Chan and Mitran
Vestigial Sideband and AM Examples

Introduction

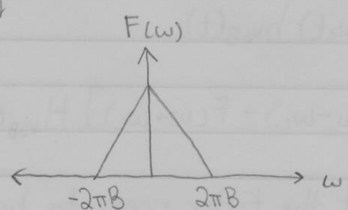
Woo, welcome to the last amplitude modulation topic, a topic with a name that sounds unfortunately like a terminal disease.

"I'm sorry sir, you've got stage 4 vestigial sidebands".

Back on topic, vestigial sideband arises from our inability to create perfect sidebands near DC, leaking spectrum from the other side.

Vestigial Sideband

So what do we do? Fuck it, dude, just let it through. I'm serious - that's exactly what happens. Vestigial sideband is the "real-life" version of single side band at low frequencies.



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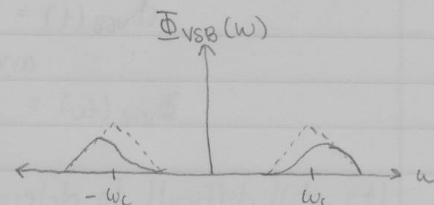
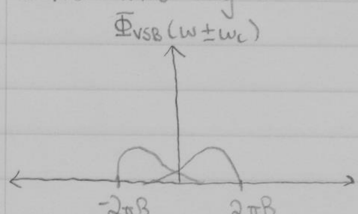


So instead of transmitting the entirety of one sideband, we transmit <u>most</u> of one, and <u>a vestige</u> of the other. If we've done this right:



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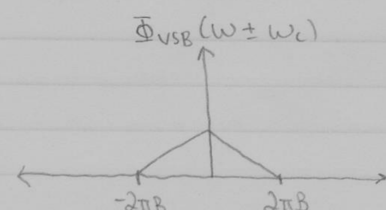


Shifting back to baseband should make the parts add back up EXACTLY to the original's spectrum

The difficulty with this form of modulation is not the shifts - we've done that maybe a trillion times now. It's figuring out what filter to use that will work for ANY signal we want to feed into it.

Generating Vestigial Sideband

The method by which we produce VSB is, unsurprisingly, basically the same as single sideband.

$$\underbrace{f(t) \rightarrow \underset{\substack{\uparrow \\ \cos(\omega_c t)}}{\otimes}}_{\text{DSB-SC}} \longrightarrow \boxed{\text{VSB filter}} \rightarrow \phi_{VSB}(t)$$

We generate double sideband suppressed carrier, and pass it through a filter, $H_{VSB}(\omega)$. So:

$$\phi_{VSB}(t) = f(t)\cos(\omega_c t)\, h_{VSB}(t)$$

and

$$\Phi_{VSB}(\omega) = \tfrac{1}{2}\left[F(\omega-\omega_c) + F(\omega+\omega_c)\right] H_{VSB}(\omega)$$

It's still difficult to determine what the filter needs to be from this equation, though. Perhaps if we look at the demodulation side it'll be easier.

Demodulating Vestigial Sideband

Unsurprisingly, we demodulate the same way as always.

$$\phi_{VSB}(t) \longrightarrow \underset{\substack{\uparrow \\ \cos(\omega_c t)}}{\otimes} \longrightarrow \text{LPF} \longrightarrow e_o(t)$$

Before the lowpass filter we have:

$$x(t) = \phi_{VSB}(t)\cos(\omega_c t)$$
$$\updownarrow$$
$$X(\omega) = \tfrac{1}{2}\Phi_{VSB}(\omega-\omega_c) + \tfrac{1}{2}\Phi_{VSB}(\omega+\omega_c)$$

We know what Φ_{VSB} is from the modulating section, so let's substitute that into $X(\omega)$.

$$\begin{aligned}
X(\omega) &= \tfrac{1}{2}\underbrace{\left[\tfrac{1}{2}F(\omega-2\omega_c) + \tfrac{1}{2}F(\omega)\right]\left[H(\omega-\omega_c)\right]}_{\Phi_{VSB}(\omega-\omega_c)} \\
&\quad + \tfrac{1}{2}\underbrace{\left[\tfrac{1}{2}F(\omega) + \tfrac{1}{2}F(\omega+2\omega_c)\right]\left[H(\omega+\omega_c)\right]}_{\Phi_{VSB}(\omega+\omega_c)} \\
&= \tfrac{1}{4}F(\omega-2\omega_c)H(\omega-\omega_c) + \tfrac{1}{4}F(\omega)H(\omega-\omega_c) \\
&\quad + \tfrac{1}{4}F(\omega)H(\omega+\omega_c) + \tfrac{1}{4}F(\omega+2\omega_c)H(\omega+\omega_c) \\
&= \tfrac{1}{4}F(\omega)\left[H(\omega-\omega_c) + H(\omega+\omega_c)\right] \\
&\quad + \tfrac{1}{4}F(\omega-2\omega_c)H(\omega-\omega_c) \\
&\quad + \tfrac{1}{4}F(\omega+2\omega_c)H(\omega+\omega_c)
\end{aligned}$$

Now, we can pass this through the filter. Both $F(\omega-2\omega_c)$ and $F(\omega+2\omega_c)$ are too far away, so they get cut.

$$E_o(\omega) = \tfrac{1}{4}F(\omega)\left[H(\omega-\omega_c) + H(\omega+\omega_c)\right]$$

We always want this last result to be some form of our original baseband spectrum: ie, (some constant) × $F(\omega)$.

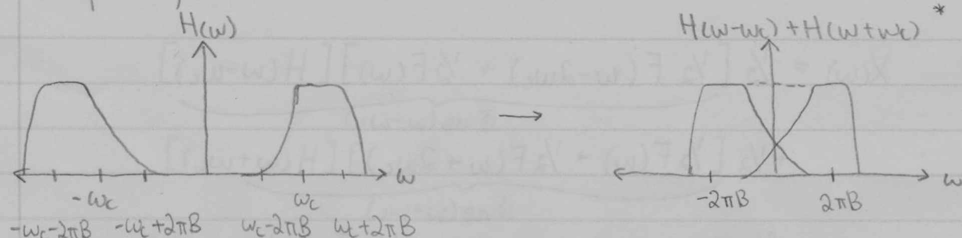
That means, in order for this to work, $H(\omega-\omega_c) + H(\omega+\omega_c)$ must be some constant as well, which we'll call k.

So for some $k = H(\omega-\omega_c) + H(\omega+\omega_c)$ for $|\omega| \le 2\pi B$,

$$E_0(\omega) = k/4\, F(\omega)$$
$$\updownarrow$$
$$e_0(t) = k/4\, f(t)$$

What that means is that we want to pick a filter that, when shifted back to baseband by $\cos(\omega_c t)$, has a constant response where our original baseband spectrum existed (which we defined to be $[-2\pi B, 2\pi B]$).

Graphically, it would look like this:



Notice that in the second plot, the sum of the two responses creates a constant value from $[-2\pi B, 2\pi B]$, ignoring my bad drawing skills.

Outside of our desired $[2\pi B, 2\pi B]$, it doesn't matter what the response is because our baseband spectrum doesn't exist outside of our defined range.

That's really it. This is a short chapter so we're going to include some examples here, too.

*this plot is missing the responses that will appear at $\pm 2\omega_c$, but since that's out of range we don't care

Example Time!

ex. A SSB signal is generated using a carrier wave with amplitude $A_c = 100$ and frequency ω_c. $f(t)$ is given as $\cos(2000\pi t) + 2\sin(2000\pi t)$.

a) Find $\hat{f}(t)$.

b) Find $\phi_-(t)$.

a) The Hilbert transform can be applied separately to terms split by $\pm$. Remember that we multiply positive frequencies by $-j$ and negative frequencies by $-j$.

$$\cos(2000\pi t) = \tfrac{1}{2}e^{j2000\pi t} + \tfrac{1}{2}e^{-j2000\pi t}$$

$$\downarrow \text{ Hilbert transform}$$

$$= -\tfrac{j}{2}e^{j2000\pi t} + \tfrac{j}{2}e^{-j2000\pi t}$$

$$= \sin(2000\pi t)$$

$$2\sin(2000\pi t) = je^{-j2000\pi t} - je^{j2000\pi t}$$

$$\downarrow \text{ Hilbert transform}$$

$$= j\left[je^{-j2000\pi t}\right] - (-j)\left[je^{j2000\pi t}\right]$$

$$= -e^{-j2000\pi t} - e^{j2000\pi t}$$

$$= -\left[e^{-j2000\pi t} + e^{j2000\pi t}\right]$$

$$= -2\cos(2000\pi t)$$

$$\therefore \hat{f}(t) = \sin(2000\pi t) - 2\cos(2000\pi t)$$

b) Remember that $\phi_\pm(t) = f(t)\cos(\omega_c t) \mp \overset{\text{important!}}{\hat{f}}(t)\sin(\omega_c t)$. So

$$\phi_-(t) = f(t)\,A_c\cos(\omega_c t) + \hat{f}(t)A_c\sin(\omega_c t)$$

$$= \left[\cos(2000\pi t) + 2\sin(2000\pi t)\right]A_c\cos(\omega_c t)$$

$$+ \left[\sin(2000\pi t) - 2\cos(2000\pi t)\right]A_c\sin(\omega_c t)$$

Expansion time!

$$\phi_-(t) = 100\cos(2000\pi t)\cos(w_c t) + 200\sin(2000\pi t)\cos(w_c t)$$
$$+ 100\sin(2000\pi t)\sin(w_c t) - 200\cos(2000\pi t)\sin(w_c t)$$

Better know your trig identities, kids.

$$\cos(2000\pi t)\cos(w_c t) \rightarrow \tfrac{1}{2}\cos((2000\pi + w_c)t) + \tfrac{1}{2}\cos((2000\pi - w_c)t)$$
$$\sin(2000\pi t)\cos(w_c t) \rightarrow \tfrac{1}{2}\sin((2000\pi + w_c)t) + \tfrac{1}{2}\sin((2000\pi - w_c)t)$$
$$\sin(2000\pi t)\sin(w_c t) \rightarrow \tfrac{1}{2}\cos((2000\pi - w_c)t) - \tfrac{1}{2}\cos((2000\pi + w_c)t)$$
$$\cos(2000\pi t)\sin(w_c t) \rightarrow \tfrac{1}{2}\sin((2000\pi + w_c)t) - \tfrac{1}{2}\sin((2000\pi - w_c)t)$$

$$\therefore\ \phi_-(t) = \cancel{50\cos[(2000\pi + w_c)t]} + 50\cos[(2000\pi - w_c)t]$$
$$+ \cancel{100\sin[(2000\pi + w_c)t]} + 100\sin[(2000\pi - w_c)t]$$
$$+ 50\cos[(2000\pi - w_c)t] - \cancel{50\cos[(2000\pi + w_c)t]}$$
$$- \cancel{100\sin[(2000\pi + w_c)t]} + 100\sin[(2000\pi - w_c)t]$$
$$= 100\cos[(2000\pi - w_c)t] + 200\sin[(2000\pi - w_c)t]$$

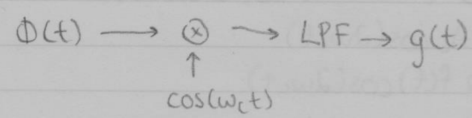
Alternatively, instead of using the definition, we can arrive at the same answer through how the signal is actually generated: DSB-SC and then filtering.

$$\phi_{DSB\text{-}SC}(t) = A_c f(t)\cos(w_c t)$$
$$= A_c\cos(w_c t)[\cos(2000\pi t) + 2\sin(2000\pi t)]$$
$$= A_c\cos(2000\pi t)\cos(w_c t) + 2A_c\sin(2000\pi t)\cos(w_c t)$$
$$= \underbrace{50\cos(2000\pi t - w_c t) + 100\sin(2000\pi t - w_c t)}_{\frac{1}{2}\phi_-(t)} + \underbrace{50\cos(2000\pi t + w_c t) + 100\sin(2000\pi t + w_c t)}_{\frac{1}{2}\phi_+(t)}$$

$$\therefore\ \phi_-(t) = 100\cos(2000\pi t - w_c t) + 200\sin(2000\pi t - w_c t)$$

Personally I like the second method better because there's less math to do. That's good on its own, and it's good because there's less opportunity to make mistakes.

ex. Let $\Phi(t)$ be an AM signal on carrier $c(t) = \cos(\omega_c t)$. $f(t)$ has bandwidth B Hz. Suppose $c(t)$ is available both at the modulator and demodulator.



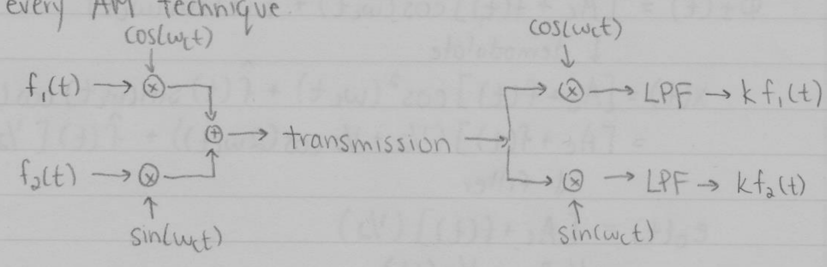
For this demodulator, which of the following:

DSB-LC	DSB-SC
SSB-LC	SSB-SC
VSB-LC	VSB-SC

will result in $g(t) = Af(t) + B$, where $A \neq 0$?

This is a bit of a trick question. The right answer is all of them. You might be inclined to say the LC uses envelope detection, which is the easiest form of demodulation, but NOT THE ONLY WAY.

Remember back in QCM we said that a QCM modulator / demodulator could do every AM technique.



As long as you demodulate with the same carrier wave and then filter, we'll get back our original signal.

Let's prove it

1) DSB - LC

$$\phi_{DSB-LC}(t) = [A_c + f(t)]\cos(\omega_c t)$$

$\downarrow$ demodulate

$$\begin{aligned} x(t) &= [A_c + f(t)]\cos^2(\omega_c t) \\ &= [A_c + f(t)]\left(1/2 + 1/2\cos(2\omega_c t)\right) \\ &= 1/2 A_c + 1/2 A_c \cos(2\omega_c t) \\ &\quad + 1/2 f(t) + 1/2 f(t)\cos(2\omega_c t) \end{aligned}$$

$\downarrow$ filter

$$e_o(t) = 1/2 A_c + 1/2 f(t)$$

2) DSB - SC

$$\phi_{DSB-SC}(t) = f(t)\cos(\omega_c t)$$

$\downarrow$ demodulate

$$\begin{aligned} x(t) &= f(t)\cos^2(\omega_c t) \\ &= f(t)\,[1/2 + 1/2\cos(2\omega_c t)] \\ &= 1/2 f(t) + 1/2 f(t)\cos(2\omega_c t) \end{aligned}$$

$\downarrow$ filter

$$e_o(t) = 1/2 f(t)$$

3) SSB - LC

$$\phi_{\pm}(t) = [A_c + f(t)]\cos(\omega_c t) + \hat{f}(t)\sin(\omega_c t)$$

$\downarrow$ demodulate

$$\begin{aligned} x(t) &= [A_c + f(t)]\cos^2(\omega_c t) + \hat{f}(t)\sin(\omega_c t)\cos(\omega_c t) \\ &= [A_c + f(t)](1/2 + 1/2\cos(2\omega_c t)) + \hat{f}(t)\,[1/2\sin(2\omega_c t)] \end{aligned}$$

$\downarrow$ filter

$$\begin{aligned} e_o(t) &= [A_c + f(t)](1/2) \\ &= 1/2 A_c + 1/2 f(t) \end{aligned}$$

4) SSB - SC

$$\Phi_{\pm}(t) = f(t)\cos(\omega_c t) + \hat{f}(t)\sin(\omega_c t)$$

$\downarrow$ demodulate

$$= f(t)\cos^2(\omega_c t) + \hat{f}(t)\sin(\omega_c t)\cos(\omega_c t)$$

$$= f(t)\left[\tfrac{1}{2} + \tfrac{1}{2}\cos(2\omega_c t)\right] + \hat{f}(t)\left[\tfrac{1}{2}\sin(2\omega_c t)\right]$$

$\downarrow$ filter

$$= \tfrac{1}{2} f(t)$$

5) VSB - LC

$$\Phi_{VSB\text{-}LC}(t) = [A_c + f(t)]\cos(\omega_c t)\, h(t)$$

$\uparrow$ = $|H(\omega)|$ and applying a* phase change $\theta(\omega)$ to the signal

$$= A_c\cos(\omega_c t)h(t) + f(t)\cos(\omega_c t)h(t)$$

* frequency-dependent

$$= A_c|H(\omega)|\cos(\omega_c t + \theta(\omega)) + f(t)\cos(\omega_c t)h(t)$$

$\downarrow$ demodulate

$$x(t) = A_c|H(\omega)|\cos(\omega_c t + \theta(\omega))\cos(\omega_c t) + f(t)\cos^2(\omega_c t)h(t)$$

The second term is by definition $k/4\, f(t)$ due to the constraints of $h(t)$ provided a few pages before.

$$= A_c|H(\omega)|\left[\tfrac{1}{2}\cos(\theta(\omega)) + \tfrac{1}{2}\cos(2\omega_c t + \theta(\omega))\right] + k/4\, f(t)$$

$\downarrow$ filter

$$= A_c|H(\omega)|\left[\tfrac{1}{2}\cos(\theta(\omega))\right] + k/4\, f(t)$$

$$= \frac{A_c|H(\omega)|}{2}\cos(\theta(\omega)) + \frac{k}{4} f(t)$$

6) VSB-SC

$$\phi_{VSB-SC}(t) = f(t)\,h(t)\cos(\omega_c t)$$
$$\downarrow \text{demodulate}$$
$$x(t) = f(t)\,h(t)\cdot\cos^2(\omega_c t)$$

I don't want to do this all over again so just go back a few pages. We don't necessarily need the LPF here, but it doesn't change our answer at all.

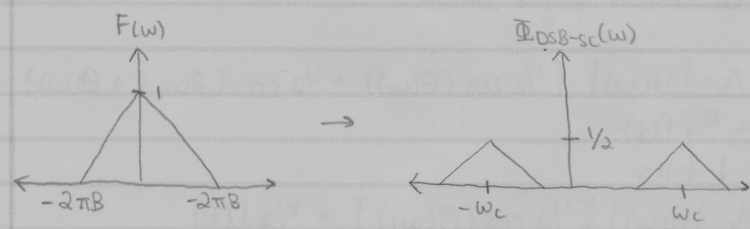
$$= k/4\, f(t)$$
$$\downarrow \text{filter}$$
$$e_0(t) = k/4\, f(t).$$

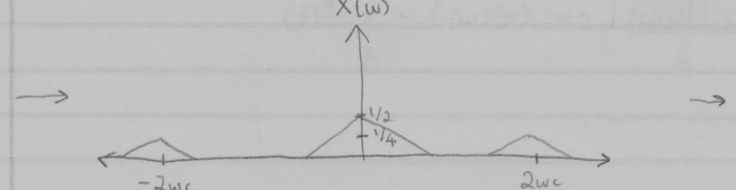
ex. Let f(t) have bandwidth B Hz, DSB-SC modulated using $\cos(\omega_c t)$. If:

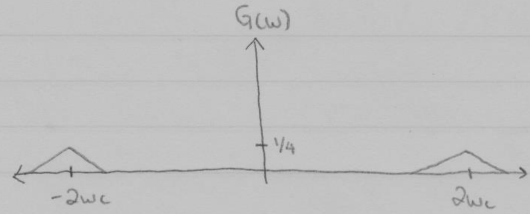
$$\phi_{DSB-SC}(t) \rightarrow \otimes \rightarrow x(t) \rightarrow \text{HPF @ } \omega_c \rightarrow g(t)$$
$$\uparrow \cos(\omega_c t)$$

Can we still get f(t) back from g(t)? If so, how?

Sure we can. Let's graph it.

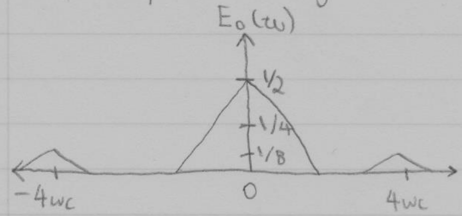






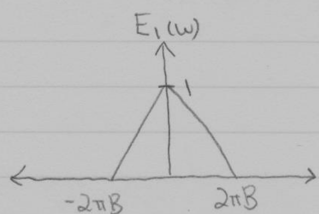
We have our shape, we just need to shift it $\pm 2\omega_c$ back to baseband!

Let's say $e_0(t) = g(t)\cos(2\omega_c t)$.



Now we just need to chop off the excess and multiply by 2.

Say $e_1(t) =$ LPF'd $2 \times e_0(t)$.



Tada! We've gotten back $F(\omega) = E_1(\omega)$.

That's all for amplitude modulation, folks! Next time: angle modulation.