Department of Automatic Control

School of Electrical Engineering

FEL3250 Network Optimization – PhD Course

Exam starts: 12:00 on the 17th March 2014
Exam ends: 14:00 on the 18th March 2014

Aids: Anything apart from Prof. Dimitri Bertsekas himself!

Attention: Write your name on the submitted answers.
Mark the total number of pages on the cover.
Write a clear answer to each question.
Each step in your solutions should be justified.
Lack of justification will result in point deductions.
To pass the exam, at least 50% of the points have to be achieved.

The exam consists of four (4) questions, 10 points each.

In charge: Associate Prof. Carlo Fischione (Email: carlofi@kth.se)

Results (Pass/Fail): Will be notified no later than March 31, 2014.

Good Luck!

1. We are given the following feasible minimum cost flow problem over the network graph $\mathcal{G} = (\mathcal{N}, \mathcal{A})$:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{(i,j)\in\mathcal{A}} a_{ij} x_{ij} \\
\text{subject to} \quad & \sum_{(i,j)\in\mathcal{A}} x_{ij} - \sum_{(j,i)\in\mathcal{A}} x_{ji} = 0 \; , \quad \forall i \in \mathcal{N} \\
& 0 \le x_{ij} \le c_{ij}, \qquad \forall (i,j) \in \mathcal{A} \; .
\end{aligned} \tag{1}
$$

Consider the algorithm constructing a sequence of flow vectors $x^0, x^1, \ldots$ as follows:

1. Let initially $x^0 = 0$;

2. Given x^k, stop if x^k is optimal, and otherwise find a simple cycle C^k that is unblocked with respect to x^k and has negative cost;

3. Increase the flow of the forward arcs of C^k by the maximum possible increment; decrease the flow of the backward arcs of C^k by the maximum possible increment;

4. $k = k + 1$ and go back to 2.

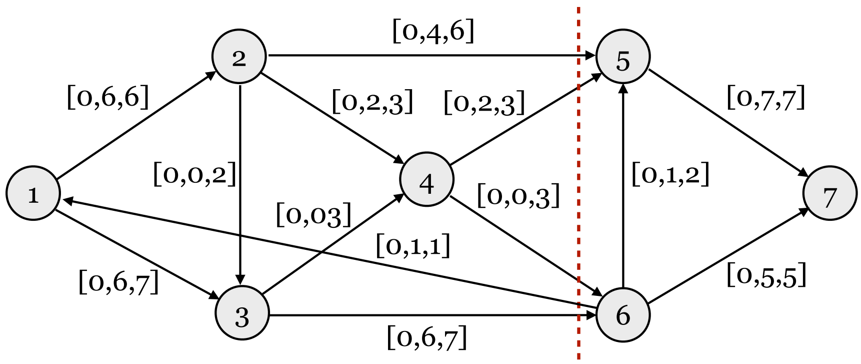
In this problem you will use this algorithm to show that, under the conditions given below, the optimal arc flows of feasible minimum cost flow problems assume *integer values*.

(a) Prove that the cost of x^{k+1} is smaller than the cost of x^k by an amount that is proportional to the cost of the cycle C^k and to the increment of the corresponding flow change. (3p)

(b) Make the fundamental assumption that the flow increment at each iteration is greater or equal to some scalar $\delta > 0$. Show that the algorithm must terminate after a finite number of iterations with an optimal flow vector. (3p)

(c) Assume that c_{ij} $\forall\, i, j$ are positive integers. Show that there exists an optimal flow vector that is integer. (*Hint*: show that the flow vectors generated by the algorithm are integer.) (4p)

2. (a) For the graph below, $[b_{ij}, x_{ij}, c_{ij}]$ provides the lower bound, flow and upper bound of the arc (i,j):

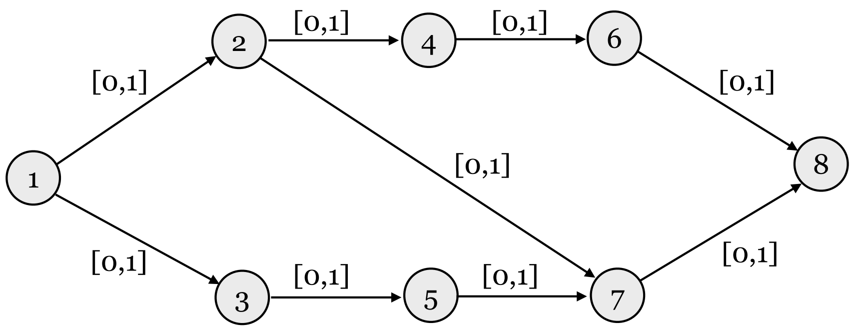


(i) Give the set $\mathcal{S}$ that defines the cut Q (given by the dotted line). (1p)

(ii) Give the sets of forward and backward arcs Q^+ and Q^-, respectively. (1p)

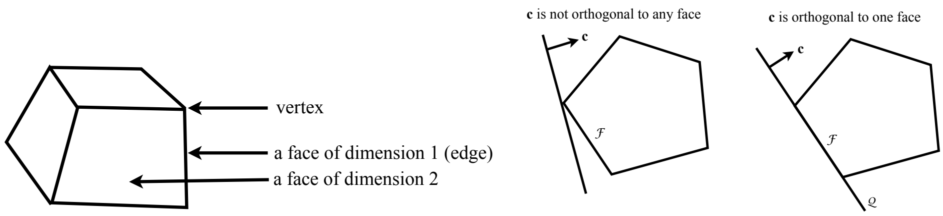
(iii) Find the flux and capacity across Q. Is it a saturated cut? (2p)

(b) Use the Ford-Fulkerson algorithm to find the the maximum flow from $s = 1$ to $t = 8$ for the following graph. Show, all the steps of the algorithm. (3p)



(c) At a conference on network flows, p research groups meet together. Each research group i has $a(i)$ participants. At the banquet they decide to sit at tables so that no two members of the same research group are at the same table, in order to increase their social interaction and foster collaborations. At the banquet there exist q tables available, each table j having a seating capacity of $b(j)$. Show how to formulate finding a seating arrangement that meets the objective as a maximum flow problem. (3p)

3. Min-Cost-Flow problem with many solutions This problem derives necessary and sufficient conditions for when a Min-Cost-Flow problem admits many solutions. Given a Min-Cost-Flow problem with *integral data* and a *non-integer* optimal solution, a method to construct an integer optimal solution is derived.



(a) (b) (c)

Figur 1: (a) faces and vertices of a polyhedron; (b) when c is not orthogonal to any face of the polyhedron, then the optimal solution is at a *unique* vertex, which is an integral; (c) when c is orthogonal to some face $\mathcal{F}$ of the polyhedron, then all the points in $\mathcal{F}$ are optimal

(a) Consider a case where we have N nodes and E edges. Given $c = [c_1, \ldots, c_E]^{\mathrm{T}} \in \mathbb{R}^E$, the cost coefficients of arcs and $b = [b_1, \ldots, b_N]^{\mathrm{T}} \in \mathbb{R}^N$, the supplies of nodes such that $\sum_{i=1}^{N} b_j = 0$. The Min-Cost-Flow problem can be written in the following compact form:

$$\begin{array}{ll} \text{minimize} & c^{\mathrm{T}}x \\ \text{subject to} & Ax = b \\ & x \geq 0 \ , \end{array} \tag{2}$$

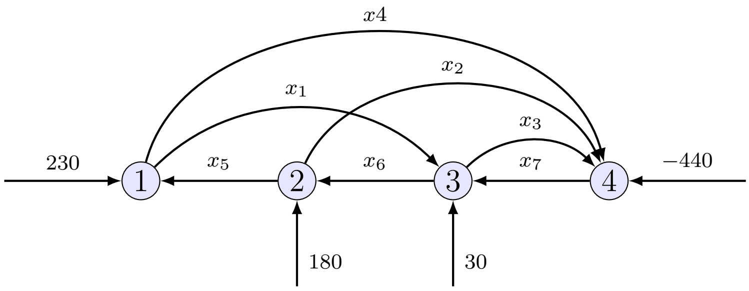
where $A \in \mathbb{R}^{N \times E}$ is the node-arc incident matrix. Suppose problem (2) has integral data, i.e., $c_j, j = 1, \ldots, E$ and $b_i, i = 1, \ldots, N$ are integers. Show that

$$\text{Problem (2) has many solutions} \iff \exists \text{ a non-integer optimal solution} \ .$$

(*Hint:* Note that the vertices of polyhedron $\mathcal{P} = \{x | Ax = b, x \geq 0\}$ are integral (or vectors of which the elements are integers). You may use that, if there are many solutions, then c is orthogonal to some face $\mathcal{F}$ of polyhedron $\mathcal{P}$. You may also use the Figure 1 above to build the intuition.) (4p)

(b) Consider the graph in Figure 2. Given $c = [1700\ 1700\ 1700\ 2200\ 0\ 0\ 0]^{\mathrm{T}}$ and $x_i \geq 0$ for all $i = 1, \ldots, 7$. Write the associated Min-Cost-Flow problem (P_{MCF}) in the form of problem (2) by explicitly computing the node-arc incidence matrix A and the supply vector b. (1p)

(c) It can be shown that the problem P_{MCF} in part (b) above has a *non-integer* optimal solution $x^\star_{\text{non-int}} = [0,\ 193.4,\ 16.6,\ 230,\ 0,\ 13.4,\ 0]^{\mathrm{T}}$, which yields the optimal value $p^\star = 863000$. Therefore, part (a) above suggests that P_{MCF} has many solutions and, thus there is a face $\mathcal{F}$ of $\mathcal{P}$ orthogonal to c. Verify this by following the steps below:



Figur 2: Network to use for Question 3-(b)

(c.1) consider the face $\mathcal{F} = \{x|Ax = b, x \geq 0, x_1 = x_5 = x_7 = 0, x_4 = 230\}$ of $\mathcal{P}$. Show that $\mathcal{F} \subset \mathcal{Q} = \{x|Ax = b, x_1 = x_5 = x_7 = 0, x_4 = 230\}$. (*Hint:* You may use the Figure 1(b)-1(c) to build the intuition.) (0.5p)

(c.2) Show that c is orthogonal to $\mathcal{Q}$. [*Hint:* You may first write $\mathcal{Q}$ in the form $\{x|Qx = d\}$. Then you may use matlab to show that $c^{\mathrm{T}}v = 0$ for all $v \in \texttt{Null}(Q)$.] (1p)

(c.3) Deduce that c is orthogonal to the face $\mathcal{F}$ considered in part (c.1). (0.5p)

(d) The key idea of a *general* method to compute an integer optimal solution $x^\star_{\text{int}}$ of P_{MCF} from a given non-integer optimal solution $x^\star_{\text{non-int}}$ is: start from $x^\star_{\text{non-int}}$ and move towards a vertex, while *being inside* $\mathcal{P}$ and *maintaining the optimality.* In particular, such a method can be summarized as follow.

`Step 1:` increase or decrease a chosen non-integer element of $x^\star_{\text{non-int}}$ by fixing the integer elements, while being inside the $\mathcal{P}$ and maintaining the optimality to get the new $x^\star_{\text{non-int}}$

`Step 2:` repeat `Step 1` until there are no more non-integer elements in the current $x^\star_{\text{non-int}}$.

By considering the non-integer optimal solution [see part (c)] of problem P_{MCF} given in part (b), `Step 1` above can be posed as a simple linear program as follows:

$$\begin{array}{ll} \text{minimize} & x_2 \\ \text{subject to} & A\underbrace{[0,\ x_2,\ x_3,\ 230,\ 0,\ x_6,\ 0]^{\mathrm{T}}}_{\bar{x}(x_2,x_3,x_6)} = b \\ & x_i \geq a,\ \ i = 2,3,6 \\ & c^{\mathrm{T}}[0,\ x_2,\ x_3,\ 230,\ 0,\ x_6,\ 0]^{\mathrm{T}} = q\ , \end{array} \tag{3}$$

where the variables are x_2, x_3, x_6. Let $x_2^\star, x_3^\star, x_6^\star$ denote the solution. What should be the values of a and q, in order to make sure that that $\bar{x}(x_2, x_3, x_6) \in \mathcal{P}$ and that the optimality of the resulting $\bar{x}(x_2^\star, x_3^\star, x_6^\star)$ is guaranteed. (1p)

(e) Note that there are other methods as well to compute an integer-optimal solution of problem P_{MCF} from a given non-integer-optimal solution $x^\star_{\text{non-int}}$. For example, one can show that the other solutions are of the form $(x^\star_{\text{non-int}}+y)$, where $y \in \mathbb{R}^E$ is any zero-cost *circulation* with non-zero elements at the positions of non-integer elements of $x^\star_{\text{non-int}}$. By inspecting the graph representation in Figure 2, [see part (b)], and in particular the simple cycle $(2, 4, 3, 2)$, one can see that for $\alpha \neq 0$, $y = [0, \alpha, -\alpha, 0, 0, \alpha, 0]^{\mathrm{T}}$ is a zero-cost circulation. Verify this. Find two values of α that corresponds to two integer solution of problem P_{MCF}. (2p)

4. Let $G = (\mathcal{N}, \mathcal{E})$ denote an undirected graph (i.e., for two nodes $i, j \in \mathcal{N}$, we have $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$). Assume that we also have a symmetric non-negative weight function $w : \mathcal{N} \times \mathcal{N} \to \mathbb{R}_+$ defined on pairs of nodes with

$$\begin{cases} w(i,j) = w(j,i) > 0 & \text{if } (i,j) \in \mathcal{E} \\ w(i,j) = 0 & \text{if } (i,j) \notin \mathcal{E} \; . \end{cases}$$

Let $|\mathcal{N}| = N$ and $|\mathcal{E}| = E$ (with the convention that if $(i, j) \in \mathcal{E}$, then (i, j) and (j, i) are only counted once in E). Recall that the $N \times E$ node-arc incidence matrix A is obtained by arbitrarily orienting each edge (e.g., say $i \to j$ or $\vec{e} = (i, j)$ if $i < j$) and setting

$$A_{i,\vec{e}} \stackrel{\text{def}}{=} \begin{cases} +1 & \text{if } \vec{e} = (i,j) \text{ for some } j \in \mathcal{N} \\ -1 & \text{if } \vec{e} = (j,i) \text{ for some } j \in \mathcal{N} \\ 0 & \text{otherwise.} \end{cases}$$

Let d_i denote the weighted degree of node i,

$$d_i \stackrel{\text{def}}{=} \sum_{j \in \mathcal{N}} w(i,j),$$

and let D denote an $N \times N$ diagonal matrix with entries $D_{i,i} = d_i$.

Finally, let B denote the (symmetric) $N \times N$ weighted adjacency matrix with entries

$$B_{i,j} = \begin{cases} w(i,j) & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise .} \end{cases}$$

(a) Recall that we defined the weighted Laplacian matrix of the graph G as the $N \times N$ matrix $L \stackrel{\text{def}}{=} AWA^T$, where W is an $E \times E$ diagonal matrix with $W_{\vec{e},\vec{e}} = w(i, j)$ for $\vec{e} = (i, j)$. Show that $L = D - B$. (*Hint*: It may help to consider the case where $w(i, j) = 1$ for all $(i, j) \in \mathcal{E}$ first.) (2p)

(b) In class we discussed that if G is connected (i.e., if there is a path from any node i to any other node j) then the Laplacian matrix has exactly one eigenvalue equal to 0 and the corresponding unit-norm eigenvector is $\mathbf{u}_1 = \frac{1}{\sqrt{N}}\mathbf{1}$; i.e., $L\mathbf{u}_1 = 0$.

We say that a graph $G = (\mathcal{N}, \mathcal{E})$ has *two connected components* if the nodes $\mathcal{N}$ can be partitioned into two disjoint sets $\mathcal{N}_1$ and $\mathcal{N}_2$ (with $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$ and $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{N}$) such that there are no edges $(i, j) \in \mathcal{E}$ with $i \in \mathcal{N}_1$ and $j \in \mathcal{N}_2$, and if we look at the graphs G_1 and G_2 restricted to those nodes in $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively, then G_1 and G_2 are each connected. (In other words, if G has two connected components then there is a path between any pair of nodes $i_1, j_1 \in \mathcal{N}_1$, and there is a path between any pair of nodes $i_2, j_2 \in \mathcal{N}_2$, but there are no paths from $i_1 \in \mathcal{N}_1$ to $j_2 \in \mathcal{N}_2$ or vice versa.) Of course, if G has two connected components then G itself is not connected.

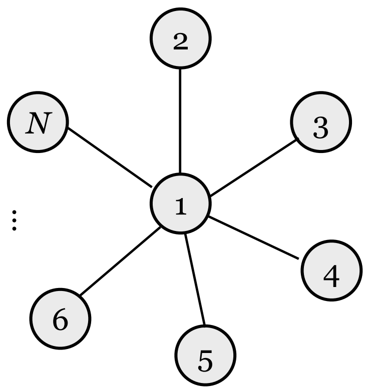
Show that if G has two connected components, then there are two unit-normed eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ of the Laplacian matrix L which satisfy the properties

- $L\mathbf{u}_1 = 0$ and $L\mathbf{u}_2 = 0$,
- $\|\mathbf{u}_1\|_2 = 1$ and $\|\mathbf{u}_2\|_2 = 1$, and
- $\mathbf{u}_1^T\mathbf{u}_2 = 0$.

Give explicit expressions for the entries of $\mathbf{u}_1$ and $\mathbf{u}_2$ in terms of $\mathcal{N}_1$ and $\mathcal{N}_2$. (4p)

(c) We say that a graph $G = (\mathcal{N}, \mathcal{E})$ is *bipartite* if the nodes $\mathcal{N}$ can be partitioned into two disjoint sets $\mathcal{N}_1$ and $\mathcal{N}_2$ such that every edge $(i, j) \in \mathcal{E}$ satisfies either $i \in \mathcal{N}_1$ and $j \in \mathcal{N}_2$, or $i \in \mathcal{N}_2$ and $j \in \mathcal{N}_1$ (i.e., every edge has one end in $\mathcal{N}_1$ and the other end in $\mathcal{N}_2$).

The star graph on N nodes (where node 1 is the "hub" and nodes $2, \dots, N$ are the "spokes"), as shown in the figure below, is an example of a bipartite graph.



Recall that the normalized Laplacian of G is defined as $\overline{L} = D^{-1/2}LD^{-1/2}$, where D is the diagonal degree matrix and L is the (unnormalized) Laplacian matrix.

Show that normalized Laplacian of the N-node star has a unit-normed eigenvector $\mathbf{u}$ with corresponding eigenvalue 2; i.e., $\overline{L}\mathbf{u} = 2\mathbf{u}$. (*Hint*: If $\mathbf{u}$ is unit-normed then it is an eigenvector of $\overline{L}$ corresponding to eigenvalue 2 if and only if $\mathbf{u}^T\overline{L}\mathbf{u} = \mathbf{u}^T(2\mathbf{u}) = 2$.) (4p)

Some useful notations

$\mathbb{R}^m$	Real m-vectors ($n \times 1$ matrices).
$\mathbb{R}^{m\times n}$	Real $m \times n$ matrices.
$\mathbb{R}_+$	Nonnegative real numbers.
X^{T}	Transpose of matrix X.
$\|x\|_2$	Euclidean (or ℓ_2-) norm of vector x.
$\|\mathcal{A}\|$	Cardinality of set $\mathcal{A}$.
$\texttt{Null}(A)$	Nullspace of matrix A.
$\mathbf{1}$	Vector with all components one.