Claim: $\mathbb{E}(\hat{\sigma}^2) = \sigma^2$

Given $\hat{\sigma}^2 = \frac{SSR}{n-p^*} \Rightarrow SSR = \hat{\sigma}^2 \cdot (n-p^*)$

Given $y = x\beta + \varepsilon$

$\Rightarrow \hat{y} = x\hat{\beta}, \quad \varepsilon = y - \hat{y}$

So $\underbrace{SSR = \sum_{i=1}^{n} \varepsilon_i^2 = \varepsilon^T\varepsilon}$ $\Rightarrow$ plug-in

So $\frac{(n-p^*)\hat{\sigma}^2}{\sigma^2} = \frac{\varepsilon^T\varepsilon}{\sigma^2}$

$\frac{(n-p^*)\hat{\sigma}^2}{\sigma^2} = \chi^2(n-p^*)$

Take expectations on both sides

$\mathbb{E}\left(\frac{(n-p^*)\hat{\sigma}^2}{\sigma^2}\right) = \mathbb{E}\left(\chi^2(n-p^*)\right)$

$\Rightarrow (n-p^*) \cdot \frac{\mathbb{E}(\hat{\sigma}^2)}{\sigma^2} = (n-p^*)$

$\Rightarrow \mathbb{E}(\hat{\sigma}^2) = \sigma^2$

Thus, $\hat{\sigma}^2$ is an unbiased estimator of σ^2

By definition, SSR follows chi-squared distribution with freedom of $(n-p)$

So $Var(SSR) = 2(n-p)$

By def, $\hat{\sigma}^2 = \frac{SSR}{n-p^*}$, $p^* = p+1$

So $Var(SSR) = 2(n-p)\sigma^4$

$$Var(\hat{\sigma}^2) = \frac{1}{(n-p^*)^2} \cdot Var(SSR)$$

$$\Rightarrow Var(\hat{\sigma}^2) = \frac{2\sigma^4(n-p)}{(n-p^*)^2}, \quad p^* = p+1 \Rightarrow n-p^* = n-p-1$$

$$\Rightarrow Var(\hat{\sigma}^2) = \frac{2\sigma^4}{n}$$

\# proved