CS6170: Randomized Algorithms
Problem Set #4

Name: Your name — Marks: 40

Roll No: Your roll number — Due: Nov 12, 23:59

Problem 1 — 13 marks

Consider the following approach to counting the number of solutions to the knapsack problem. Given items with sized $a_1, a_2, \ldots, a_n > 0$ and an integer $b > 0$, find the number of vectors $(x_1, x_2, \ldots, x_n) \in \{0,1\}^n$ such that $\sum_{i=1}^n a_i x_i \le b$.

(a) (3 marks) Consider the following direct Monte-Carlo algorithm for counting the number of solutions: Choose $(x_1, x_2, \ldots, x_n) \in \{0,1\}^n$ uniformly at random and check if satisfies $\sum_{i=1}^n a_i x_i \le b$. Let f be the fraction of samples that satisfies the equation. Output $f \cdot 2^n$ as the estimate of the number of solutions.

Show that this will not yield an FPRAS, by arguing with respect to an instance when $a_i = 1$ for every i and $b = \sqrt{n}$.

Solution: Type your solution here.

(b) (5 marks) Consider a Markov chain M on the state space $\{0,1\}^n$. From a state $X_j = (x_1, x_2, \ldots, x_n)$, M chooses an i uniformly at random. If $x_i = 1$, then X_{j+1} is obtained by setting x_i to 0. If $x_i = 0$, and setting $x_i = 1$ gives a feasible solution, then x_i is set to 1 in X_{j+1}. Otherwise $X_{j+1} = X_j$.

Suppose that $\sum_{i=1}^n a_i > b$. Show that M is irreducible and aperiodic, and that the stationary distribution is the uniform distribution.

Solution: Type your solution here.

(c) (5 marks) Argue that an FPAUS for the knapsack problem yields and FPRAS for it by proceeding as follows: Let $a_1 \le a_2 \le \cdots \le a_n$. Let $b_i = \sum_{j=1}^i a_i$. Let k be the smallest index such that $b_k \ge b$, and let $\Omega(b_i)$ denote the set of vectors $(x_1, x_2, \ldots, x_n)$ such that $\sum_{j=1}^n a_j x_j \le b_i$. Use the equation,

$$|\Omega(b)| = \frac{|\Omega(b)|}{|\Omega(b_{k-1})|} \frac{|\Omega(b_{k-1})|}{|\Omega(b_{k-2})|} \cdots \frac{|\Omega(b_2)|}{|\Omega(b_1)|} |\Omega(b_1)|.$$

Prove that $|\Omega(b_i)| \le (n+1)|\Omega(b_{i-1})|$ and use the approach discussed in class. Give all the details with the correct parameters.

Solution: Type your solution here.

Problem 2 — 5 marks

Consider a graph $G(V, E)$ on n vertices. Design a Markov chain whose state space is the set of all independent sets, except the empty independent set and whose stationary distribution is such that for an independent set I, $\pi_I = |I|/B$, for some value B.

Solution: Type your solution here.

Problem 3 — 7 marks

In class, to bound the mixing time we used d_t denote the difference between the states in the coupling at time t, and upper-bounded $\mathbb{E}[d_{t+1}|d_t] \le \beta d_t$, for $\beta < 1$.

(a) (3 marks) If $\mathbb{E}[d_{t+1}|d_t] \le \beta d_t$, for $\beta < 1$, give an upper bound on $\tau(\epsilon)$ in terms of β and d^* where d^* is the maximum difference among all pairs of states.

Solution: Type your solution here.

(b) (4 marks) Suppose we show that $\mathbb{E}[d_{t+1}|d_t] \le d_t$, and that $d_{t+1} \in \{d_t - 1, d_t, d_t + 1\}$, and $\Pr[d_{t+1} \ne d_t] \ge \gamma$. Give an upper bound for $\tau(\epsilon)$ that is polynomial in d^* and $1/\gamma$.

Solution: Type your solution here.

Problem 4 — 15 marks

Consider the following way to shuffle cards: choose two cards independently and uniformly at random from the deck, and swap them. If the two cards chosen are the same, then no change occurs to the deck.

(a) (3 marks) Show that the following is an equivalent process: Choose a card at random, and a position i at random. Swap the chosen card with the card at position i.

Solution: Type your solution here.

(b) (3 marks) Consider the coupling where the choices of the card and the position is identical for both copies of the chain. Let X_t be the number of cards whose positions are different in the two copies of the chain. Show that X_t is non-increasing.

Solution: Type your solution here.

(c) (5 marks) Show that

$$\Pr[X_{t+1} \le X_t - 1 | X_t > 0] \ge \left(\frac{X_t}{n}\right)^2.$$

Solution: Type your solution here.

(d) (4 marks) Show that the expected time until X_t is 0 is $O(n^2)$, regardless of the starting state of the chains.

Solution: Type your solution here.