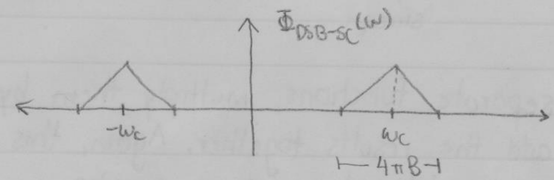
Chan and Mitran

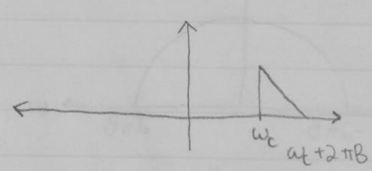
Quadrature Carrier Multiplexing

Introduction

As the end of the last unit, we came up with a modulation method that is completely power-efficient. But what it lacks is spectrum efficiency.



The upper sideband and lower sidebands are simply mirror images of each other. We're sending a signal with a bandwidth of $4\pi B$, but in order to reconstruct the baseband, we only need to send $2\pi B$:



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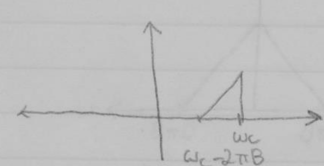
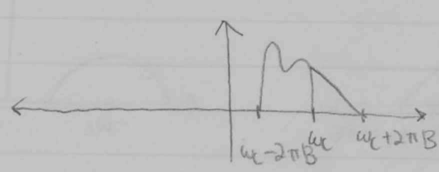


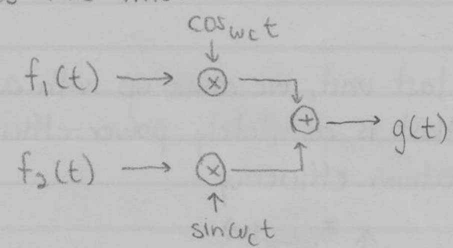
This way, we have 100% spectrum efficiency. Alternatively, we could send two separate signals, each one using either the upper and lower sidebands, ALSO achieving spectrum efficiency:



The first option is called Single-Sideband, which we'll cover next unit. This unit, we'll focus on the 2-in-1: quadrature carrier multiplexing.

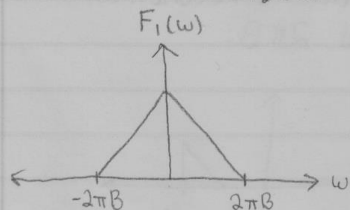
Quadrature Carrier Multiplexing

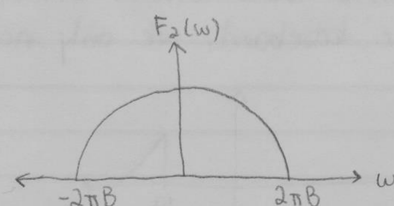
QCM looks like this:



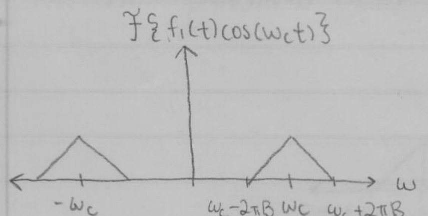
We take two separate functions, multiply them by sin and cos, and add the results together. Again, this is hard to visualize, so let's draw some graphs.

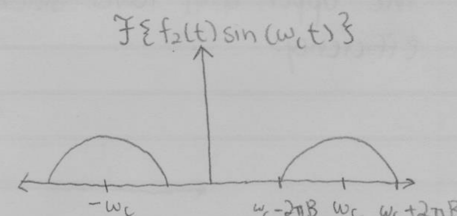
Say we have $f_1(t) \longleftrightarrow F_1(\omega)$, each with bandwidth
$f_2(t) \longleftrightarrow F_2(\omega)$ B Hz.



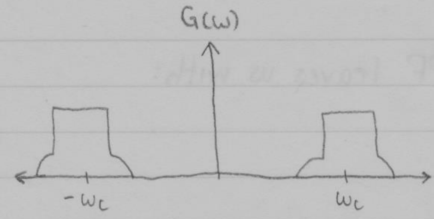


Then, we multiply by either sin or cos, both of which split up the spectrum and shift them right/left by their respective frequency.





Finally, we add the results together



Since both graphs originally had bandwidths of B Hz, this result still has a band width of 2B Hz, or 4πB rads.

Now, this resulting spectrum looks pretty whack. How do we recover our original signals? In engineering, we like to reuse existing technology, so let's try using the receiver from DSB-SC:

$$g(t) \longrightarrow \otimes \longrightarrow \text{LPF} \rightarrow d_1(t) \text{ or } d_2(t)$$

$$\uparrow$$

$$\cos(\omega_c t) \text{ or } \sin(\omega_c t)$$

$$g(t)\cos(\omega_c t) = [f_1(t)\cos(\omega_c t) + f_2(t)\sin(\omega_c t)]\cos(\omega_c t)$$
$$= f_1(t)\cos^2(\omega_c t) + f_2(t)\sin(\omega_c t)\cos(\omega_c t)$$

We can then split these terms up using trig identities.

$$= \tfrac{1}{2} f_1(t) + \tfrac{1}{2} f_1(t)\cos(2\omega_c t) + \tfrac{1}{2} f_2(t)\sin(2\omega_c t)$$

Then, we need to pass this through a low pass filter. The first term is a low frequency and gets let through. $2\omega_c$ is pretty high, so it gets rejected.

$$\therefore \; d_1(t) = \tfrac{1}{2} f_1(t)$$

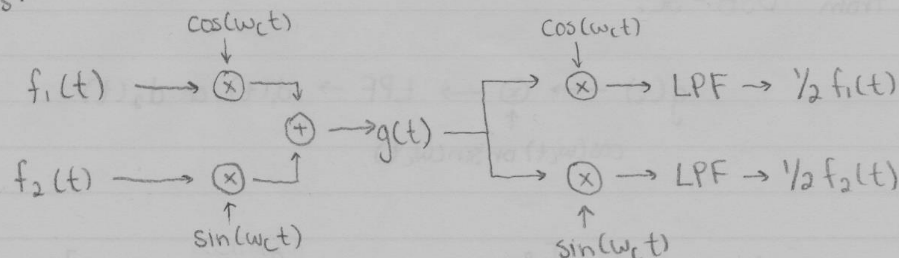
Let's do the same for $\sin(\omega_c t)$:

$$g(t)\sin(\omega_c t) = [f_1(t)\cos(\omega_c t) + f_2(t)\sin(\omega_c t)]\sin(\omega_c t)$$
$$= f_1(t)\cos(\omega_c t)\sin(\omega_c t) + f_2(t)\sin^2(\omega_c t)$$
$$= \tfrac{1}{2} f_1(t)\sin(2\omega_c t) + \tfrac{1}{2} f_2(t) + \tfrac{1}{2} f_2(t)\cos(2\omega_c t)$$

Again, passing through the LPF leaves us with:

$$d_2(t) = \tfrac{1}{2} f_2(t)$$

So thankfully, we can demodulate using DSB-SC's technique, and the entire transmitter/receiver structure looks like this:



Even though we've jumbled up the spectrum of $f_1(t)$ and $f_2(t)$ into $g(t)$, we can recover our original signals EXACTLY just by multiplying by $\cos(\omega_c t)$ or $\sin(\omega_c t)$ again.

<u>Frequency and Phase Error</u>

Once again, we're relying on the assumption that we can produce two (or four, in this case) sinusoids that are exactly in sync.

Let's explore what happens when we have a sinusoid with a frequency error $\Delta\omega$ and a phase error θ.

$$g(t)\cos[(\omega_c + \Delta\omega)t + \theta] = [f_1(t)\cos(\omega_c t) + f_2(t)\sin(\omega_c t)] \times \cos[(\omega_c + \Delta\omega)t + \theta]$$

$$= f_1(t)\cos(\omega_c t)\cos[(\omega_c+\Delta\omega)t+\theta]$$
$$+ f_2(t)\sin(\omega_c t)\cos[(\omega_c+\Delta\omega)t+\theta]$$

↓ trig identity

$$= f_1(t)\left[\tfrac{1}{2}\cos[(2\omega_c+\Delta\omega)t+\theta] + \tfrac{1}{2}\cos(\Delta\omega t+\theta)\right]$$
$$+ f_2(t)\left[\tfrac{1}{2}\sin[(2\omega_c+\Delta\omega)t+\theta] + \tfrac{1}{2}\sin(\Delta\omega t+\theta)\right]$$
$$= \tfrac{1}{2}f_1(t)\cos[(2\omega_c+\Delta\omega)t+\theta] + \tfrac{1}{2}f_1(t)\cos(\Delta\omega t+\theta)$$
$$+ \tfrac{1}{2}f_2(t)\sin[(2\omega_c+\Delta\omega)t+\theta] + \tfrac{1}{2}f_2(t)\sin(\Delta\omega t+\theta)$$

Passing this through a LPF, we can reject the $2\omega_c+\Delta\omega$ and let the $\Delta\omega$ through.

$$= \tfrac{1}{2}f_1(t)\cos(\Delta\omega t+\theta) + \tfrac{1}{2}f_2(t)\sin(\Delta\omega t+\theta)$$

We multiplied by $\cos[(\omega_c+\Delta\omega)t+\theta]$ hoping to retrieve the entirety of our $f_1(t)$ signal, but what do we actually get?

Some distortion/beats with the $\cos(\Delta\omega t+\theta)$. We ALSO get a completely unrelated signal $f_2(t)$. As if it wasn't important enough to get our carrier sinusoids synchronized, we basically completely fuck ourselves when using QCM.

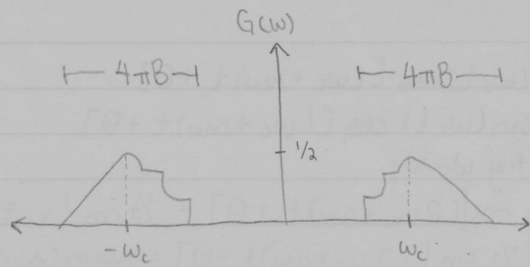
Arbitrary Spectrum

In all the modulation techniques we've looked at so far, the upper and lower sidebands are always mirror images - that is, complex conjugates.

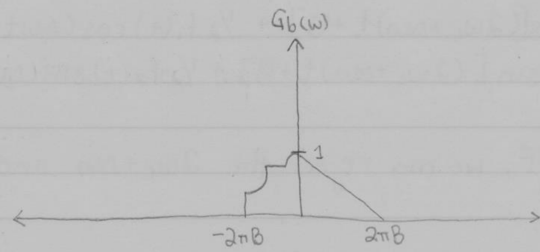
QCM allows us to create ANY arbitrary bandpass spectrum. Which really means we can make signals with any shape we want for the spectrum.*

Let's prove it.

* Not actually ANY, but any reasonable 1-to-1 shape.



Here we have some arbitrary spectrum $G(\omega)$.



Next, we have $G_b(\omega)$ which we get by shifting $G(\omega)$ down to DC.

As such, we can define $G(\omega)$ one of three ways:

$$① \quad G(\omega) = \tfrac{1}{2} G_b(\omega - \omega_c), \quad \omega > 0$$

This first definition is your regular multiply-by-cos shift.

$$② \quad G(\omega + \omega_c) = \tfrac{1}{2} G_b(\omega), \quad \omega + \omega_c > 0$$

In this case, $G(\omega)$ is shifted to the left by ω_c, bringing the spectrum to DC. $G_b(\omega)$ is shrunk by $1/2$ to accommodate for the height

$$③ \quad 2G(\omega + \omega_c) = G_b(\omega), \quad \omega > -\omega_c$$

Honestly, it's the same thing. It's math, guys.

Now, we can say that $g_b(t)$, the complex baseband equivalent is the time-domain function that creates $G_b(\omega)$.

Okay, but so what? How does that help?

Remember that $G(\omega)$ is the spectrum of some function $g(t)$. The positive frequencies of $G(\omega)$ allow us to completely reconstruct $g(t)$ - that's what the inverse Fourier does.

So it makes sense that $G_b(\omega)$ allows us to completely reconstruct $g_b(t)$.

We managed to CREATE $G_b(\omega)$ by shifting things by ω_c and modifying the amplitudes of our spectrum. In other words, by the definition

$$G(\omega) = \tfrac{1}{2} G_b(\omega - \omega_c) \ , \ \omega > 0,$$

$G_b(\omega)$ in combination with ω_c COMPLETELY SPECIFY G(ω). specifies lol
By that exact same argument, $g_b(t)$ in combination with ω_c COMPLETELY SPECIFIES $g(t)$.

Let's get into the math.

How do we relate $g_b(t)$ and ω_c with $g(t)$?

$$g(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\longleftrightarrow}} G(\omega)$$

$$G(\omega) = \begin{cases} G(\omega) & \omega > 0 \\ G(-\omega)^* & \omega < 0 \end{cases}$$

as $g(t)$ is a real-valued signal. As before, we can express $G(\omega)$ in terms of $G_b(\omega)$.

$$= \begin{cases} \tfrac{1}{2} G_b(\omega - \omega_c) & \omega > 0 \\ \tfrac{1}{2} G_b(-\omega - \omega_c)^* & \omega < 0 \end{cases}$$

Since the first part is 0 when ω is negative, and the second part is 0 when ω is positive, we can just add them.

$$G(\omega) = \tfrac{1}{2} G_b(\omega - \omega_c) + \tfrac{1}{2} G_b(-\omega - \omega_c)^*$$

To get $g(t)$ from $G(\omega)$ we take the inverse Fourier.

$$g(t) = \mathcal{F}^{-1}\{\tfrac{1}{2} G_b(\omega-\omega_c) + \tfrac{1}{2} G_b(\omega-\omega_c)^*\}$$
$$= \tfrac{1}{2}\mathcal{F}^{-1}\{G_b(\omega-\omega_c)\} + \tfrac{1}{2}\mathcal{F}^{-1}\{G_b(-\omega-\omega_c)^*\}$$

We know $\mathcal{F}^{-1}\{G_b(\omega-\omega_c)\}$ is $g_b(t)e^{j\omega_c t}$ from the transform properties in the first couple of units. But what happens to $\mathcal{F}^{-1}\{G_b(-\omega-\omega_c)^*\}$? We'll have to do this the old-fashioned way: by definition.

$$\mathcal{F}^{-1}\{G_b(-\omega-\omega_c)^*\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_b(-\omega-\omega_c)^* e^{j\omega t}\,d\omega$$

We'll have to integrate by substitution:

$$\text{Let } u = -\omega - \omega_c \qquad \omega = -u - \omega_c$$
$$du = -d\omega \quad \therefore \quad d\omega = -du$$

We also need to change the bounds of integration.

$$\omega = -\infty \rightarrow -\infty = -u - \omega_c$$
$$-\infty + \omega_c = -u$$
$$u = \infty$$

$$\omega = \infty \rightarrow \infty = -u - \omega_c$$
$$\infty + \omega_c = -u$$
$$u = -\infty$$

$$\mathcal{F}^{-1}\{G_b(-\omega-\omega_c)^*\} = \frac{1}{2\pi}\int_{\infty}^{-\infty} G_b(u)^* e^{j(-u-\omega_c)t} (-du)$$

The $-du$ is gross, so we'll get rid of the $-$ to flip the bounds of integration again.

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_b(u)^* e^{j(-u-\omega_c)t} du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_b(u)^* e^{-jut} \cdot e^{-j\omega_c t} du$$

no u terms, constant

$$= \frac{e^{-j\omega_c t}}{2\pi}\int_{-\infty}^{\infty} G_b(u)^* e^{-jut} du$$

We'll pull out the complex conjugate

$$= \frac{e^{-j\omega_c t}}{2\pi} \underbrace{\left(\int_{-\infty}^{\infty} G_b(u) e^{-jut} du\right)^*}_{\text{literally just } 2\pi \times \text{ the inverse Fourier transform}}$$

$$= \frac{e^{-j\omega_c t}}{2\pi}\left[2\pi\, g_b(t)\right]^*$$

$$= g_b(t)^* e^{-j\omega_c t}$$

So overall,

$$g(t) = \frac{g_b(t) e^{j\omega_c t}}{2} + \frac{g_b(t)^* e^{-j\omega_c t}}{2}$$

Already, this might look vaguely familiar. Let's extract the complex conjugate.

note now missing −

$$= \tfrac{1}{2} g_b(t) e^{j\omega_c t} + \left(\tfrac{1}{2} g_b(t) e^{j\omega_c t}\right)^*$$

Remember that anything + its complex conjugate is $2 \times \operatorname{Re}\{\text{thing}\}$.

$$g(t) = 2\,\mathrm{Re}\{\tfrac{1}{2} g_b(t) e^{j\omega_c t}\}$$
$$= \mathrm{Re}\{g_b(t) e^{j\omega_c t}\}$$

Alright, we're about to get FUNKY. Check these two expansions.

$$g_b(t) = \mathrm{Re}\{g_b(t)\} + j\,\mathrm{Im}\{g_b(t)\}$$

$$e^{j\omega_c t} = \cos(\omega_c t) + j\sin(\omega_c t)$$

Everything is a sum of its real and imaginary parts. The exponent has been expanded to its sum of sinusoids.

$$\therefore g(t) = \mathrm{Re}\left[\mathrm{Re}\{g_b(t)\} + j\,\mathrm{Im}\{g_b(t)\} \times (\cos(\omega_c t) + j\sin(\omega_c t))\right]$$

Let's cross-multiply.

$$= \mathrm{Re}\Big[\ \mathrm{Re}\{g_b(t)\}\cos(\omega_c t) + \mathrm{Re}\{g_b(t)\}\,j\sin(\omega_c t) + j\,\mathrm{Im}\{g_b(t)\}\cos(\omega_c t) + j^2\,\mathrm{Im}\{g_b(t)\}\sin(\omega_c t)\ \Big]$$

The purely real terms are the ones at the top and bottom, so we keep them and toss the rest.

$$= \mathrm{Re}\{g_b(t)\}\cos(\omega_c t) - \mathrm{Im}\{g_b(t)\}\sin(\omega_c t)$$

Now let's define: $f_1(t) = \mathrm{Re}\{g_b(t)\}$
$f_2(t) = -\mathrm{Im}\{g_b(t)\}$

$$\therefore g(t) = f_1(t)\cos(\omega_c t) + f_2(t)\sin(\omega_c t)$$

If this doesn't look familiar, you haven't been paying enough attention.

This is the EXACT definition for QCM. We've proved that if we want to generate some arbitrary passband signal $G_b(\omega)$, we just need to choose $f_1(t)$ and $f_2(t)$:

$$f_1(t) = \mathrm{Re}\{g_b(t)\}$$
$$f_2(t) = -\mathrm{Im}\{g_b(t)\}$$

$f_1(t)$ is known as the "in phase" component and $f_2(t)$ is known as the "quadrature" component.

Let's summarize.

Summary

1) $g(t)$ is some real-valued signal that occupies the frequency band $[\omega_c - 2\pi B, \omega_c + 2\pi B]$, a bandwidth of $4\pi B$.

2) $G_b(\omega)$ is the positive shape of the spectrum we want to create. It is shifted down to DC and as such, occupies $[2\pi B, 2\pi B]$. This bandwidth is $2\pi B$ (positive only).

3) $g_b(t)$ is the complex baseband equivalent model (or complex envelope) of $G_b(\omega)$. Obviously, as they're the same signal, $g_b(t)$'s bandwidth is $2\pi B$.

4) We select $f_1(t)$ as $\mathrm{Re}\{g_b(t)\}$. We select $f_2(t)$ as $-\mathrm{Im}\{g_b(t)\}$.

5) We can then create $G_b(\omega)$ using QCM by generating $g(t) = f_1(t)\cos(\omega_c t) + f_2(t)\sin(\omega_c t)$.

EVERY real-valued passband signal can be generated using QCM. It can do the same thing as every technique out there - the only question left is whether it's the simplest method for our problem.

For example - what if we don't have two messages to send, and we don't need to construct some arbitrary spectrum?

Coming up next - single side band.