Chan and Mitran
Single Sideband

Introduction

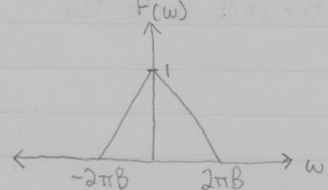
As we mentioned before, both double sideband modulation techniques waste spectrum by sending redundant information. For some signal with bandwidth B Hz, DSB uses 2B Hz.

We've started to figure out ways to alleviate the problem, notably, with QCM. But how about when we don't have two signals to feed into a QCM modulator?

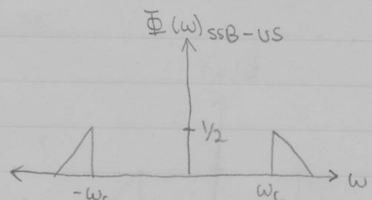
That's where single sideband comes in.

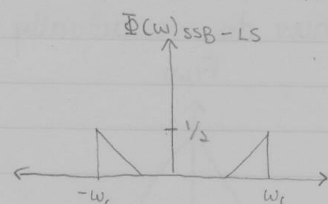
Single - Sideband

We'll talk about how SSB works in the frequency domain first.



As always, we have our trusty triangle spectrum to begin with. Since both lower and upper sidebands exactly describe each other, we can choose to transmit either.



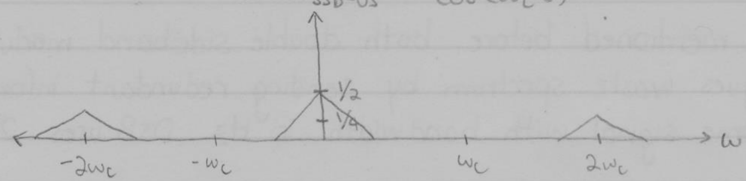


We can transmit either the upper or the lower sideband. So how do we reconstruct the original baseband?

Unsurprisingly, in the exact same way as any other case so far: multiply by cos.

We'll draw out the graph for the upper sideband, but obviously this technique works regardless of which.

$\Phi(\omega)_{SSB-US} \times \cos(\omega_c t)$



Each part of the spectrum shifts $\pm\omega_c$ AGAIN. So:

$-\omega_c$ portion: half amplitude, shifted to $-2\omega_c$
half amplitude, shifted to 0

creates $\frac{1}{2}F(\omega)$

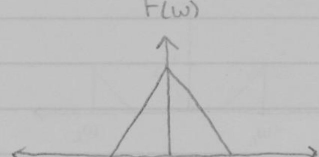
$+\omega_c$ portion: half amplitude, shifted to 0
half amplitude, shifted to $2\omega_c$

So we end up with $\frac{1}{2}F(\omega)$ back about DC. Then, we can use a low-pass filter to cut off the stuff that's far from DC.

<u>Creating Single Sideband</u>

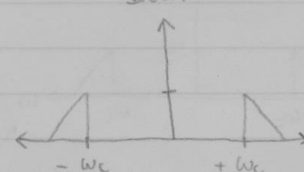
So how do we actually go from

$F(\omega)$



to

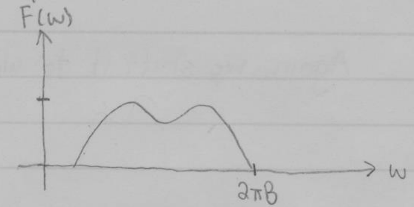
$\Phi(\omega)$



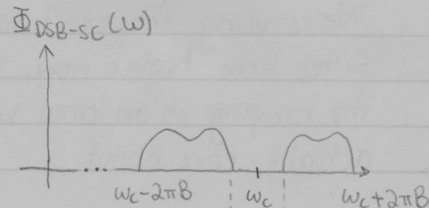
?

The answer is two steps: ① DSB-SC modulation
② Filters

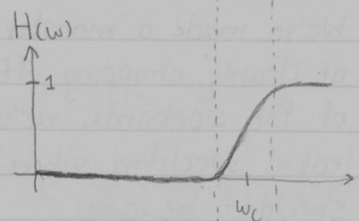
Let's walk through this with a more believable spectrum. We're only going to look at the positive ω axis because anything real has a mirror image in $-\omega$. It'll just save us space.



This is our baseband spectrum.

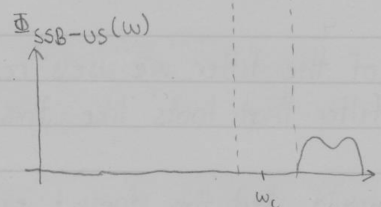


By multiplying $f(t)$ by $\cos(\omega_c t)$, we've shifted the spectrum to ω_c, and shrunk its amplitude by $1/2$.



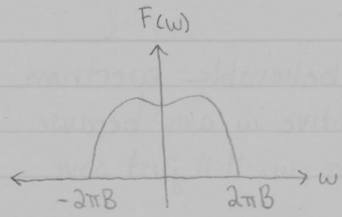
This is the filter we'll be using. Remember that in the frequency spectrum, using a filter makes the new spectrum:

$$\Phi_{new} = \Phi_{old} H(\omega)$$

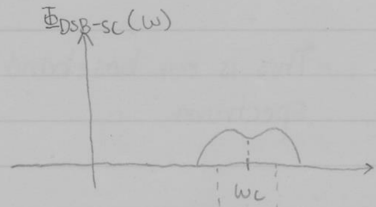


So wherever our filter is 0, our spectrum just dies out. The bit of space around ω_c gives the filter time to "ramp up" to 1, so it doesn't actually matter what the filter does in the space where our spectrum is zero.

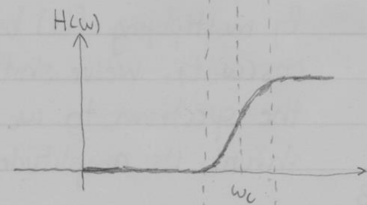
But what happens if we have some important spectrum at DC? Let's try this again.



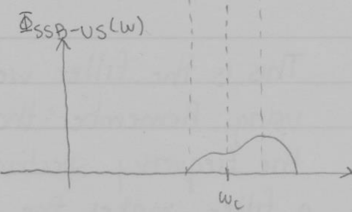
Here we've got the same shape spectrum we had before, except now, it's centered at DC.



Again, we shift it to ω_c.

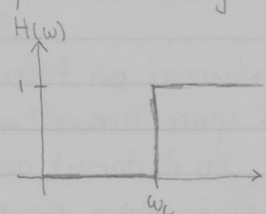


We're using the exact same filter here. Notice now, though, it's ramping in an area we actually care about.



We've made a monster. Instead of cleanly chopping off one of the sidebands, we've leaked spectrum where there shouldn't be any.

This is because of the nature of the filter we use's response. So why don't we just use a filter that looks like this?



We would, but this doesn't exist in real life.

This shortcoming will be addressed later, but for now, it's enough to conclude that you can't use SSB for signals near DC: the spectrum has to be "far enough" away from DC to allow the filter time to ramp up.

Mathematical Representation of Single Sideband

I still haven't provided an equation for $\Phi_{SSB}(\omega)$. This is because we're going to solve for it ourselves: starting from QCM.

Remember that it's possible to generate any arbitrary-shaped spectrum using QCM. It follows that we can also generate single sideband. Let's quickly go over the principle.

$$g(t) = f_1(t)\cos(\omega_c t) + f_2(t)\sin(\omega_c t)$$

where

$$f_1(t) = \mathrm{Re}\{g_b(t)\}$$

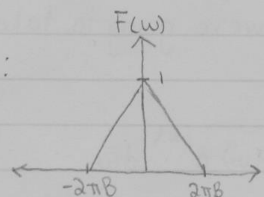
$$f_2(t) = -\mathrm{Im}\{g_b(t)\}$$

where

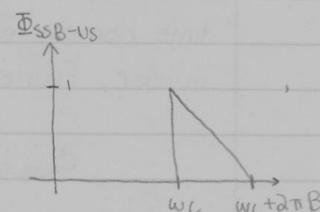
$g_b(t)$ is the complex baseband equivalent model

How do we get $g_b(t)$? By taking the inverse Fourier transform of $G_b(\omega)$, where $G_b(\omega)$ is the arbitrarily shaped spectrum we want shifted down to DC. Phew. So if:

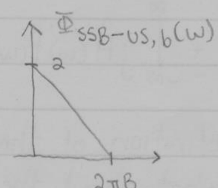
we have:



and we want



the complex baseband equivalent of $\Phi_{SSB\text{-}US}$ is:



So we can say that

$$\Phi_{SSB\text{-}US,b}(\omega) = \begin{cases} 2F(\omega) & \omega \geq 0 \\ 0 & \omega < 0 \end{cases}$$

We're going to define something new here.

$$\hat{H}(\omega) \triangleq \begin{cases} -j, & \omega \geq 0 \\ j & \omega < 0 \end{cases}$$

I don't actually know what this is called. In any case, we can express $\Phi_{SSB-US,b}(\omega)$ in terms of this.

$$\Phi_{SSB-US,b}(\omega) = F(\omega) + j\hat{H}(\omega)F(\omega)$$

$$\hookrightarrow = \left(\begin{cases} -j & \omega \geq 0 \\ j & \omega < 0 \end{cases}\right) jF(\omega)$$

$$= \begin{pmatrix} 1 & \omega \geq 0 \\ -1 & \omega < 0 \end{pmatrix} F(\omega)$$

$$= \begin{matrix} F(\omega) & \omega \geq 0 \\ -F(\omega) & \omega < 0 \end{matrix}$$

$$= \begin{cases} 2F(\omega) & \omega \geq 0 \\ 0 & \omega < 0 \end{cases}$$

Time for my least favourite part: actual math. We need the time domain equivalent of this so we're going to take the inverse Fourier transform.

$$\phi_{SSB-US,b}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_{SSB-US,b}(\omega)e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}[F(\omega) + j\hat{H}(\omega)F(\omega)]e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega + \int_{-\infty}^{\infty} j\hat{H}(\omega)F(\omega)e^{j\omega t}\,d\omega\right]$$

The first part, $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega$ is the definition of the inverse Fourier, which simply produces $f(t)$. But what about the next term?

Since we're already playing with random things we've just defined, we'll do the same thing here.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} jH(\omega)F(\omega)e^{j\omega t}\,d\omega = \frac{j}{2\pi}\int_{-\infty}^{\infty} H(\omega)F(\omega)e^{j\omega t}\,d\omega$$
$$= j\hat{f}(t)$$

We're going to call $\hat{f}(t)$ the Hilbert transform of f, and it is defined by:

$$\hat{F}(\omega) = \begin{cases} -jF(\omega) & \omega > 0 \\ jF(\omega) & \omega < 0 \end{cases}$$

So, back to the original topic.

$$\therefore \Phi_{SSB\text{-}US,b}(t) = f(t) + j\hat{f}(t)$$

In order to get the time domain equivalent, we simply plug it back into QCM's definition.

$$\begin{aligned}\Phi_{SSB\text{-}US}(t) &= Re\{f(t) + j\hat{f}(t)\}\cos(\omega_c t) \\ &\quad + (-Im\{f(t) + j\hat{f}(t)\})\sin(\omega_c t) \\ &= f(t)\cos(\omega_c t) - \hat{f}(t)\sin(\omega_c t)\end{aligned}$$

To get the lower sideband instead, we just flip the sign of the second term.

$$\Phi_{SSB\text{-}LS}(t) = f(t)\cos(\omega_c t) + \hat{f}(t)\sin(\omega_c t)$$

How exactly do we get this? To solve, remember that adding up both sidebands creates DSB-SC: specifically, DSB-SC with double the amplitude.

$$2f(t)\cos(\omega_c t) = \Phi_{SSB-US}(t) + \Phi_{SSB-LS}(t)$$
$$= f(t)\cos(\omega_c t) - \hat{f}(t)\sin(\omega_c t) + \Phi_{SSB-LS}(t)$$
$$\Phi_{SSB-LS}(t) = f(t)\cos(\omega_c t) + \hat{f}(t)\sin(\omega_c t)$$

Pretty simple, no complex math here. So in each $\Phi_{SSB}(t)$, we have the original $f(t)\cos(\omega_c t)$, and an extra term, $\mp \hat{f}(t)\sin(\omega_c t)$, that kills off either the lower or the upper sideband.

The Hilbert Transform

In trying to understand the Hilbert transform, I would avoid Wikipedia - its explanation is way too fucking complex. Instead, let's look at this mathematically, with an example.

What is the Hilbert transform of $f(t) = \cos(\omega_c t)$?

$$f(t) = \tfrac{1}{2}e^{j\omega_c t} + \tfrac{1}{2}e^{-j\omega_c t}$$

where

$$\hat{f}(t) = \begin{cases} -j\,F(\omega), & \omega > 0 \\ j\,F(\omega), & \omega < 0 \end{cases}$$

So the transform multiplies all POSITIVE frequencies by $-j$
NEGATIVE j

A simple sinusoid like a cosine wave consists of two parts. Its positive frequency, represented in the time domain by $\tfrac{1}{2}e^{j\omega_c t}$, and its equal but opposite negative frequency, $\tfrac{1}{2}e^{-j\omega_c t}$.

$$\therefore \hat{f}(t) = {-j}/{2}\, e^{j\omega_c t} + {j}/{2}\, e^{-j\omega_c t}$$
$$= \frac{1}{2j}e^{j\omega_c t} - \frac{1}{2j}e^{-j\omega_c t} \quad \text{(multiply by } j/j\text{)}$$

Does that look familiar? It should: it's the exponential form of sin.

$$\therefore \hat{f}(t) = \sin(\omega_c t)$$

The Hilbert transform is a filter. It takes every individual frequency of $f(t)$, and applies some transformation to it. In this instance, we've shifted the wave in the time domain $-90°$.

We can confirm this is true by finding $\hat{f}(t)$ for $f(t) = \sin(\omega_c t)$.

$$f(t) = \frac{1}{2j} e^{j\omega_c t} - \frac{1}{2j} e^{-j\omega_c t}$$

$$\hat{f}(t) = -\frac{j}{2j} e^{j\omega_c t} - \frac{j}{2j} e^{-j\omega_c t}$$

$$= -\tfrac{1}{2} e^{j\omega_c t} - \tfrac{1}{2} e^{-j\omega_c t}$$

$$= -\left[\tfrac{1}{2} e^{j\omega_c t} + \tfrac{1}{2} e^{j\omega_c t} \right]$$

$$= -\cos(\omega_c t)$$

As one last example, we'll find $\phi_{SSB-US}(t)$ of $f(t) = A_m \cos(\omega_m t)$.

$$\phi_{SSB-US}(t) = f(t)\cos(\omega_c t) - \hat{f}(t)\sin(\omega_c t)$$

$$= A_m \cos(\omega_c t)\cos(\omega_m t) - A_m \sin(\omega_m t)\cos(\omega_c t)$$

We could go further with this by expanding using trig identities, but let's not, that's a pain.

From now on, we'll use $\phi_+(t)$ for $\phi_{SSB-US}(t)$ and
$\phi_-(t)$ for $\phi_{SSB-LS}(t)$
because I'm tired of writing all of those letters.

Demodulation of SSB

We used QCM to construct our modulated signal. So would it not make sense to demodulate in a similar way as we do in QCM?

There is a difference, though - QCM demodulation gives us two separate signals. We only want one. So what do we do? The same thing we do every night, Pinky... TAKE OVER THE WORLD!

Just kidding. I mean multiply by $\cos(\omega_c t)$. You probably could have guessed that by now.

$$\phi_\pm(t)\cos(\omega_c t) = \left[f(t)\cos(\omega_c t) \mp \hat{f}(t)\sin(\omega_c t)\right]\cos(\omega_c t)$$
$$= f(t)\cos^2(\omega_c t) \mp \hat{f}(t)\sin(\omega_c t)\cos(\omega_c t)$$

$\downarrow$ by the power of trig identities

$$= f(t)\left[\tfrac{1}{2} + \tfrac{1}{2}\cos(2\omega_c t)\right]$$
$$\mp \hat{f}(t)\left[0 + \tfrac{1}{2}\sin(2\omega_c t)\right]$$
$$= \tfrac{1}{2}f(t) + \tfrac{1}{2}\cos(2\omega_c t)$$
$$\mp \tfrac{1}{2}\hat{f}(t)\sin(2\omega_c t)$$

So we get back $\frac{1}{2}f(t)$ which is nice. But we also have two extra terms, shifted by $2\omega_c$ away from baseband. How do we rid ourselves of them? Simple - a low pass filter.

That'll pass $\frac{1}{2}f(t)$ and reject the other terms, leaving us with just the original signal.

Effects of Frequency/Phase Error

This should be standard procedure by now. Let's say the demodulating wave is:

$$c(t) = \cos[(\underset{\text{freq err}}{\omega_c + \Delta\omega})t + \underset{\text{phase error}}{\theta}]$$

So after demodulation:

$$\begin{aligned}
x(t) &= \phi_{\pm}(t)\cos[(\omega_c+\Delta\omega)t+\theta] \\
&= [f(t)\cos(\omega_c t) \mp \hat{f}(t)\sin(\omega_c t)]\cos[(\omega_c+\Delta\omega)t+\theta] \\
&= f(t)\cos(\omega_c t)\cos[(\omega_c+\Delta\omega)t+\theta] \\
&\quad \mp \hat{f}(t)\sin(\omega_c t)\cos[(\omega_c+\Delta\omega)t+\theta] \\
&\qquad \downarrow \text{ trig identities woo} \\
&= f(t)\,[\tfrac{1}{2}\cos(\Delta\omega t+\theta) + \tfrac{1}{2}\cos[(2\omega_c+\Delta\omega)t+\theta]] \\
&\quad \mp \hat{f}(t)\,[\tfrac{1}{2}\sin(\Delta\omega t+\theta) + \tfrac{1}{2}\sin[(2\omega_c+\Delta\omega)t+\theta]] \\
&= \tfrac{1}{2}f(t)\cos(\Delta\omega t+\theta) + \tfrac{1}{2}f(t)\cos[(2\omega_c+\Delta\omega)t+\theta] \\
&\quad \mp \tfrac{1}{2}\hat{f}(t)\sin(\Delta\omega t+\theta) \mp \tfrac{1}{2}\hat{f}(t)\cos[(2\omega_c+\Delta\omega)t+\theta]
\end{aligned}$$

Applying the same low-pass filter from before leaves us with

$$= \tfrac{1}{2}f(t)\cos(\Delta\omega t+\theta) \mp \tfrac{1}{2}\hat{f}(t)\sin(\Delta\omega t+\theta)$$

Obviously, if $\Delta\omega$ and θ were both zero, we'd get

$$= \tfrac{1}{2}f(t)$$

If we ONLY have frequency error, we get

$$= \tfrac{1}{2}f(t)\underset{\text{beats}}{\cos(\Delta\omega t)} \mp \tfrac{1}{2}\underset{\text{distortion}}{\hat{f}(t)\sin(\Delta\omega t)}$$

If we only have phase error, we get

$$= \tfrac{1}{2} f(t) \cos\theta \mp \hat{f}(t) \sin\theta$$

attenuation ↑ ↖ still distortion

Single Sideband - Large Carrier

Here we are, answering the question no one is asking: can we use large carrier modulation on single sideband so we can use an envelope detector at the receiver?

Why yes, yes we can.

$$\Phi_{\pm}(t) = [A_c + f(t)] \cos(\omega_c t) + \hat{f}(t) \sin(\omega_c t)$$

↖ carrier component

So what, mathematically, is the envelope of this wave? Let's generalize a bit.

$$\Phi_{\pm}(t) = x(t) \cos(\omega_c t) + y(t) \sin(\omega_c t)$$

We can change the $\sin(\omega_c t)$ to a $\cos(\omega_c t)$ by adding a phase shift, whose value is dependent on $\omega_c t$.

$$\Phi_{\pm}(t) = x(t) \cos(\omega_c t) + y(t) \cos(\omega_c t + \theta)$$

$$= \sqrt{x^2(t) + y^2(t)} \cos(\omega_c t + \theta(t))$$

$$\theta(t) = \tan^{-1}\left(\frac{y(t)}{x(t)}\right)$$

The term $\sqrt{x^2(t) + y^2(t)}$ is the envelope itself. I wish I could tell you exactly why but to be honest I've got a rough time understanding exactly what the deal is here

In any case, we can apply our newfound learnings to $\phi_\pm(t)$, where

$$x(t) = A_c + f(t)$$
$$y(t) = \hat{f}(t)$$

$$\therefore \sqrt{x^2(t) + y^2(t)} = \sqrt{(A_c + f(t))^2 \mp \hat{f}^2(t)}$$

$$= \sqrt{A_c^2 + 2A_c f(t) + f^2(t) \mp \hat{f}^2(t)}$$

$$= A_c \sqrt{A_c^2/A_c^2 + 2A_c f(t)/A_c^2 + f^2(t)/A_c^2 \mp \hat{f}^2(t)/A_c^2}$$

$$= A_c \sqrt{1 + \frac{2f(t)}{A_c} + \frac{f^2(t) \mp \hat{f}^2(t)}{A_c^2}}$$

We're going to invoke two assumptions here

1. $A_c \gg f(t)$ or $\hat{f}(t)$: fairly believable, we need to shift $f(t)$ so it doesn't dip into the negatives
2. $\sqrt{1 + 2\Delta x} \approx 1 + \Delta x$ for $\Delta x \ll 1$: just try it out, it's pretty accurate.

Invoking 1:

$$= A_c \sqrt{1 + \frac{2f(t)}{A_c}}$$ ← we can't throw away too much or else the envelope doesn't depend on t anymore!

Invoking 2:

$$= A_c \left[1 + f(t)/A_c\right]$$
$$= A_c + f(t)$$

Which confirms it's A-Ok to use SSB-LC. Next time, vestigial sideband.