0

$$\Leftrightarrow a_{ij}\delta_{ij} = C \Leftrightarrow a_{ij}\delta_{ij} - C = 0$$

$$\Leftrightarrow a_{ij}\delta_{ij} - \frac{C}{N}\delta_{ij} = 0$$

$$\Rightarrow \left(a_{ij} - \frac{C}{N}\right)\delta_{ij} = 0$$

$\Rightarrow$ como $G_{ij} = \frac{1}{2}a_{ij} + \frac{1}{2}b_{ij} = \frac{1}{2}\left(a_{ij} - \frac{C}{N}\delta_{ij}\right) + \frac{1}{2}b_{ij} + \underbrace{\frac{C}{N}\delta_{ij}}_{\lambda_{ij} \text{ isótropo}}$

$$\boxed{\Rightarrow G_{ij} = \underbrace{\lambda_{ij}}_{\text{isótropo}} + \underbrace{A_{ij}}_{\text{simetrica de tr nula}} + \underbrace{B_{ij}}_{\text{antisimetrica}}}$$

Ejercicio 3

i) $\underline{u}\cdot(\underline{v}\times\underline{w}) = u_i\hat{e}_i\cdot(\varepsilon_{ijk} v_j w_k \hat{e}_i) =$

$= u_i(\cancel{v_j w_k} - v_k w_j) = u_1(v_2w_3 - v_3w_2) + u_2(-v_1w_3 + v_3w_1) + u_3(v_1w_2 - w_2v_1)$

$= v_1(w_2u_3 - w_3u_2) + v_2(w_3u_1 - w_1u_3) + v_3(w_1u_2 - w_2u_1)$

$= \boxed{v_i\hat{e}_i\cdot(\varepsilon_{ijk} w_j u_k \hat{e}_i) = w_i\hat{e}_i(\varepsilon_{ijk} u_j v_k)\hat{e}_i}$

ii) $\underline{u}\times(\underline{v}\times\underline{w}) = \varepsilon_{ijk} u_j (\underline{v}\times\underline{w})_k \hat{e}_i = \varepsilon_{ijk} u_j \varepsilon_{klm} v_l w_m \hat{e}_i$

$= \varepsilon_{ijk}\varepsilon_{lmk} u_j v_l w_m \hat{e}_i$

$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(u_j v_l w_m \hat{e}_i)$

$= [u_j v_i w_j - u_j v_j w_i]\hat{e}_i = (v_i(u_j w_j) - w_i(u_j v_j))_i$

$= \boxed{\underline{v}(\underline{u}\cdot\underline{w}) - \underline{w}(\underline{u}\cdot\underline{v})}$

iii) $[(\underline{u}\times\underline{v})\cdot(\underline{w}\times\underline{s})]_i = \varepsilon_{ijk} u_j v_k \varepsilon_{ilm} w_l s_m$

$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) u_j v_k w_l s_m$

$= u_j w_j v_k s_k - u_j s_j v_k w_k = [(\underline{u}\cdot\underline{w})(\underline{v}\cdot\underline{s}) - (\underline{u}\cdot\underline{s})(\underline{v}\cdot\underline{w})]_i$

$\therefore \boxed{(\underline{u}\times\underline{v})\cdot(\underline{w}\times\underline{s}) = (\underline{u}\cdot\underline{w})(\underline{v}\cdot\underline{s}) - (\underline{u}\cdot\underline{s})(\underline{v}\cdot\underline{w})}$

iv) $\underline{\nabla}\cdot\underline{r} = 3 \quad (\underline{r} = (x, y, z))$

$$\partial_i r_i = \frac{\partial r_i}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial x_i}{\partial x_i} = 3$$

v) $\underline{\nabla}\times\underline{r} = 0$

$$\nabla\times\underline{r} = \varepsilon_{ijk}\partial_j r_k = \varepsilon_{ijk}\delta_{jk} = 0$$

$$\frac{\partial r_k}{\partial x_j} = \frac{\partial x_k}{\partial x_j} = 0 \quad \text{si } k \neq j$$

vi) $\underline{\nabla} r = \partial_i r = \frac{\partial}{\partial x_i}\left(\sqrt{x_i^2+x_j^2+x_k^2}\right)\hat{e}_i = +\frac{1}{2}\frac{2x_i}{\sqrt{x_i^2+x_j^2+x_k^2}}\hat{e}_i = \frac{\overline{x+y+z}}{\|(x,y,z)\|}$

$= \frac{\underline{r}}{r}$

vii) $\underline{\nabla}\cdot\left(\frac{1}{r}\right) = \frac{\partial}{\partial x_i}\left[\frac{1}{\sqrt{x_i^2+x_j^2+x_k^2}}\right]\hat{e}_i = -\frac{1}{2}\cdot\frac{2x_i}{\sqrt{x_i^2+x_j^2+x_k^2}^{\,3}}\hat{e}_i = -\frac{\underline{r}}{r^3}$

viii) $\underline{\nabla}\times\underline{\nabla}\phi = 0$ $\qquad \underbrace{\frac{\partial(\partial\phi_k)}{\partial j}}_{=0} - \frac{\partial}{\partial k}(\partial\phi_j) = 0$

$(\underline{\nabla}\times\underline{\nabla}\phi)_i = \varepsilon_{ijk}\,\partial_j(\partial_k\phi) = \partial_j(\partial_k\phi) - \partial_k\partial_j\phi = 0 \qquad \frac{\partial x_j}{\partial x_i} = 0 \;\forall j \neq i \qquad \frac{\partial y_k}{\partial j} = 0$

ix) $[\underline{\nabla}\cdot(\underline{\nabla}\times\underline{u})]_i = \partial_i\,\varepsilon_{ijk}\,\underbrace{\partial_j u_k}_{=0} = \partial_i[\partial_j u_k - \partial_k u_j] = \underbrace{\frac{\partial}{\partial i}(\partial u_{k})}_{=0} - \underbrace{\frac{\partial}{\partial i}(\partial u_{jk})}_{=0}$

x) $\underline{\nabla}^2\psi = \underline{\nabla}\cdot(\underline{\nabla}\psi) \qquad \underline{\nabla}^2\psi = \partial_i\partial_i\psi = \partial_i(\partial_i\psi) = \partial_i(\partial_j\psi)$

pues $\frac{\partial(\partial\psi_j)}{\partial i} = 0$ si $j \neq i$ $\qquad = \underline{\nabla}\cdot(\underline{\nabla}\psi)$

xi) $\underline{\nabla}^2(\phi\psi) = \partial_i\partial_i(\phi\psi) = \partial_i\left((\partial_i\phi)\psi + \phi(\partial_i\psi)\right)$

$= (\partial_i\partial_i\phi)\psi + 2(\partial_i\phi)(\partial_i\psi) + (\partial_i\partial_i\psi)\phi$

$= (\underline{\nabla}^2\phi)\psi + \phi(\underline{\nabla}^2\psi) + 2(\underline{\nabla}\phi\cdot\underline{\nabla}\psi)$

xii) $\underline{\nabla}(\phi\psi) = (\partial_i\phi)\psi + \phi(\partial_i\psi) = (\underline{\nabla}\phi)\psi + \phi(\underline{\nabla}\psi)$

xiii) $\underline{\nabla}\cdot(\underline{u}\times\underline{v}) = \partial_i\,\varepsilon_{ijk}u_j v_k = \varepsilon_{ijk}\,\partial_i(u_j v_k) = \varepsilon_{ijk}\left[(\partial_i u_j)v_k + u_j(\partial_i v_k)\right]$

$= \varepsilon_{kij}\,v_k(\partial_i u_j) + \varepsilon_{jik}\,u_j(\partial_i v_k) = \underline{v}\cdot(\underline{\nabla}\times\underline{u}) - \underline{u}\cdot(\underline{\nabla}\times\underline{v})$

xiv) $\underline{\nabla} \cdot (\phi \underline{u}) = \partial_i (\phi u_i) = (\partial_i \phi) u_i + \phi (\partial_i u_i) = (\underline{\nabla} \phi) \cdot \underline{u} + \phi (\underline{\nabla} \cdot \underline{u})$

xv) $\underline{\nabla} \times (\phi \underline{u}) = \varepsilon_{ijk} \partial_j (\phi u_k) = \varepsilon_{ijk} \left[(\partial_j \phi) u_k + \phi (\partial_j u_k) \right]$

$= (\underline{\nabla} \phi \times \underline{u}) + \phi (\underline{\nabla} \times \underline{u})$

xvi) $\underline{\nabla} \times (\underline{u} \times \underline{v}) = \varepsilon_{ijk} \partial_j \varepsilon_{klm} u_l v_m = \cancel{\varepsilon_{ijk} \varepsilon_{klm} \partial_j u_l v_m}$

$\varepsilon_{ijk} \varepsilon_{klm} \left[(\partial_j u_l) v_m + u_l (\partial_j v_m) \right] = \varepsilon_{ijk} \varepsilon_{klm} (\partial_j u_l) v_m + \varepsilon_{ilk} \varepsilon_{kjm} u_l ($

$= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) \left[(\partial_j u_l) v_m \right] - (\delta_{lj} \delta_{km} - \delta_{lm} \delta_{kj}) u_l (\partial_j v_m) \quad)$

$= (\partial_j a_j) v_k - (\partial_j u_k) v_j$

xviii) $\underline{\nabla} (\underline{u} \cdot \underline{v}) = \partial_i (u_j v_j) = (\partial_i u_j) v_j + u_j (\partial_i v_j)$

$= \frac{\partial}{\partial x_i} \delta_{jl} u_j v_l = \delta_{jl} \frac{\partial u_j}{\partial x_i} v_l + \delta_{jl} u_j \frac{\partial v_l}{\partial x_i}$

xviii) $\underline{\nabla}^2 \underline{u} = \partial_i \partial_i u_j$