Chan and Mitran

Power Spectral Density

Introduction

This isn't really a separate section - my bad But it's too late now, you'll have to deal with it.

Remember that we have the following, Parseval's theorem:

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$

which applies ONLY for energy signals. We interpret the $|F(\omega)|^2$ as energy spectral density, which is expressed in energy per frequency.

Now, what if $f(t)$ is a power signal? What we want is an expression for power spectral density, power per frequency.

Definition

Specifically, in the mathy jargon way, we want an expression for $S_f(\omega)$, such that power due to the range $\omega_1 \leq \omega \leq \omega_2$ is:

$$\frac{1}{2\pi} \int_{\omega_1}^{\omega_2} S_f(\omega) \, d\omega$$

where $S_f(\omega)$ is the power spectral density itself. Notice the parallel to how average energy is calculated:

$$\frac{1}{2\pi} \int_{\omega_1}^{\omega_2} |F(\omega)|^2 \, d\omega$$

Now, normally, this is fairly difficult to calculate what exactly $S_f(\omega)$ is. There are cases that are useful to us where it becomes trivial to calculate, though. We've explored this already, where $f(t)$ is a signal that can be decomposed into

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

The n^{th} term of this summation, $F_n e^{jn\omega_0 t}$, is the component due to the frequency $n\omega_0$. Remember that according to Fourier theory, components where $n \neq$ an integer are always 0.

This component has average power given by,

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |F_n e^{jn\omega_0 t}|^2 \, dt$$

which is a fairly regular averaging formula - adding up power due to each time instance and dividing it by the number of instances.

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |F_n|^2 \, |e^{jn\omega_0 t}|^2 \, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |F_n|^2 \, (1) \, dt$$

since the magnitude of $e^{j\theta}$ is simply 1.

$$= \lim_{T\to\infty} \frac{1}{T} \left(|F_n|^2\right) \int_{-T/2}^{T/2} dt$$

$$= |F_n|^2$$

Which again, concludes what we've kind of understood before - that the power due to a specific frequency

$n\omega_0$ can be given simply as the magnitude of that particular n's Fourier coefficient: $|F_n|^2$.

Now, we still don't know what $S_f(\omega)$ is. Normally, I would try to work towards solving the answer, but this time I'm instead, going to make a claim, and then explain why the claim makes sense.

The claim is as such:

$$S_f(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |F_n|^2 \delta(\omega - n\omega_0)$$

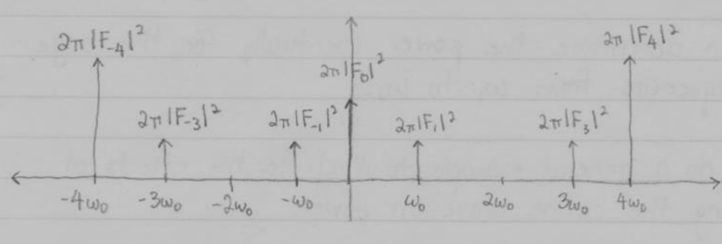
Let's break this down.

$\delta(\omega - n\omega_0)$ is an impulse function that occurs at the frequency $n\omega_0$, so one impulse function of a certain magnitude at EVERY multiple of $n\omega_0$.

That magnitude is $|F_n|^2$. The 2π is there because of the original definition we wanted - that the average power is

$$\frac{1}{2\pi} \int_{\omega_1}^{\omega_2} S_f(\omega)\, d\omega$$

If we were to graph $S_f(\omega)$, it would look like this:



(In this case, $|F_{-2}|^2 = |F_2|^2 = 0$)

Now, remember what happens when you integrate the delta function. The result is completely dependent on the range of integration

For the following integral,

$$\int_a^b A\,\delta(t-t_0)\,dt$$

If the integration is over WHERE the impulse occurs, ie. $a \le t_0 \le b$, the result is simply A.

If the integration is NOT over the impulse, ie. $t_0 < a$ or $b < t_0$, the result is simply 0.

So by integrating over $S_f(\omega)$, which is the summation of a whole bunch of delta functions, we simply get 2π multiplied by the sum of all the powers due to each distinct frequency.

$$= \frac{1}{2\pi}\int S_f(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\left[2\pi \sum_n |F_n|^2\right]$$

$$= \sum_n |F_n|^2$$

So by restricting our integral's bounds on $[\omega_1, \omega_2]$,

$$= \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} S_f(\omega)\,d\omega$$

We can determine the power specifically for the range of frequencies from ω_1 to ω_2.

Let's do a general example to illustrate the effects of changing the cosine wave on power.

ex. Find $S_f(\omega)$ for $f(t) = A\cos(\omega_0 t + \theta)$.

By expanding f(t) into its exponential form, we can skip calculating the Fourier coefficients directly.

$$f(t) = A\left[\frac{e^{j(\omega_0 t+\theta)} + e^{-j(\omega_0 t+\theta)}}{2}\right]$$

$$= \frac{A}{2} e^{j(\omega_0 t+\theta)} + \frac{A}{2} e^{-j(\omega_0 t+\theta)}$$

$$= \underbrace{\frac{A}{2} e^{j\theta}}_{F_1} \underbrace{e^{j\omega_0 t}}_{\substack{e^{jn\omega_0 t} \\ \text{where} \\ n=1}} + \underbrace{\frac{A}{2} e^{-j\theta}}_{F_{-1}} \underbrace{e^{-j\omega_0 t}}_{\substack{e^{jn\omega_0 t} \\ \text{where} \\ n=-1}}$$

It's fairly clear that cos only has two non-zero F_ns, at $n = 1, -1$. Let's find the magnitudes.

$$|F_1|^2 = \left|\frac{A}{2} e^{j\theta}\right|^2 = \frac{A^2}{4}(1) = \frac{A^2}{4}$$

$$|F_{-1}|^2 = \left|\frac{A}{2} e^{-j\theta}\right|^2 = \frac{A^2}{4}(1) = \frac{A^2}{4}$$

We didn't actually need to calculate both - the magnitudes for $n = -n$ are the same, I just forgot lol. So if our equation for $S_f(\omega) = 2\pi \sum |F_n|^2 \delta(\omega - n\omega_0)$,

$$S_f(\omega) = 2\pi\left[\frac{A^2}{4}\delta(\omega-(1)\omega_0) + \frac{A^2}{4}\delta(\omega-(-1)\omega_0)\right]$$
$$= 2\pi\,(A^2/4)\,\delta(\omega-\omega_0) + 2\pi\,(A^2/4)\,\delta(\omega+\omega_0)$$

Notice the A^2 - you might remember power is the square of the amplitude. Also that θ simply doesn't exist - phase has no effect on power.

Power Spectral Density and Linear Time-Invariant Systems

Remember that LTI system transforms the amplitude and phase of a signal. However, phase has no effect on power.

As such, it makes sense that only the amplitude transformation, $|H(\omega)|^2$, applies to the power spectral density.

This relationship is very simple. If:

$$f(t) \text{ has PSD } S_f(\omega)$$

$$f(t) \longrightarrow \overset{h(t)}{\text{LTI System}} \longrightarrow g(t)$$

$$g(t) \text{ has PSD } |H(\omega)|^2 S_f(\omega)$$

That's it! Next time, actual course material.