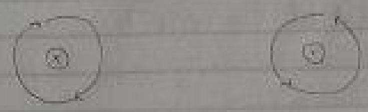
MAGNETIC FIELD STRENGTH

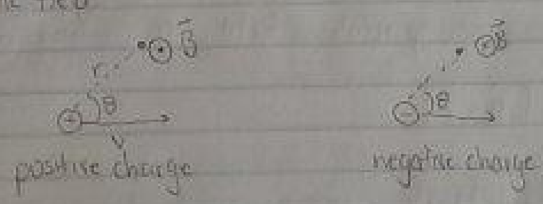
Magnetic Fields



Biot-Savart Law: $\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{qv\sin\theta}{r^2}\right]$, direction given by right hand rule

This is the magnetic field generated by a point charge, q being the magnitude and v being the velocity of it. r is the distance away from the point charge, where we want to find the magnetic field

$\mu_0 = 1.257 \times 10^{-6}\ \frac{Tm}{A}$



These fields look the same as the ones above - tangent circles along the path of motion. With multiple charges, $\vec{B}_{total} = \vec{B}_1 + \vec{B}_2 + \ldots + \vec{B}_n$

The Biot-Savart Law can be written as

$$\vec{B} = \frac{\mu_0 q\left[\vec{v} \times \vec{r}\right]}{4\pi r^3}$$ with the cross product.

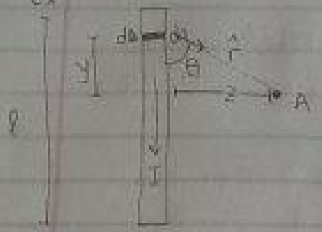
Magnetic Fields of Currents

We're really much more interested in describing the magnetic fields of currents. By using the $\frac{\Delta Q}{\Delta t}$ definition of current, we can say:

$$\vec{B}_{\substack{current\\segment}} = \frac{\mu_0 I\,[\vec{\Delta s} \times \hat{r}]}{4\pi r^2}$$

where $\vec{\Delta s}$ is a very short segment of the current.
Let's do some examples

ex



Let this be a long line of charge. What is the magnetic field strength and direction at A?

We can assume A is some distance z away from the conductor. Let's take a differential element of dq, length dℓ, which generates a field dB pointing out

$$dB = \frac{\mu_0 I}{4\pi r^2}[\vec{d\ell} \times \hat{r}]$$

$$= \frac{\mu_0 I\, d\ell\, \cancel{|\hat{r}|} \sin\theta}{4\pi r^2} \quad \text{where } \sin\theta = \frac{z}{\sqrt{z^2+y^2}} \text{ and } r = \sqrt{z^2+y^2}$$

$$= \frac{\mu_0 I\, d\ell\, z}{4\pi (z^2+y^2)^{1/2} (z^2+y^2)}$$

$$= \frac{\mu_0 I\, d\ell\, z}{4\pi (z^2+y^2)^{3/2}} \quad \text{where } d\ell \text{ is just } dy$$

$$= \frac{\mu_0 I\, dy\, z}{4\pi (z^2+y^2)^{3/2}}$$

$$\Rightarrow B = \int_{line} dB = \frac{\mu_0 I z}{4\pi} \int \frac{1}{(z^2+y^2)^{3/2}}\, dy$$

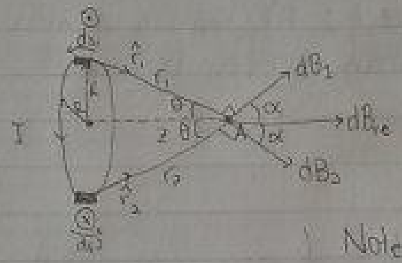
$$= \frac{\mu_0 I z}{4\pi}\left[\frac{y}{z^2\sqrt{z^2+y^2}}\right]$$

$$= \frac{\mu_0 I}{4\pi z}\, \frac{y}{\sqrt{z^2+y^2}}$$

$$\lim_{y \to \infty} = \frac{\mu_0 I}{2\pi z} \qquad \frac{L/2}{\sqrt{z^2+\frac{L^2}{4}}}$$

Process Summary - Set up differential element of length $d\ell$ which creates field dB, some distance r away
- Express cross product in expanded form
- Realize that there is no movement in θ, so replace $\sin\theta$ with equivalent z/y expression
- Replace $d\ell$ with dy (both are just elements of length)
- Integrate, and solve for limit as y becomes infinity, growing the length of the conductor

ex.



Ring of current. What is magnetic field at A?

Note that $\theta + \alpha = 90° = \pi/2$, and $ds_1 \perp r_1$, $ds_2 \perp r_2$

$$dB_1 = \frac{\mu_0 I\,[\vec{ds_1} \times \hat{r}_1]}{4\pi r_1^2} = \frac{\mu_0 I |\vec{ds_1}|\,|\hat{r}_1|^{1} \sin 90°^{1}}{4\pi r_1^2}$$

$$= \frac{\mu_0 I\, ds_1}{4\pi r_1^2}$$

$$dB_2 = \frac{\mu_0 I\, ds_2}{4\pi r_2^2}$$

$$\Rightarrow \quad dB_{re} = \frac{\mu_0 I\, ds}{4\pi r^2} \cos\alpha = \frac{\mu_0 I\, ds}{4\pi r^2} \sin\theta$$

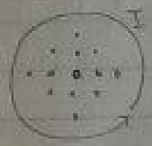
$$= \frac{\mu_0 I\, ds\, R}{4\pi (z^2+R^2)(z^2+R^2)^{1/2}}$$

$$= \frac{\mu_0 I R\, ds}{4\pi (z^2+R^2)^{3/2}}$$

$$\Rightarrow B_z = \frac{\mu_0 I R}{4\pi (z^2+R^2)^{3/2}} \int ds = 2\pi R \frac{\mu_0 I R}{2 \cdot 4\pi (z^2+R^2)^{3/2}} = \frac{\mu_0 I R^2}{2(z^2+R^2)^{3/2}} \hat{z}$$

Process Summary: - Set up differential element with length ds which creates field dB some distance r away
- Use expanded Biot-Savart Law to solve for magnitude of dB
- Realize the symmetry of the problem means resultant dB is only in the z direction
- convert $\cos\alpha$ to $\sin\theta$ since magnetic field is perpendicular to r, which means $180 - 90° + \theta + \alpha$
- Express $\sin\theta$ as (z, R), leaving us with just $\int ds$, which is circumference of the circle

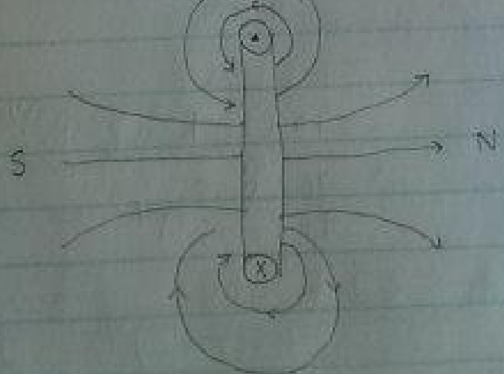
<u>Magnetic Fields and Rings</u>



field in a CCW ring

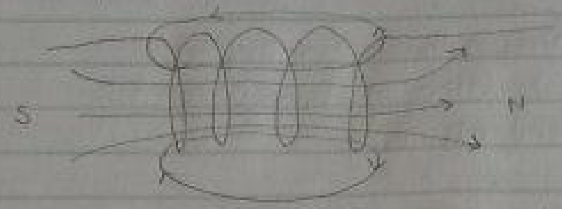
field inside a CW ring

Notice that from the previous example, the field strength has no relation to where in the loop it is: it's constant through the middle of the loop.



The field lines will always exit one side and enter the other

This symmetry can be extended to solenoids:



There will be a constant field inside the volume of the solenoid. Ideal solenoids will not have any magnetic field lines that loop completely around

$$B_{\substack{coil\\center}} = \frac{\mu_0 N I}{2R} \text{, where } N \text{ is the number of turns}$$

$$= \frac{\mu_0 I}{2R}\left(\frac{nd}{l}\right) \text{, where } \frac{n}{l} \text{ is turns per unit length}$$

and d is length of the solenoid.

Current loops and solenoids are known as magnetic dipoles because there are fixed entry points and exit points for magnetic field lines.

Magnetic Dipole Moment

From the equations we've already derived for magnetic dipoles, given that $z \gg R$

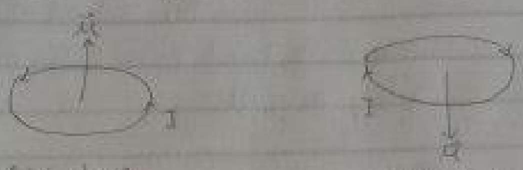
$$B_{loop} \approx \frac{\mu_0 I R^2}{2z^3} = \frac{\mu_0}{4\pi}\,\frac{2(\pi R^2)I}{z^3} = \frac{\mu_0 2AI}{4\pi z^3}$$

that we can take into account the entire area of the loop. However, we can do this more elegantly by expanding to current loops of any shape, as the distant field (NOT THE ONE CLOSE BY) only depends on area.

So let's define $\vec{\mu}$ as the magnetic dipole moment.

$\vec{\mu} = AI$ (from south pole $\rightarrow$ north pole)

which is is in the same direction as $\vec{B}$, and can be found using the right hand rule along.



CCW loop CW loop

$$\Rightarrow \vec{B}_{dipole} = \frac{\mu_0 2\vec{\mu}}{4\pi z^3}$$

<u>Ampere's Law and Solenoids</u>

Much like Gauss's Law of Electric Fields, we have Ampere's Law, which allows us to abuse the symmetry of problems to calculate magnetic fields.

Let's introduce the concept of line integrals: essentially we subdivide a line into an infintesimal number of pieces, and integrate over those pieces.

$$\int_i^f \vec{B} \cdot d\vec{s} = \text{line integral from } i \text{ to } f$$

when $\vec{B}$ is perpendicular to line

$$\int \vec{B} \cdot ds = 0$$

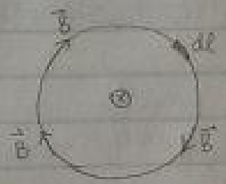
$\vec{B}$ is tangent to line

$$\int \vec{B} \cdot ds = (\vec{B})(\ell)$$

So what is Ampere's Law? Ampere's Law states that the line integral is equal to μ_0 times the current running enclosed by that line.

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

This can be used in cases where there is symmetry in the strength of $\vec{B}$. It can be derived as such:

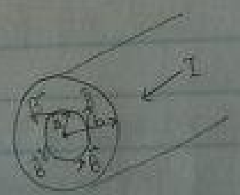


$$\begin{aligned} \oint \vec{B} \cdot d\vec{l} &= \oint |\vec{B}||d\vec{l}| \cos\theta \\ &= |\vec{B}| \oint dL \\ &= \left[\frac{\mu_0 I}{2\pi r}\right]\left[2\pi r\right] \\ &= \mu_0 I_{enc} \end{aligned}$$

Obviously, picking a loop that reflects the symmetry of the problem is key in using Ampere's Law. Had we used, say, a square, $\vec{B}$ would vary at almost every point, rendering the loop useless for solving our problem.

Note that a) the loop is completely arbitrary and it doesn't need to be a physical surface or boundary, and b) direction of current plays a part in I_{enc}. Let's do some examples.

ex.



This is a current-carrying wire of radius b. What is the field strength at radius a?

Draw an Amperian loop of radius a.

$I_{enc} = J A_{circle} = \pi a^2 J$ (J is current density)

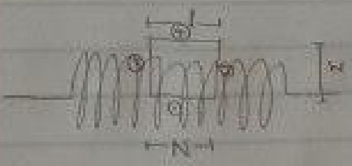
since $J = \frac{I}{A} \Rightarrow I_{enc} = \frac{a^2}{b^2} I$

$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \frac{\mu_0 a^2 I}{b^2} \qquad \Rightarrow B = \frac{\mu_0 I a}{2\pi b^2}$$

$$B\, 2\pi a = \frac{\mu_0 a^2 I}{b^2}$$

Magnetic Field of Solenoids

Ideally, field in the very center of the solenoid is almost maximum while field outside is zero. Here, we'll derive the field strength inside the solenoid using Ampere's Law.



Ideal solenoid: n turns per unit length. We'll draw an Amperian loop (a square) of length l and width w around N turns of the solenoid.

$$\oint \vec{B}\cdot dl = \mu_0 I_{enc}$$
$$= \oint_1 \vec{B}\cdot d\vec{l} + \oint_2 \vec{B}\cdot d\vec{l} + \oint_3 \vec{B}\cdot d\vec{l} + \overbrace{\oint_4 \vec{B}\cdot d\vec{l}}^{0,\ \text{outside}}$$
$$= \oint_1 Bdl\cos 0^\circ + \oint_2 Bdl\cos 90^\circ + \oint_3 Bdl\cos 90^\circ$$

$$B\oint dl = \mu_0 NI$$
$$Bl = \mu_0 NI$$
$$B = \frac{\mu_0 NI}{l} = \mu_0 nI$$

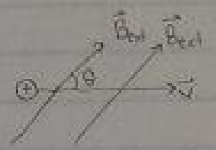
MAGNETIC FORCES

The force on a moving charge from a magnetic field is given by

$$\vec{F}_{on\ q} = q(\vec{v} \times \vec{B})$$
$$= q(vB\sin\theta)$$

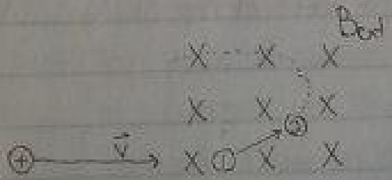
The magnetic force is perpendicular to both the field and the velocity of the charge Note that the force is dependent on the polarity of the charge. Let's do an example.

ex.



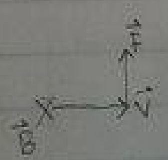
Which direction will this charge get pushed?
⊙ according to right hand rule.

ex.



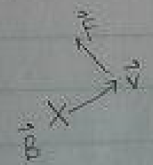
What will happen as the proton moves into this magnetic field?

①



resultant $\vec{v}$: ↗

②



resutant $\vec{v}$: ↑

Eventually, the proton will be forced in the other direction and exit the field

This lets us move into the next topic: cyclotron motion. We can now conclude that a particle that moves perpendicular to the field its in will undergo uniform circular motion.

As such: $\vec{F} = qvB = ma_r = \frac{mv^2}{r}$

And also: $r = \frac{mv}{qB}$, the radius of the circular motion.

We can also determine the frequency of this motion by rearranging $f = \frac{v}{2\pi r} \rightarrow f_{cyc} = \frac{qB}{2\pi m}$

Note that this depends on charge-to-mass ratio but not velocity

We can use conservation of energy to determine any of the variables.

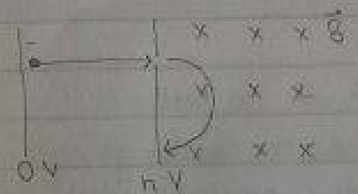
voltage difference

$$K_f + qV_f = K_i + qV_i$$

$$\frac{1}{2}mv_f^2 + qV_f = \frac{1}{2}mv_i^2 + qV_i$$

Let's do an example.

ex.



A particle of charge q is accelerated from rest to a speed of v_f at $V_f = nV$. Find v_f.

$$\frac{1}{2}mv_f^2 + qV_f = 0 + 0$$

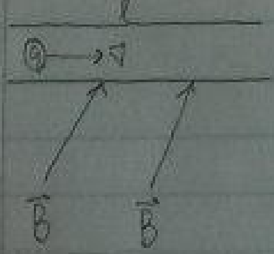
$$mv_f^2 = -2qV_f$$

$$v_f = \sqrt{\frac{2qV_f}{m}}$$

Magnetic Force on Current-Carrying Wires

$$\vec{F} = IlB\sin\theta = I[\vec{l} \times \vec{B}]$$

the force is perpendicular to both the length of the wire and the magnetic field. It can be derived as such:



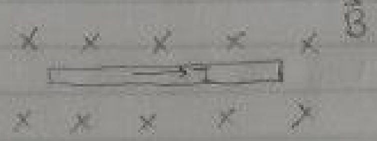
$$\vec{F} = q(\vec{v} \times \vec{B})$$

$$= q\left(\frac{\vec{l}}{t} \times \vec{B}\right)$$

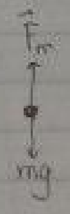
$$= I(\vec{l} \times \vec{B})$$

Let's do an example.

ex



Conductor of mass m in an electric field with I running through length l. What I is needed so that the conductor will levitate?

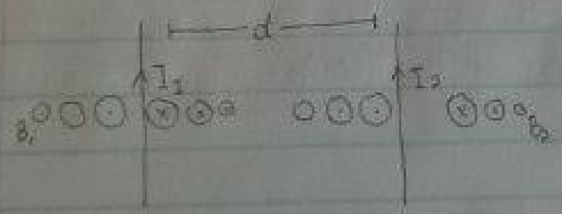


$$\vec{F}_m = \vec{F}_g$$
$$I(\vec{l} \times \vec{B}) = mg$$
$$I = \frac{mg}{(\vec{l} \times \vec{B})} = \frac{mg}{lB\sin 90^\circ} = \frac{mg}{lB}$$

Force Between Parallel Wires

$\vec{F}_{21}$ is read as the force applied on 1 by 2. The forces felt by parallel wires are decided by the direction of flow of current. If they are flowing in the same direction, they attract. Flow in opposite directions mean they repel.

ex

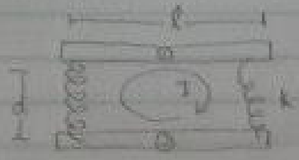


What is F_{21}?

$$\vec{F}_{21} = I_1 [\vec{l}_1 \times \vec{B}_2]$$
$$= I_1 l_1 B_2 \sin 90^\circ$$
$$= I_1 l_1 \left[\frac{\mu_0 I_2}{2\pi d}\right]$$
$$= \frac{\mu_0 I_1 I_2 l_1}{2\pi d}$$

If we assume that these are of an unknown length, we can also have

$$\frac{\vec{F}_{21}}{l} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

Obviously this magnitude is the same for opposite flow, just direction of the force is reversed ex. 

Parallel wires of length ℓ.
Springs of spring constant k, unstretched length x. How much current to keep this stretched by distance d?

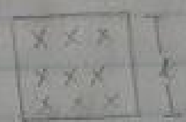
$$F_{21} = \frac{I_1 I_2 \mu_0 \ell}{2\pi d} \Rightarrow \text{Total force} = \frac{I^2 \mu_0 \ell}{\pi d}$$

$$\vec{F}_{mag} = \vec{F}_{spring}$$

$$\frac{I^2 \mu_0 \ell}{\pi d} = 2k(d-x)$$

$$I = \sqrt{\frac{2\pi d k(d-x)}{\mu_0 \ell}}$$

Forces and Torques on Current Loops



Remember that $\tau = 2Fd$

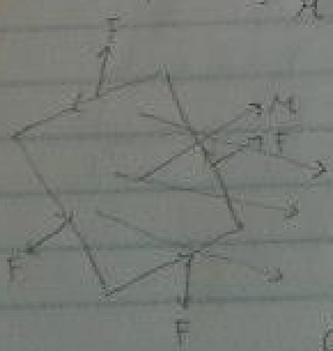
$$= 2(I\ell B)(\tfrac{1}{2}\ell\sin\theta)$$

$$= \underbrace{I\ell^2}\, B\sin\theta$$

$$= \mu B\sin\theta \text{ , } \mu \text{ being the loop's magnetic dipole mom}$$

(μ is perpendicular to the area of the loop)

$$= \vec{\mu} \times \vec{B}$$



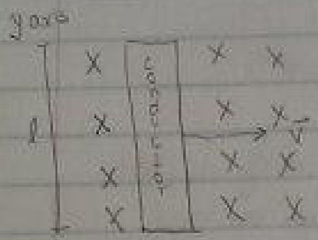
Note that $\tau = 0$ when $\vec{\mu}$ is aligned with the $\vec{B}$.

Also, usually it's easier to determine direction of rotation through calculating forces on each side of the loop as opposed to using the $\vec{\mu} \times \vec{B}$ definition

Electromagnetic Induction

The current in a circuit due to a changing magnetic field passing through the coil is known as an induced current

Motional EMF - we'll look at fixed magnetic fields, first, where the circuit moves within it.



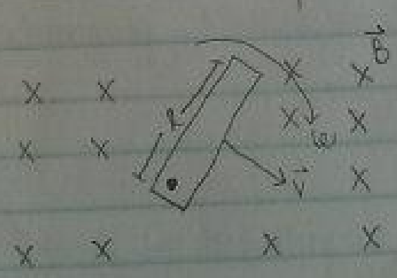
Moving this conductor means that the qs within it are subject to a force upwards dictated by $\vec{v} \times \vec{B}$, in other words, a current upwards.

This current flows until $\vec{F_B} = qE \Rightarrow qvB = qE \Rightarrow \vec{E} = vB$, or a field with magnitude vB is generated inside the conductor. This electric field in turn generates a potential difference of $\Delta V = -\int_0^{\ell} E_y \, dy = v\ell B$.

$$\therefore \quad \mathcal{E} = v\ell B$$

Let's do an example, but throw in a trick.

ex.

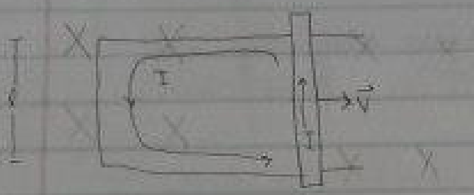


A metal bar of length ℓ is rotating about a point with angular velocity ω. What is the potential difference between ends of the bar?

Remember that $\vec{v}$ at a particular point $= \omega r$, and $\vec{E} = \omega r B$

$$\Delta V = V_{tip} - V_{pivot} = -\int_0^{\ell} E_r \, dr$$
$$= -\int_0^{\ell} (-\omega r B) \, dr$$
$$= \omega B \int_0^{\ell} r \, dr$$
$$= \frac{1}{2} \omega \ell^2 B$$

Induced Current in a Circuit



Now, what happens if our bar is on a track? The same current gets generated due to the EMF generated. HOWEVER, now we must worry about the resistance of the circuit.

$$I = \frac{\mathcal{E}}{R} = \frac{vlB}{R}$$

Now, let's imagine the $\vec{v}$ is caused by a force being applied to the bar. It only makes sense that our F_{pull} is countered by an equal and opposite F_{mag}.

$$\therefore \; F_{pull} = F_{mag} = IlB = \left(\frac{vlB}{R}\right)lB = \frac{vl^2B^2}{R}$$

This isn't the only way to arrive at this result. Next, we'll be discussing:

Magnetic Flux and Lenz's Law

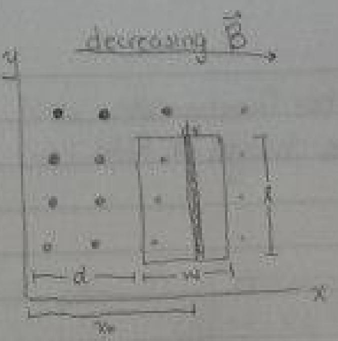
Magnetic flux can be expressed as

$$\Phi_m = \int_{loop} \vec{B} \cdot d\vec{A}$$

$$= \int_{loop} B \, dA \cos\theta$$

But this is a rather general equation, and its use isn't very obvious so let's do an example to illustrate how it's used.

ex.



There is a loop of length ℓ and width w, d away from the y axis in a field of decreasing $\vec{B}$. What is Φ_m?

Since $\vec{B}$ varies only with x, we will create a differential strip of length ℓ and width dx, x_0 away.

$$
\begin{aligned}
d\Phi_m &= \vec{B}\cdot d\vec{A} \\
&= \vec{B}\cdot(\ell\, dx) \\
&= B\ell \cos 0 \; dx \\
\Phi_m &= \int B\ell\, dx \\
&= \frac{\mu_0 I \ell}{2\pi}\int_d^{d+w} \frac{1}{x_0}\, dx \qquad \text{(since the loop goes from } d \to d+w\text{)} \\
&= \frac{\mu_0 I \ell}{2\pi}\left[\ln(d+w) - \ln(d)\right] \\
&= \frac{\mu_0 I \ell}{2\pi}\ln\left[\frac{d+w}{d}\right]
\end{aligned}
$$

Now, what is Lenz's Law? Lenz's Law is an extension of Faraday's Law that dictates how current flows depending on how flux is changing. Mathematically, it's given as

$$\mathcal{E} = -\frac{d\Phi_m}{dt}$$

though in actuality it's somewhat more complicated than that. Flux can change in three ways:

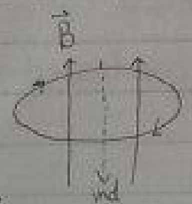
- magnetic field strength ↑ or ↓
- loop changes area or angle
- loop moves in a magnetic field

And in each way, Lenz's Law states that the current must flow in such a way to OPPOSE the change in flux, by generating its own magnetic field.

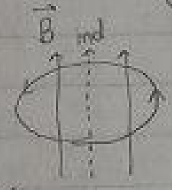
Decreasing flux → induces a current that generates a magnetic field that INCREASES flux

Increasing flux → induces a current that generates a magnetic field that DECREASES flux

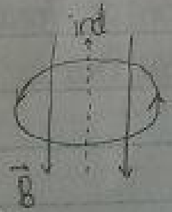
Here are four basic cases where $\vec{B}$ is increasing/decreasing.



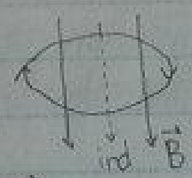
$\vec{B}$ up + increasing
ind ↓ to oppose $\Delta\Phi_m$
∴ current CW



$\vec{B}$ up + decreasing
ind ↑ to oppose $\Delta\Phi_m$
∴ current CCW



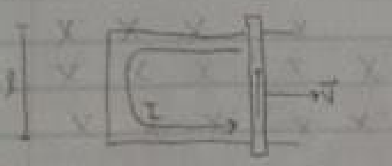
$\vec{B}$ down + increasing
ind ↑ to oppose $\Delta\Phi_m$
∴ current CCW



$\vec{B}$ down + decreasing
ind ↓ to oppose $\Delta\Phi_m$
∴ current CW

Now, remember that bar on a circuit question we did?
Let's do it again, using Faraday and Lenz's Laws.

ex.



Find I_{ind} by moving bar right.

By doing this we are increasing the amount of Φ, letting in more downward $\vec{B}$. As such, we need to generate an UPWARD $\vec{B}_{ind}$ to counteract our movement of the bar.

As such, I_{ind} must flow ↺ CCW. Let's find the magnitude, now.

$$I_{ind} = \frac{\mathcal{E}_{ind}}{R} = \left(\frac{d\Phi}{dt}\right)\left(\frac{1}{R}\right)$$

$$= \frac{d\,[BA\cos\theta]}{dt}\left(\frac{1}{R}\right)$$

$$= \frac{d\,(BA)}{dt}\left(\frac{1}{R}\right)$$

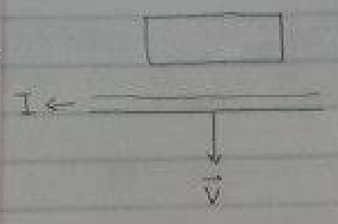
$$= \frac{d\,(Blx)}{dt}\left(\frac{1}{R}\right) \quad \text{since we are changing } x.$$

$$= Bl\left(\frac{dx}{dt}\right)\left(\frac{1}{R}\right)$$

$$= \frac{Blv}{R}$$

Which, uncoincidentally, is the same thing we derived from before!
Neat, eh?

ex.



A current carrying wire is pulled away from the loop as shown. Is there an induced current in the loop? If so, in what direction?

Remember that by the right hand rule, we will have a continuously decreasing $\vec{B}$ pointing into the loop. Moving the wire away means we are DECREASING the amount of downward $\vec{B}$, so we need a current that will generate an DOWNWARD $\vec{B}$ to compensate

As such, I_{ind} will be CW

Faraday's Law

We won't spend too much time on this, as it's relatively straight forward. Faraday's Law simply expresses the potential difference as the change in flux with respect to time.

$$\mathcal{E} = \left|\frac{d\Phi}{dt}\right| = N\left|\frac{d\Phi}{dt}\right|$$

Induced Fields

So we've spent all this time talking about induced currents that generate induced magnetic fields. But we've glossed over a fairly important point - what CAUSES the current? What force pushes the electrons against the resistance of the material?

Magnetic forces can explain motional EMFs, how do we explain currents in stationary loops...?

Lenz's Law states that something HAS to be there to move charge, so the only thing we can assume is that

CHANGING MAGNETIC FIELDS INDUCE ELECTRIC FIELDS

But this field is different than what we're used to. No longer do we have a field from ⊕→⊖, the electric field is present simply due to increasing or decreasing $\vec{B}$.

AN ELECTRIC FIELD DOES NOT NEED CHARGES TO EXIST.

Calculating the Induced Field

The induced electric field is non-conservative, so net work done on a closed path is not zero. Knowing this, we also can't associate the idea of the induced electric field representing a potential, because it is simply not true. However, we CAN associate it with EMF:

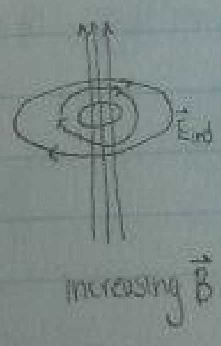
$$EMF = \frac{W}{q} \quad \text{where} \quad W_{loop} = q\oint \vec{E}\cdot d\vec{s}$$

$$\therefore \quad EMF = \oint \vec{E}\cdot d\vec{s}$$

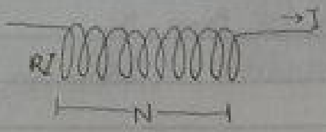
If we restrict ouselves to situations where our loop is perpendicular to the changing magnetic field,

$$EMF = \oint \vec{E}\cdot d\vec{s} = A\left|\frac{dB}{dt}\right|$$

Let's do an example. Note that



ex.



A current of amplitude I_0 oscillating at frequency f travels through a solenoid of n turns per meter, of radius R. What is the maximum induced electric field strength? (inside)

Remember that $\vec{B}_{solenoid} = n\mu_0 I$
and that $I = I_0 \sin\omega t$
where $\omega = 2\pi f$

$$\oint \vec{E} \cdot d\vec{s} = A\left|\frac{dB}{dt}\right|$$

$$E 2\pi r = \pi r^2 \left|\frac{dB}{dt}\right|$$ (where r is some radius $<R$)

$$E = \frac{r}{2} \frac{d(\mu_0 n I_0 \sin\omega t)}{dt}$$

$$E = \left(\frac{r}{2}\right)\left(\mu_0 n I_0\right) \frac{d(\sin\omega t)}{dt}$$

$$= \frac{r \mu_0 n I_0 \omega \cos\omega t}{2}$$

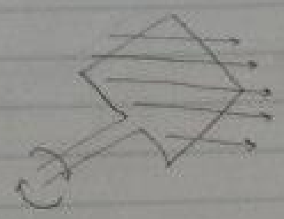
$$\Rightarrow E_{max} = \frac{R \mu_0 n I \omega}{2}$$

So E in the solenoid is maximum at the coils and when $I = I_0 \sin\omega t$ is maximum.

Generators / Transformers / Inductors

All three of these are applications of induced currents. Generators create current. Transformers either increase "step up" or decrease "step down" voltages, and inductors store energy.

ex. Generators



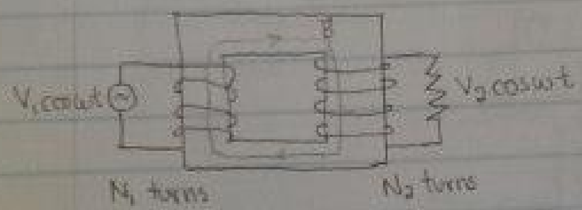
$$\mathcal{E}_{coil} = -N\left(\frac{d\Phi}{dt}\right)$$

$$= -N\,BA\,\frac{d(\cos\omega t)}{dt}$$

$$= N\,BA\,\omega\sin\omega t$$

$$\therefore\ I = \frac{NBA\omega\sin\omega t}{R}$$

ex. Transformers



Changing magnetic field is inversely proportional to turns in 1 (due to inductance. we'll talk about this soon). According to Faraday's Law, $\mathcal{E}_2 \propto N_2$. Combine the two and you get the proportionality for this transformer.

$$V_2 = \frac{N_2}{N_1}V_1$$

ex. Inductors

Inductance is the flux-to-current ratio:

$$L = \frac{\Phi_m}{I}$$

Determining inductance of a solenoid is fairly straight forw

$$B_{solenoid} = \frac{\mu_0 N I}{l}$$

$$\Rightarrow \Phi_m = N\Phi_{turn}$$
$$= N\frac{\mu_0 N I A}{\ell}$$

And accordingly, we can conclude that

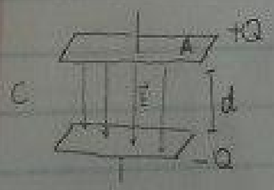
$$L_{solenoid} = \frac{\Phi_m}{I} = \frac{\mu_0 N^2 A}{\ell}$$

We can also calculate EMF induced in a coil.

$$\mathcal{E}_{coil} = N\left|\frac{d\Phi_{turn}}{dt}\right| = \left|\frac{d\Phi_m}{dt}\right|$$
$$= \left|\frac{d(LI)}{dt}\right|$$
$$= L\left|\frac{dI}{dt}\right|$$

Now, before we head into our final units of LC/LR circuits, let's go over capacitors and dielectrics again.

Capacitors and Dielectrics



A capacitor's potential difference is expressed as

$$\Delta V_c = Ed$$

and the electric field is given by

$$E = \frac{Q}{\varepsilon_0 A}$$

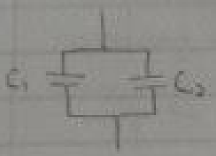
Combining these, we can conclude that

$$Q = \underbrace{\boxed{\frac{\varepsilon_0 A}{d}}}_{\text{capacitance}} \Delta V_c$$

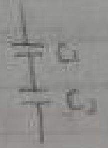
and as such, capacitance is just the charge to potential ratio. Which gives us the king of all capacitance formulae:

$$Q = C\Delta V$$

Series and parallel capacitors act simply like the opposite of series and parallel resistors:



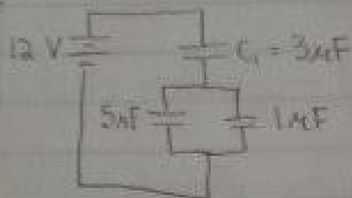
$C_{eq} = C_1 + C_2$



$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}$

Let's do an example.

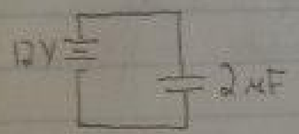
ex.



Find potential difference across all three capacitors.

Find equivalent capacitance.

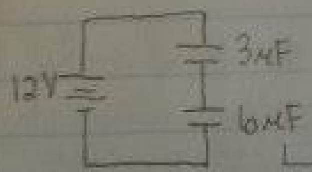
⇓



$$Q = C\Delta V$$
$$= (2\mu F)(12V)$$
$$= 24\mu C$$

⇓

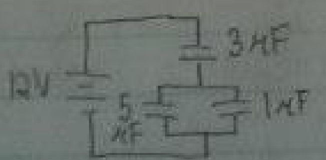
Then split them up one by one.



$3\mu F \longrightarrow \Delta V_1 = Q_1 / C_1 = 8V$

$6\mu F \longrightarrow \therefore \Delta V_{23} = 4V$

⇓



⟶ Parallel, ∴ both have 4 V across them.

The CHARGE on multiple capacitors is the same ⟷ in series

different ⟷ in parallel

Energy Stored in Capacitors

At times it's useful to know exactly how much energy is stored in a capacitor. Total energy transfer is as such:

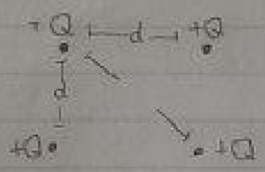
$$U_c = \frac{1}{C}\int_0^Q q\,dq = \frac{Q^2}{2C}$$

Which can be written as the handy-dandy

$$U_c = \frac{1}{2}CV^2$$

Let's do a quick example.

Ex.



Three charges are held in place. One of them is free to move. How fast will it be going at $d=\infty$, if they all have mass m?

$F = \dfrac{Q_1 Q_2}{r^2} = \dfrac{Q^2}{d^2}$ (horiz/vertical) $\quad F = \dfrac{Q^2}{(\sqrt{d^2+d^2})^2}$ (diagonal)

$$\Delta ME = 0$$
$$\Delta K + \Delta U = 0$$
$$\Delta K = -\Delta U$$
$$= -\Delta V$$
$$= -(U_f - U_i)$$

$$\frac{1}{2}mv^2 = -\left[2\left[\frac{Q^2}{4\pi\varepsilon_0 d}\right] + \left[\frac{Q^2}{4\pi\varepsilon_0\sqrt{2}d}\right]\right] - \left[\frac{4Q^2}{4\pi\varepsilon_0 d} + 2\left[\frac{Q^2}{4\pi\varepsilon_0\sqrt{2}d}\right]\right]$$

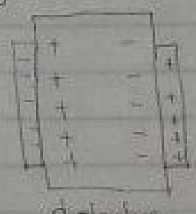
$$= \frac{2Q^2}{4\pi\varepsilon_0 d} + \frac{Q^2}{4\pi\varepsilon_0\sqrt{2}d}$$

$$mv^2 = \frac{Q^2}{\pi\varepsilon_0 d} + \frac{Q^2}{2\pi\varepsilon_0\sqrt{2}d}$$

$$v = \sqrt{\frac{\frac{Q^2}{\pi\varepsilon_0 d}\left[1 + \frac{1}{2\sqrt{2}}\right]}{m}}$$

Dielectrics

Instead of a vacuum in between the plates of the capacitor, we can have a dielectric, which is just some insulating material



As you can see, the dielectric weakens the electric field between the plates.

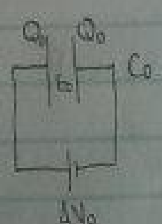
Let's define $k = \frac{E_0}{E}$, the dielectric constant. Alternatively, field strength inside the dielectric is given by $E = \frac{E_0}{k}$

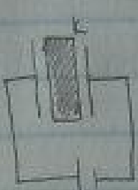
$$\Rightarrow \Delta V_c = (E)(d) = \frac{E_0}{k} d = \frac{(\Delta V_c)_0}{k}$$

So ΔV is weakened by the same ratio. Further,

$$C = \frac{Q}{\Delta V_c} = \frac{Q_0}{(\Delta V_c)_0/k} = k\frac{Q_0}{(\Delta V_c)_0} = kC_0$$

Which means our capacitance increases by a factor equal to the dielectric constant. Let's do an example.

ex. ① 

② 

③ 

Capacitor is charged to Q_0 on each plate by $\mathcal{E} = \Delta V_0$.

Dielectric inserted. Find C, Q, ΔV, E and work done to insert dielectric

Find $C, Q, \Delta V$, and E.

② $C = kC_0$

$\mathcal{E} = \Delta V_0$ because battery is still connected and providing emf

$E = \frac{E_0}{k}$ $\quad$ $Q = (kC_0)\Delta V$

$$W = U_k - U_0$$
$$= \frac{1}{2}(kC_0)\Delta V^2 - \frac{1}{2}C_0\Delta V^2$$
$$= \frac{1}{2}C_0\Delta V^2(k-1)$$

③ $\Delta V = \frac{\Delta V_0}{k} \Rightarrow Q = (kC_0)\left(\frac{\Delta V_0}{k}\right)$

$$= C_0\Delta V_0$$

Gauss's Law and Dielectrics

Likely not exam, we'll review quickly

Since $\mathcal{E} = k\mathcal{E}_0$ we can place this into $\int \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\mathcal{E}_0}$

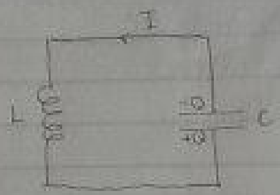
Which becomes $Q_{enc} = \mathcal{E}\int \vec{E} \cdot d\vec{A}$

and where $\vec{D} = \mathcal{E}\vec{E}$ (are these supposed to be $\mathcal{E}$s?)

$\Rightarrow Q_{enc} = \int \vec{D} \cdot d\vec{A}$

Remember that $\vec{E} = \frac{\vec{E}_0}{k}$. Note that $\vec{D}$ of a capacitor stays the same regardless of whether a dielectric is present.

LC Circuits



A circuit that consists of capacitors and inductors.

According to KVL:

$$\Delta V_c + \Delta V_L = 0$$

which can be written as

$$\frac{Q}{C} + L\left(\frac{dI}{dt}\right) = 0$$

Having two unknowns in the equation (Q, I), we can rewrite this two different ways so we only have one unknown.

① $I = \frac{dq}{dt}$

$$\Rightarrow \frac{dq}{dt} = -\frac{dQ}{dt}$$

charge into inductor   charge out of capacitor

$$\therefore I = \frac{-dQ}{dt}$$

And as such,

$$I = \underbrace{\omega Q_0}\sin\omega t$$
$$= I_{max}\sin\omega t$$

② Substitute ① in:

$$\frac{Q}{C} + L\frac{d\left(-\frac{dQ}{dt}\right)}{dt} = 0$$

$$\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0$$

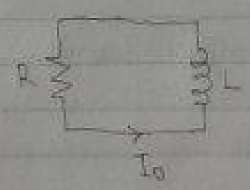
$$\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q$$

Which we can solve by analogy:

$$\leftarrow Q(t) = Q_0\cos\omega t$$
$$= Q_0\cos\left(\sqrt{\frac{1}{LC}}\,t\right)$$

An LC circuit is an electric oscillator that oscillates at a frequency of $\omega/2\pi$, with Q and I 90° out of phase. Current is zero when capacitor is fully charged, charge zero when current is maximum.

LR Circuits



A circuit with an inductor and a resistor (maybe plus a battery). Let's start again with KVL.

$$\Delta V_R + \Delta V_L = 0$$

Which once again can be written as

$$-RI - L\frac{dI}{dt} = 0$$

Now, we want to find current as a function of time.

$$\frac{dI}{I} = -\frac{R}{L}\,dt$$

$$\frac{dI}{I} = -\frac{1}{(L/R)}\,dt$$

$$\int_{I_0}^{I} \frac{dI}{I} = -\frac{1}{(L/R)}\int_0^t dt$$

$$\ln\left(\frac{I}{I_0}\right) = -\frac{t}{(L/R)}$$

$$\therefore\ I = I_0 e^{-t/(L/R)} \quad \text{or}$$

$$= I_0 e^{-Rt/L}$$