CS6170: Randomized Algorithms
Problem Set #4

Name: Chathurvedhi Talapaneni
Roll No: CS20B021

Marks: 40
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Problem 1 **13 marks**

Consider the following approach to counting the number of solutions to the knapsack problem. Given items with sized $a_1, a_2, \ldots, a_n > 0$ and an integer $b > 0$, find the number of vectors $(x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$ such that $\sum_{i=1}^{n} a_i x_i \leq b$.

(a) (3 marks) Consider the following direct Monte-Carlo algorithm for counting the number of solutions: Choose $(x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$ uniformly at random and check if satisfies $\sum_{i=1}^{n} a_i x_i \leq b$. Let f be the fraction of samples that satisfies the equation. Output $f \cdot 2^n$ as the estimate of the number of solutions.

Show that this will not yield an FPRAS, by arguing with respect to an instance when $a_i = 1$ for every i and $b = \sqrt{n}$.

Solution:

Let us run the Monte Carlo r times and $Y_i = 1$ for satisfying the equation.

$\sum Y_i = Y = f.r$

$Pr(|Y - V| \geq t) \leq exp(-\frac{2t^2}{r})$

On attempting to convert the given problem into

$Pr(|X - \mu| \geq \epsilon\mu) \leq \delta$

$Pr(|X - \mu| \geq 2^n t/r) \leq exp(-\frac{2t^2}{r})$ and $2^n t/r = \epsilon\mu$

$r = ln(1/\delta)(1/2\epsilon^2)(1/f^2)$

$f = (\Sigma_0^{\sqrt{n}} \binom{n}{i})/2^n$

$f \leq \frac{n!}{(n-\sqrt{n})!} \sum_0^{\sqrt{n}} (1/i!)/2^n$ Opening $\binom{n}{k}$

$f \leq n^{\sqrt{n}}/(e2^n)$

Thus $1/f^2$ is exponential disproving FPRAS.

(b) (5 marks) Consider a Markov chain M on the state space $\{0, 1\}^n$. From a state $X_j = (x_1, x_2, \ldots, x_n)$, M chooses an i uniformly at random. If $x_i = 1$, then X_{j+1} is obtained by setting x_i to 0. If $x_i = 0$, and setting $x_i = 1$ gives a feasible solution, then x_i is set to 1 in X_{j+1}. Otherwise $X_{j+1} = X_j$.

Suppose that $\sum_{i=1}^{n} a_i > b$. Show that M is irreducible and aperiodic, and that the stationary distribution is the uniform distribution.

Solution:

Irreducible: There is a possibility of going from a given vector to a zero vector by removing all $x_i = 1$ and from there we can build the necessary feasible vector.

Aperiodic: The possibility of a self-loop makes it non-bipartite.

Let us take 2 solutions x and y which differ by one bit. From the MC rules $P_{x,y} = P_{y,x} = 1/N$ and distribution function is uniform and $\pi = 1/M$ where M is the number of solutions, then $\pi_x P_{x,y} = 1/MN = \pi_x P_{x,y}$ proving uniform stationary distribution.

(c) (5 marks) Argue that an FPAUS for the knapsack problem yields and FPRAS for it by proceeding as follows: Let $a_1 \leq a_2 \leq \cdots \leq a_n$. Let $b_i = \sum_{j=1}^{i} a_i$. Let k be the smallest index such that $b_k \geq b$, and let $\Omega(b_i)$ denote the set of vectors $(x_1, x_2, \ldots, x_n)$ such that $\sum_{j=1}^{n} a_j x_j \leq b_i$. Use the equation,

$$|\Omega(b)| = \frac{|\Omega(b)|}{|\Omega(b_{k-1})|} \frac{|\Omega(b_{k-1})|}{|\Omega(b_{k-2})|} \cdots \frac{|\Omega(b_2)|}{|\Omega(b_1)|} |\Omega(b_1)|.$$

Prove that $|\Omega(b_i)| \leq (n+1)|\Omega(b_{i-1})|$ and use the approach discussed in class. Give all the details with the correct parameters.

Solution:

We know that $\Omega(b_{i-1}) \subset \Omega(b_i)$

So let us take a vector $v \in \Omega(b_i)/\Omega(b_{i-1})$

$\sum_{j=1}^{i-1} a_j = b_{i-1} \leq \sum_{j=1}^{n} a_j v_j \leq b_i = \sum_{j=1}^{i} a_j$

There must be a value $p > i$ such that $v_p = 1$. If we set it to 0 and obtain v_0. $v_0 \in \Omega(b_{i-1})$. There are n possible values for p. Thus $|\Omega(b_i)| \leq (n+1)|\Omega(b_{i-1})|$

Now similar to the procedure in class $r_i = |\Omega(b_i)|/|\Omega(b_{i-1})|$ and $\tilde{x_i}$ be the approximation of r_i

Using a $\epsilon/6n$ sampler on $\Omega(b_i)$, we obtain:

$|E[\tilde{r_i}] - r_i| \leq \epsilon/6n$

$E[\tilde{r_i}] \geq r_i - \epsilon/6n \geq 1/(n+1) - \epsilon/6n \geq (5n-1)/(6n(n+1))$

Using Chernoff Bounds(from MU), we bound the number of iterations:

$M \geq 3ln(2n/\delta)/(\epsilon/12n)^2.6n(n+1)/(5n-1)$

$M \geq cn^3(n+1)/(5n-1).ln(2n/\delta)$

Eq 1: $Pr(|\tilde{r_i} - E[\tilde{r_i}]| \geq \epsilon/12nE[\tilde{r_i}]) \leq \delta/n$

$1 - \epsilon/12n \leq \tilde{r_i}/E[\tilde{r_i}] \leq 1 + \epsilon/12n$ wp $1 - \delta/n$

Using the $\epsilon/6n$ sampler Eq 2:

$1 - \epsilon(n+1)/6n \leq E[\tilde{r_i}]/r_i \leq 1 + \epsilon(n+1)/6n$

Using Eq 1 and Eq 2

$1 - \epsilon(2n+3)/12n \leq \tilde{r_i}/r_i \leq 1 + \epsilon(2n+3)/12n$ wp $1 - \delta/m$

Problem 2 — 5 marks

Consider a graph $G(V, E)$ on n vertices. Design a Markov chain whose state space is the set of all independent sets, except the empty independent set, and whose stationary distribution is such that for an independent set I, $\pi_I = |I|/B$, for some value B.

Solution:

Let us first define the uniform stationary distribution. Let M be all the independent sets

Let X_i be a given node in the MC. We choose a random $u \in V$

if $u \in X_i$ then $X_{i+1} = X_i/u$

if $u \notin X_i$ and $X_i \cup u \in M$ then $X_{i+1} = X_i \cup u$

else $X_{i+1} = X_i/$

Irreducible: There is a possibility of going from a given set to an empty set by removing all $u \in X_i$ and from there we can build to any independent set.

Aperiodic: The possibility of a self-loop makes it non-bipartite.

Let us take 2 solutions x and y which differ by one vertex. From the MC rules $P_{x,y} = P_{y,x} = 1/N$ and distribution function is uniform and $\pi = 1/M$, then $\pi_x P_{x,y} = 1/MN = \pi_x P_{x,y}$ proving uniform stationary distribution.

Now we use the metropolis algorithm to edit the $P_{x,y}$ rules to get the necessary distribution.

For a given independent set x, let the neighbors $N(x)$ be all the sets differing by one vertex removal or inclusion holding the independent set property.

Uniform:

$P_{x,y} = 1/N$ if $y \in N(x)$

$P_{x,y} = 0$ if $y \notin N(x)$

$P_{x,y} = 1 - \sum_{y \in N(x)} P_{x,y}$

For given distribution $\pi_x = |x|/B$

$P_{x,y} = 1/N[min(1, |y|/|x|)]$ if $y \in N(x)$

and the remaining rules remain the same to build the MC with given distribution.

Problem 3 — 7 marks

In class, to bound the mixing time we used d_t denote the difference between the states in the coupling at time t, and upper-bounded $\mathbb{E}[d_{t+1}|d_t] \leq \beta d_t$, for $\beta < 1$.

(a) (3 marks) If $\mathbb{E}[d_{t+1}|d_t] \leq \beta d_t$, for $\beta < 1$, give an upper bound on $\tau(\epsilon)$ in terms of β and d^* where d^* is the maximum difference among all pairs of states.

Solution:

$\mathbb{E}[d_{t+1}|d_t] \leq \beta d_t$

$\mathbb{E}[d_{t+1}] \leq \beta E[d_t]$

Multiplying the equation till t = 0

$E[d_t] \leq d_0 \beta^t$

$Pr(d_t \geq 1) \leq E[d_t] \leq d^* \beta^t$

Taking the variation distance of ϵ
$\tau(\epsilon) \leq t = ln(\frac{d^*}{\epsilon}/ln(1/\beta)$

(b) (4 marks) Suppose we show that $\mathbb{E}[d_{t+1}|d_t] \leq d_t$, and that $d_{t+1} \in \{d_t - 1, d_t, d_t + 1\}$, and $\Pr[d_{t+1} \neq d_t] \geq \gamma$. Give an upper bound for $\tau(\epsilon)$ that is polynomial in d^* and $1/\gamma$.

Solution: As $\mathbb{E}[d_{t+1}|d_t] \leq d_t$ and $\Pr[d_{t+1} \neq d_t] \geq \gamma$
We can say that $Pr[d_{t+1} = d_t - 1] \geq Pr[d_{t+1} = d_t + 1]$
$Pr[d_{t+1} = d_t - 1] \geq \gamma/2$
We can take d_t as a MC state moving to $d_t - 1$ wp $\geq \gamma/2$ and $d_t + 1$ wp $\leq 1 - \gamma/2$.
Starting from $d_0 = d^*$, we upper bound the number of moves it takes to reach $d_t = 0$

Problem 4 — 15 marks

Consider the following way to shuffle cards: choose two cards independently and uniformly at random from the deck, and swap them. If the two cards chosen are the same, then no change occurs to the deck.

(a) (3 marks) Show that the following is an equivalent process: Choose a card at random, and a position i at random. Swap the chosen card with the card at position i.

Solution:
Let us first pick the position i. The difference in the shuffling is the second card picked.
Then the probability of choosing a given card after choosing position i to swap by choosing a card at random and by choosing another position at random is equivalent.

(b) (3 marks) Consider the coupling where the choices of the card and the position is identical for both copies of the chain. Let X_t be the number of cards whose positions are different in the two copies of the chain. Show that X_t is non-increasing.

Solution:
Let P_t, Q_t be both the Markov chains representing the cards. Let $P_t(v)$ be the card for P_t at position v. Let the card be chosen as C and its position in P_t and Q_t be j and k. Let the chosen position be i
Case i: $j = k \longrightarrow X_{t+1} = X_t$;wp $1 - X_t/N$
Case ii: $j \neq k, P_t(j) = Q_t(i) \longrightarrow X_{t+1} = X_t$;wp $(X_t/N)(1 - X_t/N)$
Case iii: else $X_{t+1} = X_{t-1} - 1 - 1(if\ P_t(j) = Q_t(j)) - 1(if\ P_t(k) = Q_t(k))$
Thus X_t is non increasing.

(c) (5 marks) Show that

$$\Pr[X_{t+1} \leq X_t - 1|X_t > 0] \geq \left(\frac{X_t}{n}\right)^2.$$

Solution:
From the above cases, Pr(Case iii) = 1 - Pr(Case i) - Pr(Case ii)
$\Rightarrow 1 - (1 - (\frac{X_t}{n})^2) = (\frac{X_t}{n})^2$
Thus $Pr[X_{t+1} \leq X_t - 1 | X_t > 0] \geq (\frac{X_t}{n})^2$.

(d) (4 marks) Show that the expected time until X_t is 0 is $O(n^2)$, regardless of the starting state of the chains.

Solution: Let us take $X_t = i$ and $Pr[X_{t+1} \leq X_t - 1 | X_t > 0] \geq (\frac{X_t}{n})^2$
We can say that the expected number of steps X_t takes to go from i to i-1 is $\leq \frac{n^2}{i^2}$
Thus we can say that the total number of steps to go from a given state with $X_0 = k$ to $X_t = 0$ is
$\leq \sum_{i=0}^{k} \frac{n^2}{i^2} \leq n^2 \sum^{\infty} 1/i^2 = \pi^2 n^2/6 = O(n^2)$