Operations Research in Health Care

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September 07, 2021

Outline

Introduction

What is Operations Research?

- **Operation research is a branch of mathematics which models and solves real life problems to make optimum decisions.**

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- **Operation research is a branch of mathematics which models and solves real life problems to make optimum decisions.**
- It is scientific approach to decision making.
- It provides a set of algorithms that act as tools for effective problem solving and decision making.

History of Operations Research

Operation research got started during the Second World war.

The application of mathematics and scientific method to military applications was called operation research.

In the context of global health, operation research is used broadly to encompass cross-sectional, case-control, retrospective or prospective cohort analyses.

Phases

The principal phases for implementing OR

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1. Definition of the problem

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2. Construction of the model

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4. Validation of the model

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2. Construction of the model
3. Solution of the model
4. Validation of the model
5. Implementation of the solution

Terminology

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- **Feasible Solution:** The set of values of decision variables X_j ($j = 1, 2, \ldots n$) which satisfy all the constraints and non-negativity conditions of a linear programming problem.
- **Infeasible Solution:** The set of values of decision variables X_j ($j = 1, 2, \ldots n$) which do not satisfy all the constraints and non-negativity conditions of the problem.

Terminology

- **Basic Feasible Solution:** A feasible solution to LP problem which is also the basic solution is called the basic feasible solution. Basic feasible solutions are of two types; (a) Degenerate: (b) Non-degenerate:
- **Optimum Basic Feasible Solution:** A basic feasible solution which optimizes (maximizes or minimizes) the objective function value of the given LP problem is called an optimum basic feasible solution.

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- **Optimum Basic Feasible Solution:** A basic feasible solution which optimizes (maximizes or minimizes) the objective function value of the given LP problem is called an optimum basic feasible solution.
- **Unbounded Solution:** If the value of the objective function Z can be increased or decreased indefinitely such solutions are called unbounded solution.

Linear Programming Problem

Linear Programming Problem

Definition

"A mathematical method to allocate scarce resources to competing activities in an optimal manner when the problem can be expressed using a linear objective function and linear inequality constraints."

- Development of linear programming has been ranked among the most important scientific advances of the mid-20th century.
- Constrained optimization models are mathematical models that find the best solution with respect to some evaluation criterion from a set of alternative solutions.

Components

Constrained optimization models have three major components:

1. Decision Variable: Variables which are changeable and going to impact the decision function.
2. Objective Function: Z defines the criterion for evaluating the solution. It is a mathematical function of the decision variables that converts a solution into a numerical evaluation of that solution.
3. Constraints: There are a set of functional equalities or inequalities that represent physical, economic, technological, legal, ethical, or other restrictions on what numerical values can be assigned to the decision variables.

In constrained optimization models we find values for the decision variables that maximize or minimize the objective function and satisfy all constraints.

Example

A diet for a sick person must contain at least 4,000 units of vitamins, 50 units of minerals and 1,400 calories. Two foods A and B are available at a cost of Rs. 4 and Rs. 3 per unit, respectively. If one of A contains 200 units of vitamins, 1 unit of mineral and 40 calories and one unit of food B contains 100 units of vitamins, 2 units of minerals and 40 calories. Formulate this problem as an LP model and solve it to find combination of foods to be used to have least cost?

Food	Units Content of			Cost per
	Vitamins	Mineral	Calories	Unit (Rs.)
A	200	1	40	4
B	100	2	40	3
Minimum requirement	4,000	50	1,400	

Standard form of LPP

An algebraic representation of a generic formulation of linear programming model could be presented as following

To maximize or minimize the objective function:

$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n(\geq, \quad =, \leq)b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdot\cdot + a_{2n}x_n(\geq, \quad =, \leq)b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdot\cdot + a_{mn}x_n(\geq, \quad =, \leq)b_m$$

Standard form of LPP

The standard form of a general LPP can be written in the in following form :-

Objective function: Max or Min $= \sum_{j=1}^{n} C_j x_j$

Subject to constraints: $\sum_{j=1}^{n} a_{ij} x_j \, (\leqslant_y =, \geqslant) \, bi; \, i = 1, 2, \ldots m$ (constraints)

and $x_j \geq 0, \forall j = 1, 2, \ldots n$ (non-negative restriction) where

1. $x_1, x_2, \ldots, x_n$ in are decision variables
2. $c_1, c_2, \ldots, c_n$ on are called cost or profit coefficients
3. $a_{ij} \quad (i = 1, 2, \ldots m)$ and $(j = 1, 2, \ldots, n)$ are structural coefficients
4. $b_i \quad (i = 1, 2, \ldots n)$ are constants representing *a* variability or requirement of ith constraint

Characteristics

Standard form of LPP must have following three characteristics:

- Objective function should be of maximization type.
- All the constraints should be of equality type.
- All the decision variables should be non-negative.

Steps in Formulating the LP problem:

1. Identify and define the decision variables for the problem.
2. Define the objective function.
3. Identify and express mathematically all of the relevant constraints

Graphical Method

The steps used in graphical method are summarized as follows:
Step 1: Replace the inequality sign in each constraint by an equal to sign.
Step 2: Plot each equation on the graph, as each one with geometrically represent a straight line.
Step 3: Every point on the line will satisfy the equation of the line.
Step 4: Shade the common portion of the graph that satisfies all the constraints simultaneously drawn so far. This shaded area is called the feasible region (or solution space) of the given LP problem.
Step 5: Locate the corner point of optimal solution by calculating the value of z for each corner point.

Example

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	Vitamins	Mineral	Calories	
A	200	1	40	4
B	100	2	40	3
Minimum requirement	4,000	50	1,400	

Example

Then LP model of the given problem is **Objective function:** $\mathrm{Z} = 4x_1 + 3x_2$
Subject to the constraints

- Vitamins : $200x_1 + 100x_2 \geq 4,000,$
- Mineral : $x_1 + 2x_2 \geq 50,$
- Calories: $40x_1 + 40x_2 \geq 1,400$

$$x_1, x_2 \geq 0$$

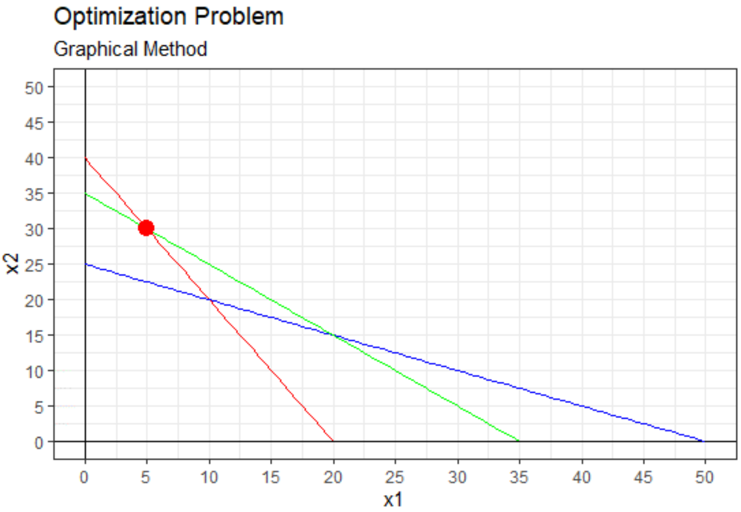
Example

```r
rm(list = ls(all=TRUE))
# Define constraints
cons.1 <- function(x)
{
  40 - 2*x
}

cons.2 <- function(x)
{
  25 - 0.5*x
}
cons.3 <- function(x)
{
  35 - x
}
# Import ggplot2 package
library(ggplot2)
```

Example

```r
# Build plot
p <- ggplot(data = data.frame(x = 0), aes(x = x)) +
  # Add axes
  geom_vline(xintercept = 0) +
  geom_hline(yintercept = 0) +
  # Add constraints lines
  stat_function(colour = "Red", fun = cons.1) +
  stat_function(colour = "Blue", fun = cons.2) +
  stat_function(colour = "green", fun = cons.3)+
  # Specify axes breaks and limits
  scale_x_continuous(breaks = seq(0, 60, 5), lim = c(0, 50)) +
  scale_y_continuous(breaks = seq(0, 50, 5), lim = c(0, 50)) +
  # Define labels
  labs(title = "Optimization Problem",
       subtitle = "Graphical Method",
       x = "x1",
       y = "x2") +

  # Add black and white theme
  theme_bw()
# Print plot
print(p)
```

Example


Optimization Problem
Graphical Method
x2
x1
0
5
10
15
20
25
30
35
40
45
50

Example

The coordinates of the extreme points of the feasible solution space are: A(0, 40), B(5, 30), C(20, 15) and D(50, 0). The value of objective function at each of these extreme points is shown in Table as below

Extreme Point	Coordinates (x_1, x_2)	Objective Function Value $Z = 4x_1 + 3x_2$
A	$(0, 40)$	$4(0) + 3(40) = 120$
B	$(5, 30)$	$4(5) + 3(30) = 110$
C	$(20, 15)$	$4(20) + 3(15) = 125$
D	$(50, 0)$	$4(50) + 3(0) = 200$

Extreme point $\mathrm{B}(5, 30)$

The minimum (optimal) value of the objective function $Z = 110$.

Simplex method

We go to this method when decision variables are more than 2
Slack variable: A non- negative variable which is added to the LHS of the constraint (if it is $<=$ type) to convert it into equation is called slack variable.
Surplus Variable: A non-negative variable which is subtracted from the LHS of the constraint (if it is $>=$ type) to convert it into equations is called surplus variable.

$$\text{Max }.Z = 13x_1 + 11x_2$$

Subject to constraints:

$$4x_1 + 5x_2 \leq 1500$$
$$5x_1 + 3x_2 \leq 1575$$
$$x_1 + 2x_2 \leq 420$$
$$x_1, x_2 \geq 0$$

Steps

Step 1: Convert all the inequality constraints into equalities by the use of slack variables.
Let $\mathrm{S}_1,\ \mathrm{S}_2,\ \mathrm{S}_3$ be three slack variables.
Model can rewritten as:

$$\mathrm{Z} - 13\mathrm{x}_1 + 11\mathrm{x}_2 = 0$$

Subject to constraints:

$$4x_1 + 5x_2 + S_1 = 1500$$
$$5x_1 + 3x_2 + S_2 = 1575$$
$$x_1 + 2x_2 + S_3 = 420$$
$$x_1, x_2, S_1, S_2, S_3$$
$$\geq 0$$

Steps

Step 2: Find the Initial BFS.
Now, $S_1,\ S_2,\ S_3$ are Basic variables. $\left[x_B = B^{-1}b\right]$.
Step 3: Set up an initial table as:

Row no.	Basic variable	Coefficients of:						sol.	Ratio
		Z	X_1	X_2	S_1	S_2	S_3		
A1	Z	1	-13	-11	0	0	0	0	
B1	S_1	0	4	5	1	0	0	1500	375
C1	S_2	0	5	3	0	1	0	1575	315
D1	S_3	0	1	2	0	0	1	420	420

Steps

Step 4: a) Choose the most negative number from row AI. Therefore, x_1 is a entering variable.
b) Calculate Ratio $=$ Sol col. $/x_1$ col. $(x_1 > 0)$
c) Choose minimum Ratio.
Step 5: x_1 becomes basic variable and S_2 becomes non basic variable.

Row no.	Basic variable	Coefficients of:						sol.	Ratio
		Z	X_1	X_2	S_1	S_2	S_3		
A1	Z	1	0	$-16/5$	0	$13/5$	0	4095	
B1	S_1	0	0	$13/5$	1	$-4/5$	0	240	92.3
CI	x_1	0	1	$3/5$	0	$1/5$	0	315	525
D1	S_3	0	0	$7/5$	0	$-1/5$	1	105	75

Steps

Row no.	Basic variable	Coefficients of:						sol.
		Z	X_1	X_2	S_1	S_2	S_3	
A1	Z	1	0	0	0	$15/7$	$16/7$	4335
B1	S_1	0	0	0	1	$-3/7$	$-13/7$	45
C1	x_1	0	1	0	0	$2/7$	$-3/7$	270
D1	x_2	0	0	1	0	$-1/7$	$5/7$	75

Optimal Solution is : $x_1 = 270, x_2 = 75, Z = 4335$

Example

```
> # Import lpSolve package
> library(lpSolve)
>
> # Set coefficients of the objective function
> f.obj <- c(13, 11)
>
> # Set matrix corresponding to coefficients of constraints by rows
> f.con <- matrix(c(4, 5,
+                   5,3,
+                   1, 2), nrow = 3, byrow = TRUE)
>
> # Set unequality/equality signs
> f.dir <- c("<=", "<=","<=")
>
> # Set right hand side coefficients
> f.rhs <- c(1500, 1575,420)
>
> # Final value (z)
>
>
> # Variables final values
> sol <- lp("max", f.obj, f.con, f.dir, f.rhs,compute.sens = TRUE)$solution
> sol
[1] 270  75
```

Two-phase method

- LPP, in which constraints may also have $\geq$ and $=$ signs, *w* introduce a new type of variable, called the artificial variable. The solution is obtained in two phases.
- In the **first phase** of this method, the sum of the artificial variables is minimized subject to the given constraints in order to get a basic feasible solution of the LP problem.
- The **second phase** minimizes the original objective function starting with the basic feasible solution obtained at the end of the first phase.

Assignment Problem

Assignment Problem

Definition

The assignment problem refers to a special class of linear programming problems that involve determining the most efficient assignment of people to projects, salespeople to territories, contracts to bidders, jobs to machines, and so on.

Characteristics of assignment problem is that only one job is assigned to one machine or one worker is assigned to one project.

This method was developed by D. Konig, a Hungarian mathematician.

One needs to know only the cost of making all the possible assignments.

Each assignment problem has a matrix associated with it.

Assignment Problem

Example:
A marketing manager has 5 medical representatives and 5 sales districts. Considering the capabilities of the medical representative and the nature of districts, the marketing manager estimates that sales per month (in thousand rupees) for each medical representative in each district would be follows:

Reprsentatives	District				
	A	B	C	D	E
1	5	7	9	11	12
2	9	15	12	6	2
3	4	7	8	5	10
4	12	9	13	11	10
5	3	6	7	4	5

Assign different district to the representatives so as to maximize the total sales.

Assignment Problem

Reprsentatives	District				
	A	B	C	D	E
1	6	8	10	12	11
2	9	15	12	6	2
3	4	7	8	5	10
4	12	9	13	11	10
5	3	6	7	4	5

Assignment Problem

Reprsentatives	District				
	A	B	C	D	E
1	6	8	10	12	11
2	9	15	12	6	2
3	4	7	8	5	10
4	12	9	13	11	10
5	3	6	7	4	5

Reprsentatives	District				
	A	B	C	D	E
1	9	7	5	3	4
2	6	0	3	9	13
3	11	8	7	10	5
4	3	6	2	4	5
5	12	9	8	11	10

Assignment Problem

Reprsentatives	District				
	A	B	C	D	E
1	9	7	5	3	4
2	6	0	3	9	13
3	11	8	7	10	5
4	3	6	2	4	5
5	12	9	8	11	10

Assignment Problem

Reprsentatives	District				
	A	B	C	D	E
1	9	7	5	3	4
2	6	0	3	9	13
3	11	8	7	10	5
4	3	6	2	4	5
5	12	9	8	11	10

Repersentatives	District				
	A	B	C	D	E
1	6	4	2	0	1
2	6	0	3	9	13
3	6	3	2	5	0
4	1	4	0	2	3
5	4	1	0	3	2

Assignment Problem

Repersentatives	District				
	A	B	C	D	E
1	6	4	2	0	1
2	6	0	3	9	13
3	6	3	2	5	0
4	1	4	0	2	3
5	4	1	0	3	2

Assignment Problem

Repersentatives	District				
	A	B	C	D	E
1	6	4	2	0	1
2	6	0	3	9	13
3	6	3	2	5	0
4	1	4	0	2	3
5	4	1	0	3	2

Repersentatives	District				
	A	B	C	D	E
1	5	4	2	0	1
2	5	0	3	9	13
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Assignment Problem

Repersentatives	District				
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1	5	4	2	0	1
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5	3	1	0	3	2

Assignment Problem

Repersentatives	District				
	A	B	C	D	E
1	5	4	2	0	1
2	5	0	3	9	13
3	5	3	2	5	0
4	0	4	0	2	3
5	3	1	0	3	2

Repersentatives	District				
	A	B	C	D	E
1	5	4	2	[0]	1
2	5	[0]	3	9	13
3	5	3	2	5	[0]
4	0	4	0	2	3
5	3	1	[0]	3	2

Assignment Problem

Repersentatives	A	B	C	D	E
			District		
1	5	4	2	[0]	1
2	5	[0]	3	9	13
3	5	3	2	5	[0]
4	0	4	0	2	3
5	3	1	[0]	3	2

Assignment Problem

Repersentatives	District				
	A	B	C	D	E
1	5	4	2	[0]	1
2	5	[0]	3	9	13
3	5	3	2	5	[0]
4	0	4	0	2	3
5	3	1	[0]	3	2

Repersentatives	District				
	A	B	C	D	E
1	5	4	2		1
2	5	[0]	3	9	13
3	5	3	2	5	[0]
4	[0]	4	0	2	3
5	3	1	[0]	3	2

Assignment Problem

Unbalanced assignment problem

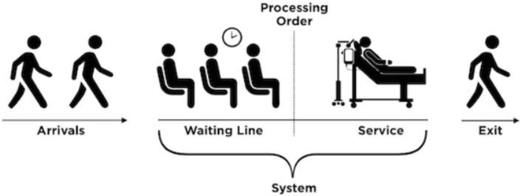
1. Number of columns are not equal to number of rows.
2. Dummy columns or rows are inserted.
3. The entries of dummy column/row are zeros.

Queuing Theory

Queuing Theory

Definition

Queuing theory is the mathematical study of queuing. A basic queuing system consists of an arrival process, the queue itself, the service process for attending to those customers, and departures from the system.



Queuing Theory

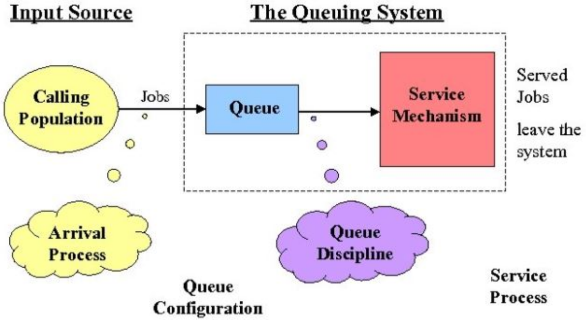
Why Queuing Theory?

Queuing theory is important because it helps describe features of the queue, like average wait time, and provides the tools for optimizing queues.

Queuing System Characteristics

- **The population source** -the population of potential customers, may be assumed to be finite or infinite.
- **Number of servers** - The number of servers that are available to serve the patients simultaneously.
- **Arrival Patterns**- Arrival rate is determined as the average arrivals for a given time period. It is denoted by $\lambda \sim \text{Pois}\,(\lambda)$
- **Service Patterns** - The service mechanism consists of one or more service facilities, each of which contains one or more parallel service channels, called servers. service times are exponentially distributed, at the rate of mu (μ) per hour.
- **Queue discipline**- Describes the form in which the patients are served after a queue has been formed

Queuing System Characteristics



Queuing Model Notations

1. λ arrival rate
2. μ service rate
3. L_q average number of customers waiting for service
4. L average number of customers in the system
5. W_q average time customers wait in line
6. W average time customers spend in the system
7. ρ system utilization
8. $1/\mu$ service time
9. P_0 probability of zero units in system
10. P_n probability of n units in system

Model Formulations

Five key relationships provide the basis for queuing formulations and are common for all infinite - source models:

1. The average number of patients in the system is the average number in line plus the average number being served.

$$L = L_q + r$$

2. The average time in line is the average number in line divided by the arrival rate.

$$W_q = \frac{L_q}{\lambda}$$

Model Formulations

3. The average time in the system is the sum of the time in line plus the service time.

$$W = W_q + \frac{1}{\mu}$$

4. System utilization is the ratio of arrival rate to service capacity.

$$\rho = \frac{\lambda}{c\mu}$$

Little's formula
Assume that λ_n is a constant λ for all n. It has been proved that in a steady-state queueing process,

$$L = \lambda W$$

Furthermore, the same proof also shows that

$$L_q = \lambda W_q$$

Single Server

The (M/M/1): (∞/ FIFO) system

We consider an infinite source population with first come first served discipline where both the inter arrival time λ and service time μ follow an exponential distribution.

Utilization $(\rho) = \frac{\lambda}{\mu}$

Probability of system is ideal $(\mathrm{P}_0) = 1 - \rho = \mathrm{P}_0 = 1 - \frac{\lambda}{\mu}$

Probability of n units in the system (waiting time and service time) $\mathrm{P_n} = \left(\frac{\lambda}{\mu}\right)^{\mathrm{n}} \mathrm{P}_0$

The expected number of customer in the system $L_s = \frac{\lambda}{\mu - \lambda}$

The expected number of customers in the queue $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$

Single Server

The average waiting time in the system $W_s = \frac{1}{\mu - \lambda}$
The average waiting time of a customer in the queue $W_q = \frac{\lambda}{\mu(\mu - \lambda)}$

The probability track that K or more than K customersin the system $\mathrm{P} \geq \mathrm{K} = \left(\frac{\lambda}{\mu}\right)^{\mathrm{k}}$

The probabilitythat more than K customers are in the system $(\mathrm{P} > \mathrm{K}) = \left(\frac{\lambda}{\mu}\right)^{\mathrm{K}+1}$

The probability that atleast one customer is standing in queue $\mathrm{P} = \mathrm{K} = \left(\frac{\lambda}{\mu}\right)^{2}$

Single Server

A hospital is exploring the level of staffing needed for a booth in the local mall, where they would test and provide information on diabetes. Previous experience has shown that, on average, every fifteen minutes a new person approaches the booth. A nurse can complete testing and answering questions, on average, in twelve minutes. If there is a single nurse at the booth, calculate system performance measures including the probability of idle time and of one or two persons waiting in the queue. What happens to the utilization rate if another workstation and nurse are added to the unit?
Arrival rate: $\lambda = 1$ (hour) $\div 15 = 60$ (minutes) $\div 15 = 4$ persons per hour.
Service rate: $\mu = 1$ (hour) $\div 12 = 60$ (minutes) $\div 12 = 5$ persons per hour.
we get:

$$\rho = \frac{\lambda}{\mu} = \frac{4}{5} = .8$$
$$L_q = \frac{4^2}{5(5-4)} = 3.2$$

Single Server

$$L = L_q + \frac{\lambda}{\mu} = 3.2 + .8 = 4 \text{ persons.}$$

$$W_q = \frac{L_q}{\lambda} = \frac{3.2}{4} = 0.8 = 48$$
$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{4}{5} = 1 - .8 = .2$$

$$P_1 = P_0 \left(\frac{\lambda}{\mu}\right)^1 = (.2)\left(\frac{4}{5}\right)^1 = (.2)(.8)^1 = (.2)(.8) = .16 \text{ or } 16\%$$

Single Server

$$P_2 = P_0 \left(\frac{\lambda}{\mu}\right)^2 = (.2)\left(\frac{4}{5}\right)^2 = (.2)(.8)^2 = (.2)(.64) = .128 \text{ or } 12.8\%$$

$$\rho = \frac{\lambda}{s\mu} = \frac{4}{1 \times 5} = 80\%$$

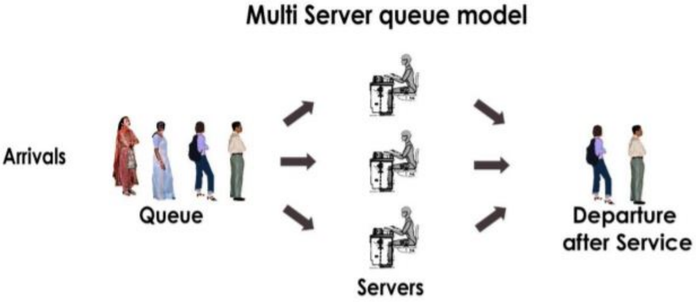
$$\rho = \frac{\lambda}{s\mu} = \frac{4}{2 \times 5} = 40\%$$

Single Server

```
> rm(list=ls())
>
> mm1_metrics <- function(lambda, mu, n, sl){
+   rho <- lambda/mu
+   Ls <- lambda/(mu-lambda)
+   Lq <- lambda**2/(mu*(mu-lambda))
+   Ws <- 1 / (mu-lambda)
+   Wq <- Lq/lambda
+   Pn <- (1-rho)*(rho)**n
+   X <- data.frame(rho, Ls, Lq, Ws, Wq, Pn)
+
+   names(X)<-c('rho','Ls','Lq','Ws','Wq','Pn')
+   return(X)
+ }
>
> mm1_metrics(15, 20, 3)
   rho Ls   Lq  Ws   Wq        Pn
1 0.75  3 2.25 0.2 0.15 0.1054688
```

Multiple Server

(M/M/C) :(∞/∞)



Multiple servers

The formulas for Multi Server model:
Utilization factor:

$$\rho = \frac{\lambda}{c\mu}$$

The probability of idle customer service:

$$P_0 = \left\{ \sum_{n=0}^{c-1} \frac{\rho^n}{n!} + \frac{\rho^c}{c!} \left(\frac{1}{1 - \rho/c} \right) \right\}^{-1}$$

The average number of customer in the queue:

$$L_q = \left[\frac{\rho^{c+1}}{(c-1)!(c-\rho)^2} \right] P_0$$

Multiple Server

The average number of customer in the system:

$$L_s = \left[\frac{\rho^{c+1}}{(c-1)!(c-\rho)^2}\right] P_0 + \rho$$

The average time a customer in the queue:

$$W_q = \frac{\rho^c}{\mu(c-1)!(c-\rho)^2} P_0$$

The average time a customer in the system:

$$W_s = \frac{\rho^c}{\mu(c-1)!(c-\rho)^2} P_0 + \frac{1}{\mu}$$

Conclusion

Operation research has the following sub areas of study

- Transportation Problem
- Reliability theory
- Network analysis
- Inventory
- Simulation

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THANK YOU