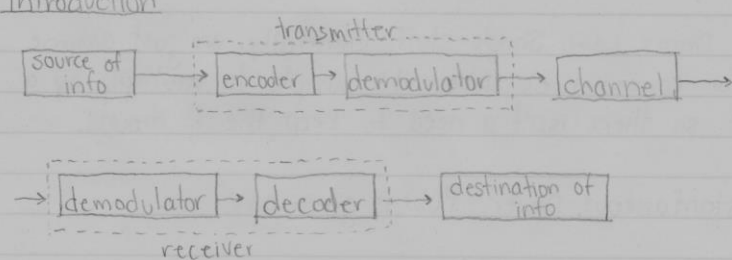
Chan and Mitran: Introduction / Review

Welcome to ECE 318: Analog / Digital Communications. Even though this section is titled introduction and review, we're just going to skim the introduction portion and jump straight into review after.

Introduction



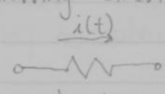
This is an entire communication system expressed as a block diagram. The encoder and decoder (compress and add extra stuff) and (decompress and remove extra stuff) respectively. The extra stuff is there so we can give ourselves a better chance to ensure the destination actually gets the message we want. The modulator and demodulator converts the signal to a method that fits the channel, or a method that fits the decoder.

That's really about it. All we really need to know is that encoders help the message not to lose its intended meaning, and modulators exist because sometimes it just makes no logical or physical sense to send a message using the source directly - for example, you can't send an electrical impulse easily directly through the air.

We don't really need detailed explanations of anything else, so let's get to review!

Energy and Power

Given the following circuit:



where resistance is R,
$v(t) = R\,i(t)$.

That's Ohm's Law. Simple stuff. Generally, we just assume $R=1$ for convenience's sake - it's simple to multiply by a constant so there isn't a need to keep the R around.

The instantaneous power, $P(t)$ is given as

$$P(t) = \frac{v^2(t)}{R} = i^2(t)\,R$$

Using our convenient $R=1$, we'll generally see power expressed as

$$P(t) = v^2(t) = i^2(t)$$

Average power and energy, in turn, are.

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 \, dt$$

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} |f(t)|^2 \, dt$$

Average power is computed by integrating over a period. Since power is the energy consumed over a certain time period, we have the $1/T$. By the repetitive nature of a periodic signal, it's not really necessary to consider more than one period, though we include the limit as it is technically part of the definition.

We define an energy signal if its energy is finite. It would then follow that an energy signal has no power

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 \, dt$$

$$= \left(\lim_{T\to\infty} \frac{1}{T}\right) \left(\lim_{T\to\infty} \int_{-T/2}^{T/2} |f(t)|^2 \, dt\right)$$

$$= \left(\lim_{T\to\infty} \frac{1}{T}\right) (\text{finite number})$$

$$= 0$$

We can define a power signal as one with power greater than 0. For that to be true, the energy, or

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} |f(t)|^2 \, dt = \infty$$

The Fourier Series

Most (for this course, all) signals $f(t)$ defined on $t \in [t_1, t_2]$ can be decomposed into a linear combination of orthogonal signals, of the form

$$\Phi_n(t) = e^{jn\omega_0 t}, \quad \omega_0 = \frac{2\pi}{t_2 - t_1}, \quad n = \text{an integer index}$$

The defintion of orthogonality is given as such: for two signals $f(t)$ and $g(t)$, they are orthogonal if their inner product,

$$\int f(t)\, g^*(t) \, dt$$

$$= \int f^*(t)\, g(t) \, dt$$

$$= 0 \quad \text{for } f \neq g$$

Here, we're going to do a quick proof that all functions of the form $\Phi_n(t)$ are orthogonal with each other.

$$\int_{t_1}^{t_2} \Phi_n(t)\, \Phi_m^*(t)\, dt$$

$$= \int_{t_1}^{t_2} e^{jnw_0t}\, e^{-jmw_0t}\, dt$$

$$= \int_{t_1}^{t_2} e^{j(n-m)w_0t}\, dt$$

If $n=m$:

$$= \int_{t_1}^{t_2} e^{j0w_0t}\, dt$$

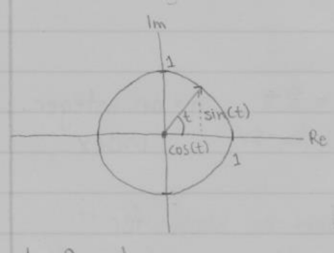
$$= \int_{t_1}^{t_2} dt$$

$$= t_2 - t_1$$

If $n \neq m$:

$$= \int_{t_1}^{t_2} e^{\frac{2\pi j(n-m)t}{t_2-t_1}}\, dt$$

This is kind of a gross integral, so let's think about this intuitively. First, we should remember Euler's Formula, which states that $\cos(t) + j\sin(t) = e^{jt}$. What this represents is a unit circle on the complex plane.



So for a specific value of t_0, we're looking at a specific point on the unit circle. Using 2π yields the same point as 0 or 4π. They are all multiples of 2π, whether that multiple is 0, 1, 2, etc.

So for the standard circle, 0 to 2π constructs one revolution. It then follows that by changing e^{jt} to $e^{2\pi jt}$,

$$e^{2\pi jt} \text{ requires } [0,1]$$

to draw the full circle. If we take this to another step,

$$e^{2\pi j t} \longrightarrow e^{\left(\frac{2\pi}{t_2-t_1}\right) j t}$$

$$[0,1] \longrightarrow [0, t_2-t_1]$$

Then, we can shift the interval forward by t_1, meaning it takes $[t_1, t_2]$ to draw the circle. In essence,

$$\int_{t_1}^{t_2} e^{j\left(\frac{2\pi}{t_2-t_1}\right)t} \, dt = 0$$

because every point on the circle has an equal and opposite point. So adding up every point creates a total of zero. n-m is always an integer, which we'll call α.

$$e^{j\left(\frac{2\pi\alpha}{t_2-t_1}\right)t} \quad \text{requires} \quad [t_1/\alpha, \ t_2/\alpha]$$

to create the circle. So all that α does is make us draw the ENTIRE circle multiple times. As such,

$$\int_{t_1}^{t_2} e^{j(n-m)\omega_0 t} = 0 \quad \text{for } n \neq m.$$

Phew, okay, we've proved $\phi_n(t)$ to be orthogonal. But what about the linear combination part? C is a LC of A and B if

$$C = aA + bB, \quad a, b \text{ are constants}$$

So how do we get those constants - the Fourier Coefficients?

$$F_n = \frac{1}{t_2-t_1} \int_{t_1}^{t_2} f(t) e^{-jn\omega_0 t} \, dt$$

Alright, so we have some indices again; where do they come from? Let's tie it all together

Most signals $f(t)$ can be decomposed such that

$$f(t) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t)\, e^{-jn\omega_0 t}\, dt \right) e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} F_n \, \Phi_n(t)$$

If it happens that $f(t)$ is periodic with period T, we can use one period instead of the entire domain the function is defined on:

$$f(t) = \sum_{-\infty}^{\infty} F_n e^{jn\omega_0 t}, \quad \omega_0 = 2\pi / T$$

$$F_n = \frac{1}{T} \int_{\substack{\text{any} \\ \text{period}}} f(t)\, e^{-jn\omega_0 t}$$

It's best to choose whatever period makes the math the easiest. In what way is this useful? We'll go back to the average power equation for periodic signals.

$$P = \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 \, dt$$

Because $|f(t)|^2 = f(t) f(t)^*$ (use $f(t) = a + jb$ if you need to confirm this is true):

$$= \frac{1}{T} \int_{-T/2}^{T/2} f(t) f(t)^* \, dt$$

We can then decompose each into its Fourier Series:

$$= \frac{1}{T} \int_{-T/2}^{T/2} \left(\sum_n F_n \Phi_n(t) \right) \left(\sum_m F_m^* \Phi_m^*(t) \right) dt$$

Then we'll extract the summations out of the integral

$$= \frac{1}{T} \sum_n \sum_m F_n F_m^* \int_{-T/2}^{T/2} \Phi_n(t)\, \Phi_m(t)\, dt$$

From orthogonality, we know in any $n \neq m$ case, it's zero, so we only care about the $n=m$ case:

$$= \frac{1}{T} \sum_n F_n F_n^* \int_{-T/2}^{T/2} \Phi_n(t)\, \Phi_n^*(t)\, dt$$

We can then convert these back to squared magnitudes:

$$= \frac{1}{T} \sum_n |F_n|^2 \int_{-T/2}^{T/2} |\Phi_n(t)|^2\, dt$$

Since $\Phi_n(t)$ refers to the unit circle, its magnitude $|\Phi_n(t)|^2$ must always be the length 1:

$$= \frac{1}{T} \sum_n |F_n|^2 \left(t \Big|_{-T/2}^{T/2} \right)$$

$$= \frac{1}{T} (T) \sum_n |F_n|^2$$

$$= \sum_n |F_n|^2$$

So we can conclude that the coefficient's magnitude, $|F_n|^2$, is the power generated at the frequency that results from the change in index: specifically this portion.

$$\frac{1}{T} \int_{-T/2}^{T/2} f(t)\, e^{-j \boxed{n\omega_0} t}$$

<u>Imaginary Frequencies</u>

You might have noticed that $\Phi_n(t)$ has a j in it, which implies it's a sinusoid with an imaginary frequency? What does that mean? What does it represent in real life?

This is a question, as a whole, that has bothered me for some time. Why do imaginary numbers exist when there is no real life analogue? How can Johnny eat $2.5j$ apples?

The short answer is that they don't represent anything. They're a mathematical convenience.

I had been thinking of it incorrectly the entire time. Imaginary numbers don't have any PHYSICAL meaning. Intuitively, I was right - Johnny truly is enable to eat 2.5j apples.

Why they actually matter is because having them will allow us to reach a meaningful, real-valued answer. Let's show this using the Fourier series.

Remember that the Fourier series is an infinite sum. So in the $\sum_{-\infty}^{\infty}$, there exists a number n, and its negative, and those are added together. Let's try it.

$$F_n e^{jnw_0t} + F_{-n} e^{-jnw_0t} = F_n e^{jnw_0t} + F_n^*(e^{jnw_0t})^*$$
$$= F_n e^{jnw_0t} + (F_n e^{jnw_0t})^*$$

Any number summed with its conjugate becomes 2 times the real value of that number

$$= 2\,\mathrm{Re}\{F_n e^{jnw_0t}\}$$

Let's convert F_n to polar form, for reasons I'll explain shortly.

$$= 2\,\mathrm{Re}\{|F_n| e^{j\angle F_n}(e^{jnw_0t})\}$$
$$= 2\,\mathrm{Re}\{|F_n| e^{j(w_0t + \angle F_n)}\}$$
$$= 2|F_n|\,\mathrm{Re}\{e^{j(w_0t+\angle F_n)}\}$$
$$= 2|F_n|\,\mathrm{Re}\{\cos(w_0t + \angle F_n) + j\sin(w_0t + \angle F_n)\}$$
$$= 2|F_n|\cos(w_0t + \angle F_n)$$

So adding two imaginary parts of the Fourier series gave us a FULLY REAL sinusoid, with a real amplitude and a real phase shift

The imaginary numbers are only mathematical conveniences that allow us to reach real-valued answers.

The Fourier Transform

For most (see: all) energy signals $f(t)$, the Fourier and inverse Fourier are defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

In the same way as before, we can find an expression for energy in the frequency domain.

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} f(t) f(t)^* dt$$

We'll express $f(t)^*$ as its transformed form:

$$= \int_{-\infty}^{\infty} f(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \right]^* dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) F(\omega)^* e^{-j\omega t} d\omega\, dt$$

Since $F(\omega)^*$ is constant with respect to t, we'll extract it

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)^* \left(\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right) d\omega$$

Hey look, it's another $F(\omega)$!

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)^* F(\omega) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

So... What did we learn?

Firstly, we can find energy by using the frequency domain, not just the time domain. Secondly, we can find the energy contributed to the total for ANY specific frequency we have in mind. This was not possible in the time domain.

$|F(\omega)|^2$ is called the energy spectral density, and it is expressed, fairly obviously, in

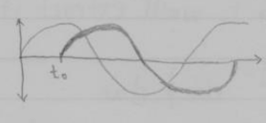
$$\frac{\text{energy}}{\text{rads/s}}$$

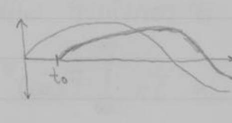
Properties of the Fourier Transform

There will be a few properties that'll be useful to us throughout the term. Instead of proving them, though, I'll do my best to explain the general reason why they work that way.

1) $\mathcal{F}\{f(t-t_0)\} = e^{-j\omega t_0} F(\omega)$

This is a delay in time, where we're delaying it by some fixed t_0.





Delaying/speeding up a signal is very much like phase shifting it. In fact, it's indistinguishable. However, shifting by a constant amount is a larger phase shift in signals with lower periods - so there's some sort of scale factor based on the signal's frequency, ω.

Turns out that it's exactly $e^{-j\omega t_0}$. Notice that its magnitude, $|e^{-j\omega t_0}|$, is equal to 1. As such it has no effect on the overall magnitude (aka amplitude) and only affects the phase.

2) $\mathcal{F}\{ \frac{d}{dt} f(t) \} = j\omega F(\omega)$

This one is fairly simple to show

$$\frac{d}{dt} \sin(\omega t) = \omega \cos(\omega t)$$
$$= \omega \sin(\omega t + \pi/2)$$

Taking the derivative gives us the same thing, except the amplitude is multiplied by ω, and the phase is shifted by $\pi/2$.

The j creates our $\pi/2$ shift. The ω is the multiplication. That's really about it.

3) $\mathcal{F}\{ f(t) e^{j\omega_0 t} \} = F(\omega - \omega_0)$

The Fourier transform makes things into complex exponentials, of the form $e^{j\alpha t}$, so $f(t) = e^{j\alpha t}$.

$$\Rightarrow f(t) e^{j\omega_0 t} = e^{j\alpha t} e^{j\omega_0 t}$$
$$= e^{j(\alpha + \omega_0)t}$$

this here shifts the frequency to the right by ω_0

Since the e term in $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$ has a negative exponential, $F(\omega) \rightarrow F(\omega - \omega_0)$, not $F(\omega + \omega_0)$.

4) $\mathcal{F}\{f(t)\cos(\omega_0 t)\} = \frac{1}{2}F(\omega-\omega_0) + \frac{1}{2}F(\omega+\omega_0)$.

This is a result of Euler's Formula.

$$\cos(\omega_0 t) = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

The half multipliers are constant, and then each $e^{j\omega_0 t}$ acts exactly the same way as rule 3) does.

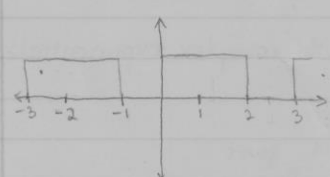
The Relationship Between Hertz and Rad/s

In almost every case, you can simply sub $\omega = 2\pi f$.
The only case this doesn't work is for the impulse function

$$\delta(ax) = \frac{1}{a}\delta(x)$$

Examples.

ex. Find the Fourier coefficients for:



This has a period of 3.

$$F_n = \frac{1}{3}\int_0^3 f(t)\,e^{-jn\omega_0 t}\,dt$$

$$\omega_0 = \frac{2\pi}{t_2 - t_1} = \frac{2\pi}{3} =$$

Since $(2,3]$ is 0, we can omit it from the integral.

$$F_n = \frac{1}{3}\int_0^2 (1)\,e^{-jn\left(\frac{2\pi}{3}\right)t}\,dt$$

Now we need to consider the cases $n=0$ and $n \neq 0$.

$$n=0: \quad F_n = \frac{1}{3}\int_0^2 (1)(1)\,dt$$
$$= \frac{1}{3}[2-0]$$
$$= 2/3$$

$$n \neq 0: \quad F_n = \frac{1}{3}\int_0^2 e^{-jn\left(\frac{2\pi}{3}\right)t}\,dt$$
$$= \frac{1}{3}\left[\frac{e^{-jn(2\pi/3)t}}{-jn\left(\frac{2\pi}{3}\right)}\right]_0^2$$
$$= \frac{1}{3}\left[\frac{e^{-jn(4\pi/3)}}{-jn(2\pi/3)} - \frac{e^0}{-jn\left(\frac{2\pi}{3}\right)}\right]$$
$$= \frac{1}{3}\left[\frac{e^{-jn\left(\frac{4\pi}{3}\right)}-1}{-jn(2\pi/3)}\right]$$

ex. Find $F(f)$ given $F(\omega) = \delta(\omega) + \frac{a^2}{a^2+\omega^2}$

$$F(\omega)\big|_{\omega=2\pi f} = \delta(2\pi f) + \frac{a^2}{a^2+(2\pi f)^2}$$
$$= \frac{1}{2\pi}\delta(f) + \frac{a^2}{a^2+(2\pi f)^2}$$

Transmission Through LTI Systems

Whoops, forgot a part.

$$\delta(t) \rightarrow \boxed{\text{LTI System}} \rightarrow h(t)$$

Sending an impulse into a system gives you $h(t)$, the impulse response. The Fourier Transform of $h(t)$, $H(\omega)$, is known as the transfer function, which dictates how functions behave when sent through that system.

$$f(t) \rightarrow \boxed{\text{LTI System}} \rightarrow g(t) = f(t) \overset{\text{convolution}}{*} h(t) = h(t) * f(t)$$
$$\updownarrow$$
$$G(\omega) = H(\omega)F(\omega)$$

Basically, $H(\omega)$ introduces a gain of $|H(\omega)|$ and a phase shift of $\angle H(\omega)$. We'll do a simple example.

ex. $5\cos(2t+3) - 7\sin(t-1) \rightarrow \boxed{I} \rightarrow$

$$5\,|H(2)|\cos(2t+3+\angle H(2)) - 7\,|H(1)|\sin(t-1+\angle H(1))$$

This is simple enough to do by inspection. Normally, you would

- find $F(\omega)$
- multiply by $H(\omega)$
- inverse transform $G(\omega) \rightarrow g(t)$

but this is much faster and is exactly the same. Don't waste your time.

That's it for the review! Next time - actual course content!