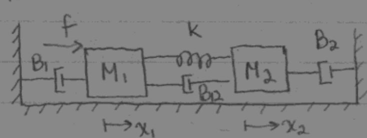
ECE 380 Midterm Solutions
Spring 2015

So apparently the profs hate us because we're all idiots, and I didn't have time to make a review sheet for this course, so hopefully this will be useful for final exam prep.

We'll introduce the question, go over the concepts required to solve them, and then actually solve the question.

1) Find the transfer function X_1/F of the mechonical system below.



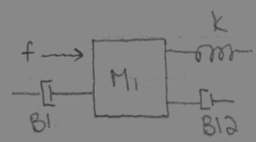
These mechanical system problems always follow the same process.

A) Construct a system of equations from free-body diagrams
B) Take the Laplace transform of all equations in the system
C) Use Cramer's Rule to solve for both required portions of the transfer function (in this case, $X_1(s)$ and $F(s)$)
D) Simplify the end result

FBD and System of Equations

One thing to remember is that dampeners (-[⊢) are always modelled as $B\vec{v}$ in this course, where B is the damping constant and $\vec{v}$ is x', or the velocity.

Let's start with M1. It's always good to define the direction of +/-, so we'll say rightward movement is +.



As with all free body diagrams, all we really care about is the stuff attached to M1 itself.

Of course, f is applied in the positive direction. The spring tries to resist this, applying a leftward force to M1. This is the same case with the B1 dampener.

Since B_{12} is attached to both blocks, we don't really know what direction it'll go in: but honestly, it doesn't matter.* Remember that if we keep our signs consistent, it'll all work out. So let's just say it acts to the right.

Applied: f — Spring: $-k(x_2 - x_1)$

B_1 damp: $B_1 x_1'$ — B_{12} damp: $B_{12}(x_2' - x_1')$

$$B_1 x_1' \longleftarrow \bullet \longrightarrow f$$
$$\longrightarrow k(x_2 - x_1)$$
$$\longrightarrow B_{12}(x_2' - x_1')$$

$$\therefore \; M_1: \quad M_1 x_1'' = f + k(x_2 - x_1) + B_{12}(x_2' - x_1') - B_1 x_1'$$

Now, for M2. Forces are equal and opposite, so the spring and B_{12} damp must act in the othe direction for M2. Then we have the B2 damp which pulls M2 to the right.

$$k(x_2 - x_1) \longleftarrow \bullet \longrightarrow B_2 x_2'$$
$$B_{12}(x_2' - x_1') \longleftarrow$$

$$\therefore \; M_2: \quad M_2 x_2'' = B_2 x_2' - k(x_2 - x_1) - B_{12}(x_2' - x_1')$$

*this is the case for all forces, we just choose directions for ease of understanding

Laplace Transform of System

There's the property of transforms that says

$$\mathcal{L}\{x'\} = sX(s) - x(0-)$$

but the mechanical system is initially at rest, so basically we can just ignore the second part when we're doing the transformation.

① $M_1 x_1'' = f + k(x_2 - x_1) + B_{12}(x_2' - x_1') - B_1 x_1'$

$$\downarrow$$

$$M_1 s^2 X_1(s) = F(s) + k(X_2(s) - X_1(s)) + B_{12}(sX_2(s) - sX_1(s)) - B_1 s X_1(s)$$

Rearrange this to group by function $(F(s), X_1(s), X_2(s))$.

$$F(s) = (M_1 s^2 + B_{12}s + k + B_1 s)X_1(s) + (-B_{12}s - k)X_2(s)$$

② $M_2 x_2'' = B_2 x_2' - k(x_2 - x_1) - B_{12}(x_2' - x_1')$

$$\downarrow$$

$$M_2 s^2 X_2(s) = B_2 s X_2(s) - k(X_2(s) - X_1(s)) - B_{12}(sX_2(s) - sX_1(s))$$

Again, we rearrange:

$$0 = (-B_{12}s - k)X_1(s) + (M_2 s^2 + B_2 s + k + B_{12}s)X_2(s)$$

So now we have our system in the Laplace form (aka s domain)

Apply Cramer's Rule

Basically, Cramer's Rule say this.

$$a\,X_1(s) + b\,X_2(s) = F(s)$$
$$c\,X_1(s) + d\,X_2(s) = 0$$

is some system of equations. We can construct this:

$$\underbrace{\begin{bmatrix} \overset{\text{coeff of } X_1(s)}{a} & \overset{\text{coeff of } X_2(s)}{b} \\ c & d \end{bmatrix}}_{\text{coefficients}} \underbrace{\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}}_{\text{functions}} = \underbrace{\begin{bmatrix} F(s) \\ 0 \end{bmatrix}}_{\text{what they're equal to}},$$

a system of matrices. If we want to find $X_1(s)$, the first element of the functions matrix, we replace the first column of the coefficients with "what they're equal to"*. Then,

$$X_1(s) = \text{determinant of } \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}$$
$$= 1d - 0b$$
$$= d$$

$F(s)$ itself is the determinant of the coefficients matrix. So let's apply this.

$$\begin{bmatrix} M_1s^2 + B_{12}s + B_1s + k & -B_{12}s - k \\ -B_{12}s - k & M_2s^2 + B_{12}s + B_2s + k \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F(s) \\ 0 \end{bmatrix}$$

$$\therefore\ F(s) = \det[\text{coeff matrix}]$$
$$= (M_1s^2 + B_{12}s + B_1s + k)(M_2s^2 + B_{12}s + B_2s + k) - (-B_{12}s - k)^2$$

* where $F(s)$ is set equal to 1

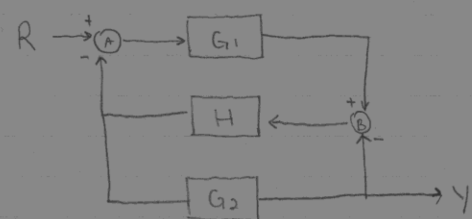
$$\therefore X_1(s) = \det \begin{bmatrix} 1 & -B_{12}s - k \\ 0 & M_2s^2 + B_{12}s + B_2s + k \end{bmatrix}$$
$$= 1(M_2s^2 + B_{12}s + B_2s + k) - 0(-B_{12}s - k)$$
$$= M_2s^2 + B_{12}s + B_2s + k$$

Putting it all together,

$$\frac{X_1(s)}{F(s)} = \frac{M_2s^2 + B_{12}s + B_2s + K}{(M_1s^2 + B_{12}s + B_1s + k)(M_2s^2 + B_{12}s + B_2s + k) - (-B_{12}s - k)^2}$$

And simplifying is just kind of an optional step, so we'll leave it as is.

2) Find the transfer function Y/R of the block diagram below.



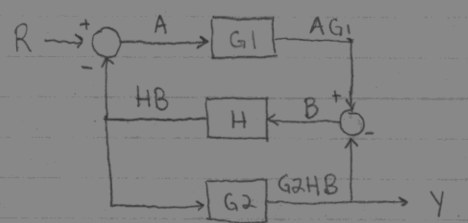
I highly dislike the graphical way of solving block diagrams so I'm going to use the algebraic method. There's a process that will prevent us from going around in circles forever.

A) Find an expression for the signal coming out of all summing junctions
B) Find an expression for the output signal
C) Set all equations to equal zero or the input, and apply Cramer's Rule to solve for R and Y.

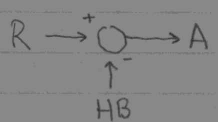
These aren't actually very difficult problems as long as you take care to follow the process in order.

Expressions for Summing Junctions

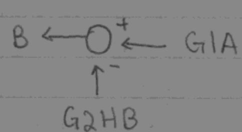
It's very useful to label the outputs of summing junctions as new signals, so we can kind of ignore the details of where they came from.



Now, let's break it into parts according to summing junction input / outputs.



$\therefore \quad A = R - HB$



$\therefore \quad B = G1A - G2HB = G1A - Y$

Expressions for Output

This one's straight forward.

$$Y = G2HB$$

Apply Cramer's Rule

First we set our equations equal to input / zero.

$$R = (1)A + (H)B + (0)Y$$
$$0 = (G1)A - (1)B - (1)Y$$
$$0 = (0)A + (G2H)B - (1)Y$$

$$\rightarrow \begin{bmatrix} 1 & H & 0 \\ G1 & -1 & -1 \\ 0 & G2H & -1 \end{bmatrix} \begin{bmatrix} A \\ B \\ Y \end{bmatrix} = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore Y = \det \begin{bmatrix} 1 & H & R \\ G1 & -1 & 0 \\ 0 & G2H & 0 \end{bmatrix}$$ (we're using $\begin{smallmatrix} R \\ 0 \\ 0 \end{smallmatrix}$ as the reference column)

$$= R[(G1)(G2H) - (0)(-1)] - 0[\text{who cares}] + 0[\text{who cares}]$$
$$= RG1G2H$$

$$\therefore R = \det \begin{bmatrix} 1 & H & 0 \\ G1 & -1 & -1 \\ 0 & G2H & -1 \end{bmatrix}$$ (we're using $1\ H\ 0$ as the reference row)

$$= 1[(-1)(-1) - (G2H)(-1)] - H[(G1)(-1) - (0)(-1)] + 0[\text{who cares}]$$
$$= [1 + G2H] - H[-G1]$$
$$= 1 + G2H + G1H$$

$$\therefore \frac{Y}{R} = \frac{RG1G2H}{1 + HG2 + HG1}$$

That's it! Obviously, depending on which transfer function we're solving for, we'd need to muck around a bit with our matrix so we can solve for the proper values.

3) We have a system

$$\text{①} \quad Mr'' = \overset{\text{constant}}{f_r} + \overset{\text{constant}}{Mr\omega^2}$$
$$\text{②} \quad Mr\omega' = \underset{\text{constant}}{f_\theta} + 2Mr'\omega$$

a) Find an operating point where $\omega = 0$.
b) Linearize the equation about the found operating point.

So all of these equations have independent variables r and ω, and f_r for ①, and f_θ for ②. As such, we can represent them like this:

$$\text{①} \quad g(f_r, r, \omega) = 0 \quad , \text{ where } r(t) = \text{some } r_0$$
$$\text{②} \quad h(f_\theta, r, \omega) = 0 \qquad\qquad \omega(t) = \text{some } \omega_0$$

constants

We're given that $\omega_0 = 0$. Taking the derivative of 0 is still 0, meaning that $\omega' = 0$. We can plug these into our equations:

$$\text{①} \quad Mr'' = f_r + 0$$
$$\text{②} \quad 0 = f_\theta + 0 \longrightarrow f_\theta = 0$$

So all that's left over is that we don't know what r_0 is, nor f_r. Since $r(t)$ = a constant, $r'(t)$ MUST equal zero, by definition. It then follows that $r''(t)$ MUST also equal zero, $\therefore r'' = 0$.

$$\text{①} \quad 0 = f_r$$

Now, all that is remaining is - what is r_0? The thing is: any constant will satisfy our equation for r_0, so we can pick literally anything.

0 is an easy choice here, because why the fuck not.

$\therefore$ the operating points (f_r, r_0, ω_0)
$(f_\theta, r_0, \omega_0)$

are just $(0, 0, 0)$.

b) Now that we have the operating point, we want to linearize both equations about that point.

$$① \quad g(f_r, r, \omega) = Mr'' - f_r - Mr\omega^2 = 0$$
$$② \quad h(f_\theta, r, \omega) = Mr\omega' - f_\theta - 2Mr'\omega = 0$$

Again, there's a process we should follow, and it's useful to keep in mind exactly what we're trying to do: create equations that behave in a linear fashion near the operating point.

With that being said:

A) Take the partial derivative of the equation with respect to each independent variable and any derivative of it, replacing it with the Δ of itself.

This represents the small movements away from the operating point.

B) Take the Laplace transform.

C) Re-order to find the transfer function of the linearized equation.

Let's start with ①.

A) $$g(f_r, r, \omega) = Mr'' - f_r - Mr\omega^2 = 0$$

$$\rightarrow 0 = \frac{\partial g}{\partial r''}\bigg|_{op} \Delta r'' + \frac{\partial g}{\partial f_r}\bigg|_{op} \Delta f_r + \frac{\partial g}{\partial r}\bigg|_{op} \Delta r + \frac{\partial g}{\partial \omega}\bigg|_{op} \Delta\omega$$

$$= (M|_{op})\Delta r'' + (1|_{op})\Delta f_r + (M\omega^2|_{op})\Delta r + (2Mr\omega|_{op})\Delta\omega$$
$$= (M)\Delta r'' + (-1)\Delta f_r + (-M(0)^2|_{op})\Delta r + (-2M(0)(0))\Delta\omega$$
$$= M\Delta r'' - \Delta f_r$$

Now, we've conveniently reduced a bunch of terms to zero. Remember that each Δsomething represents an independent function

B) $$\mathcal{L}\{0\} = \mathcal{L}\{M\Delta r'' - \Delta f_r\}$$
$$0 = Ms^2 R(s) - F_r(s)$$
$$\frac{R(s)}{F_r(s)} = \frac{1}{Ms^2}$$

Let's repeat the process.

A) $$h(f_\theta, r, \omega) = Mr\omega' - f_\theta - 2Mr'\omega = 0$$

$$\rightarrow 0 = \frac{\partial h}{\partial r}\bigg|_{op} \Delta r + \frac{\partial h}{\partial \omega'}\bigg|_{op} \Delta\omega' + \frac{\partial h}{\partial f_\theta}\bigg|_{op} \Delta f_\theta + \frac{\partial h}{\partial r'}\bigg|_{op} \Delta r' + \frac{\partial h}{\partial \omega}\bigg|_{op} \Delta\omega$$
$$= (M\omega'|_{op})\Delta r + (Mr|_{op})\Delta\omega' + (-1|_{op})\Delta f_\theta$$
$$+ (-2M\omega|_{op})\Delta r' + (-2Mr'|_{op})\Delta\omega$$
$$= (M(0))\Delta r + (M(0))\Delta\omega' + (-1)\Delta f_\theta$$
$$+ (-2M(0))\Delta r' + (-2M(0))\Delta\omega$$
$$= -\Delta f_\theta$$

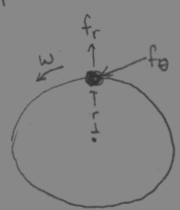
B) $$\mathcal{L}\{0\} = \mathcal{L}\{-\Delta f_\theta\}$$
$$0 = -F_\theta(s)$$
$$F_\theta(s) = 0$$

And at this stage we have expressions for all of our independent variables in the frequency domain. For exam purposes, we're done. But I'd like to take a closer look.

$$R(s) = \frac{F_r(s)}{Ms^2} \qquad F_r(s) = Ms^2 R(s) \qquad W(s) = 0 \qquad F_\theta(s) = 0$$

↑distance from center of rotation — ↑applied force away from center — ↑angular speed — ↑applied force tangential to rotation

These equations model the rotation of a particle. We've specified the angular speed to be 0.



If we actually look at this system, it makes a ton of sense. $W(s)$ is zero because we've specified it to be so. $F_\theta(s)$ MUST be zero, or else the particle starts rotating.

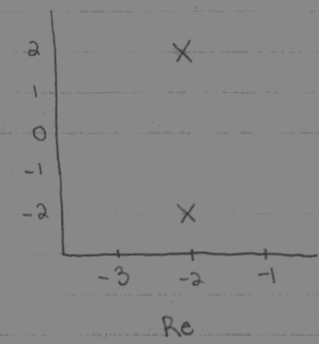
However, we can be any distance away ($R(s)$), and apply any amount of force away ($F_r(s)$), but still not have the particle rotate. Physics!

4) Two linear, time-invariant systems, with DC gains of 1. Using their pole-zero plots, find their peak times, percentage overshoots, and 5% settling times.

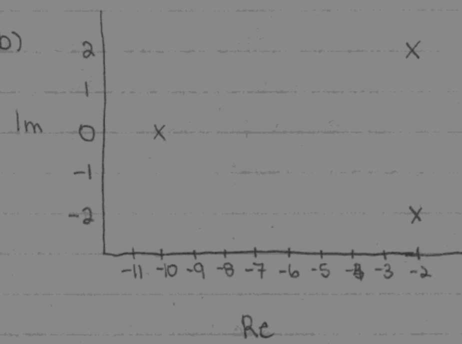
You're given the following:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}, \quad OS = \left(e^{-\zeta\pi/\sqrt{1-\zeta^2}}\right)100\%, \quad T_s(5\%) = \frac{3}{\zeta\omega_n}.$$

a)



b)



The first thing to know about these graphs are that they MUST be symmetrical about the line Imaginary = 0.

ω_n is given as the distance from the closest pole to the point (0,0) on the real-imaginary plot.

The distance from the closest point to the line Real = 0 is equal to $\zeta\omega_n$.

a) The closest point is (-2, 2).

$$\therefore \quad \omega_n = \sqrt{(-2-0)^2 + (2-0)^2}$$
$$= \sqrt{(-2)^2 + (2)^2}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$

We now know that:

$$2 = \zeta\omega_n$$
$$\zeta = \frac{2}{\omega_n}$$
$$= \frac{2}{2\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}$$

We can now just plug and play.

$$T_p = \frac{\pi}{2\sqrt{2}\sqrt{1-(1/\sqrt{2})^2}}$$
$$= \frac{\pi}{2\sqrt{2}\sqrt{1/2}}$$
$$= \frac{\pi}{2} \text{ seconds}$$

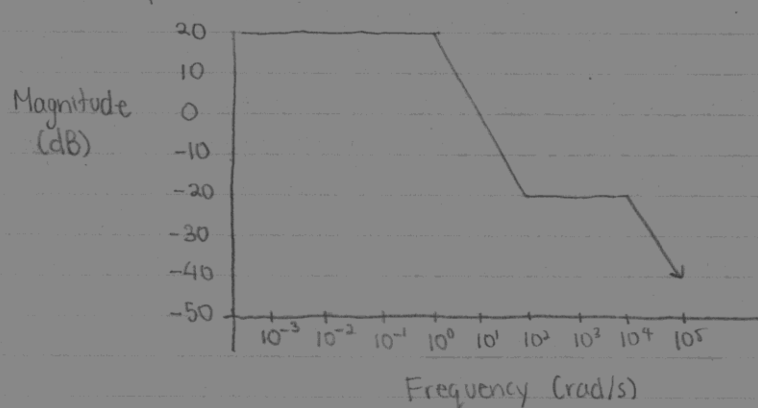
$$OS = 100e^{-\pi(1/\sqrt{2})/\sqrt{1-(1/\sqrt{2})^2}}$$
$$= 100e^{-\pi} \%$$

$$T_s = \frac{3}{\zeta\omega_n}$$
$$= \frac{3}{2} \text{ seconds.}$$

Just repeat for the next question.

b) Closest point is still $(-2,2)$, so everything is the same.

5) A linear, time-invariant system has the following Bode plot:



(I didn't draw the phase plot because it's unnecessary - all we need to know here is that the plot starts at $0°$, meaning the transfer function is positive)

What is the transfer function? Give a reasonable approximation of the system's step response.

Remember that transfer functions look like this:

$$H(s) = (\text{Low f Gain}) \frac{(1+s/\text{zero}_1)(1+s/\text{zero}_2)\ldots}{(1+s/\text{pole}_1)(1+s/\text{pole}_2)\ldots}$$

Poles increase the rate of change by 20 dB/decade while zeros decrease it by 20 dB/decade. The plot starts at 20 dB, therefore our gain must be

$$20\log(x) = 20$$
$$\log(x) = 1$$
$$x = 10$$

And we can see on our graph we have poles at 10^0 and 10^4, and a zero at 10^2.

$$\therefore \quad H(s) = \frac{10 \quad (1 + s/10^2)}{(1+s/10^0)(1+s/10^4)}$$

Now, the approximation means just to pick out the dominant pole/zero - the one with the lowest frequency, ie. $(1+s/10^0)$.

So if we ignore the others, we're left with

$$H(s) = \frac{10}{1+s}$$

A step function is given as $1/s$ in the frequency domain so the step response of $H(s)$ would be

$$\frac{H(s)}{s} = \frac{10}{s(1+s)}$$

At which point we can take the inverse Laplace to get the response in the time domain.

$$\mathcal{L}^{-1}\left\{\frac{10}{s(1+s)}\right\} = 10\,\mathcal{L}^{-1}\left\{\frac{1}{s}\;\frac{1}{(s+1)}\right\}$$

$$= 10\,u(t)\,[1-e^{-t}]$$

6) A unity-feedback control system has controller and plant equations

$$C(s) = \frac{5(s+5)}{s+10} \qquad P(s) = \frac{10}{s(s+2)}$$

a) Find the steady-state error for input $r(t) = 5t+2,\ t \geq 0$

b) " " $r(t) = 10\,u_{-1}(t-5)$

There's a super simple trick to remember for these steady-state error questions. It looks like this:

type of system ↓	Step	Ramp	Parabola ← type of input
0	$1/1+k_p$	1	1
1	0	$1/k_v$	1
2	0	0	$1/k_a$

↖ steady state error

where
$$k_p = \lim_{s\to 0} C(s)P(s)$$
$$k_v = \lim_{s\to 0} s\,C(s)P(s)$$
$$k_a = \lim_{s\to 0} s^2\,C(s)P(s)$$

To determine the type of system, simply combine $C(s)P(s)$ and see how many poles we have <u>at the origin</u>. This number is the type of the system

$$C(s)P(s) = \frac{50(s+5)}{s(s+2)(s+10)}$$

↑one pole @ origin, ∴ type 1

a) Since our input is of type ramp*, our steady state error is:

$$e_{ss} = \frac{1}{\lim_{s\to 0} s^2 C(s)P(s)}$$

$$= \frac{1}{\lim_{s\to 0} s \frac{50(s+5)}{s(s+2)(s+10)}}$$

$$= \frac{1}{\frac{50(0+5)}{(0+2)(0+10)}}$$

$$= \frac{1}{12.5}$$

$^*\ \underset{\text{ramp}}{5t} + \underset{\text{step, } \therefore\ e_{ss}=0}{2}$

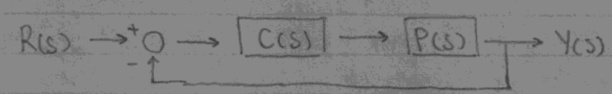
However, we need to account for the magnitude of our input ($5t$ as opposed to t)

$$\therefore \quad e_{ss} = \frac{5}{12.5}$$

b) The input is a step function, meaning our error is simply zero.

7) A unity-feedback control system has

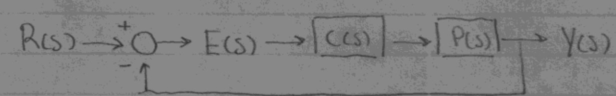
$$C(s) = \frac{5(s+5)}{s+10} \qquad P(s) = \frac{1}{s^2+100}$$



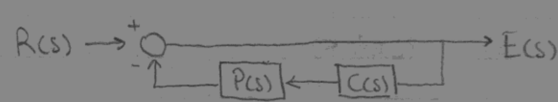
a) Find the transfer function from R(s) to E(s), the error. It is stable.

b) Find the steady-state error for input $r(t) = \sin 10(t),\ t \geq 0$

a) First thing to realize is that the error is right after the summation junction.



Let's shift this around a bit



Doesn't this look like a regular feedback system again, with $A = 1$ and $B = C(s)P(s)$? We can now say

$$\frac{E(s)}{R(s)} = \frac{A}{1+AB} = \frac{1}{1+C(s)P(s)}$$

$$= \frac{1}{1 + \frac{5(s+5)}{(s+10)(s^2+100)}}$$

$$= \frac{(s+10)(s^2+100)}{(s+10)(s^2+100) + 5(s+5)}$$

b) Now, we want to find the error itself. We're told that the error is stable, meaning that it converges upon some final value.

We can reiterate our proposition based on the final value theorem here:

If $\lim_{t\to\infty} y(t)$ has a finite limit,

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,Y(s)$$

which we can apply here.

$$\therefore \quad e_{ss} = \lim_{t\to\infty} e(t)$$
$$= \lim_{s\to 0} s\,E(s)$$
$$= \lim_{s\to 0} s \left(\frac{(s+10)(s^2+100)}{(s+10)(s^2+100)+5(s+5)} \right) R(s)$$

where $R(s) = \mathcal{L}\{\sin 10t\}$
$$= \frac{10}{s^2+10^2}$$
$$= \frac{10}{s^2+100}$$

$$\therefore \quad e_{ss} = \lim_{s\to 0} s \left(\frac{10(s+10)}{(s+10)(s^2+100)+5(s+5)} \right)$$
$$= (0)\left(\frac{10(10)}{(10)(100)+5(5)} \right)$$
$$= 0$$

So the steady-state error is zero.