ISSN 1063-780X, Plasma Physics Reports, 2020, Vol. 46, No. 7, pp. 724–731. © Pleiades Publishing, Ltd., 2020.
Russian Text © The Author(s), 2020, published in Fizika Plazmy, 2020, Vol. 46, No. 7, pp. 658–666.

METHODOLOGICAL NOTES

Drift Motion of Charged Particles in Inhomogeneous Magnetic and Strong Electric Fields

N. A. Marusov[a, b, c], **E. A. Sorokina**[a, c, *], **and V. I. Ilgisonis**[d]

[a] *National Research Centre "Kurchatov Institute," Moscow, 123182 Russia*
[b] *Moscow Institute of Physics and Technology, Dolgoprudny, Moscow oblast, 141701 Russia*
[c] *Peoples' Friendship University of Russia (RUDN University), Moscow, 117198 Russia*
[d] *State Atomic Energy Corporation ROSATOM, Moscow, 119017 Russia*
**e-mail: sorokina.ekaterina@gmail.com*

Received December 19, 2019; revised February 13, 2020; accepted February 20, 2020

Abstract—Distinctive features of the drift motion of a nonrelativistic charged particle in slowly varying magnetic and strong electric fields, for which the assumption that the electric drift velocity is small as compared to the total particle velocity is non-applicable, are studied. The variational principles of the drift motion are extended to the case of the strong electric field. The generalized Littlejohn's Lagrangian is obtained and the extended set of drift equations is derived. The possibility of particle acceleration due to the drift motion along the strong electric field is demonstrated.

Key words: charged particle motion, drift theory, electric drift, crossed electric and magnetic fields

DOI: 10.1134/S1063780X20070065

1. INTRODUCTION

The drift theory is a classical method for studying the charged particle motion in the adiabatic electric and magnetic fields. Like any other model describing the time evolution of a conservative mechanical system, the drift theory can be mathematically formalized in the framework of the Lagrange mechanics [1]. For the first time, for the time-constant electric **E** and magnetic **B** fields, the possibility of using the Lagrange equations for searching the integrals of motion of the drift equations, associated with the field spatial symmetry, was demonstrated by Morozov and Solov'ev in [2]. The Lagrange function L^* obtained by the authors has no rigorous physical interpretation and is formally introduced as the Lagrangian of a zero-mass particle moving in the effective magnetic field $\mathbf{B}^* = \nabla \times \mathbf{A}^*$:

$$L^* = \mathbf{A}^* \cdot \dot{\mathbf{R}}; \quad \mathbf{A}^* = \mathbf{A} + \frac{v_{\parallel}}{\Omega}\mathbf{B}. \tag{1}$$

Here, $\dot{\mathbf{R}}$ is the time derivative of the radius-vector of the guiding center; $\Omega = ZeB/mc$ is the cyclotron frequency, m and Ze are the particle mass and charge, respectively; c is the speed of light; and $v_{\parallel}$ is the particle velocity along the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. We note that, for deriving Eq. (1), the conditions of conservation of energy and magnetic moment are used a priori and do not directly follow from the Lagrange equations. Later, in [3], Littlejohn has constructed the Hamilton theory of the guiding center motion (see references in review [4]). The Littlejohn's Lagrangian is derived by averaging the Lagrange function of a charged particle over the cyclotron period. For the drift equations following from the variational principle the laws of conservation of the phase volume and energy (for the time-constant fields) are strictly observed. These results generalize and develop the results of Morozov and Solov'ev obtained previously.

It is convenient to represent the charged particle motion in the slowly varying electric and magnetic fields as a superposition of the longitudinal and transverse (with respect to the direction of the magnetic field) motions. Within the conventional approach of the classical drift theory, the transverse charged particle motion is divided into the fast rotation along the Larmor circle and the slow displacement of its guiding center, which is called the *drift motion*. The drift motion occurs due to the action of some force (different from the Lorentz one) on the charged particle, as well as due to the space-time inhomogeneities of the electric and magnetic fields. Obviously, the above distinguishing between the "fast" and "slow" motions is valid if changes in the fields over the Larmor length and the cyclotron period are relatively small. At that, the notion "drift" is usually attributed to the time intervals of motion exceeding the period of the particle rotation over the Larmor circle.

It is well known that, in the degenerate case of the charged particle motion in the homogeneous magnetic field under the effect of the constant force, the dividing of particle motion into the Larmor rotation and the displacement of its guiding center is *exact*. Therefore, it allows us to extend the standard drift theory to the case of the strong electric fields, for which the electric drift velocity $\mathbf{V}_{\text{E}} = c[\mathbf{E}\times\mathbf{B}]/B^2$ is of the same order as the total particle velocity v. At the same time, in the case of nonrelativistic motion, we assume that the ratio of the electric field to the magnetic field is small: $E/B \sim v/c \ll 1$. These conditions are fulfilled, for example, in the devices for plasma acceleration in the crossed fields, the principle of operation of which is based on the natural distinguishing between the Larmor radius scales of different plasma components in the relatively weak magnetic field: the Penning discharges, magnetron discharges, the Morozov's stationary plasma thrusters (SPT), ion sources with the closed electron drift, etc. [5]. The extension of the drift theory to the case of the strong electric field results in the appearance of new terms in the equation of drift motion [6, 7], which are the combinations of the electric field and the inhomogeneities of the electric and magnetic fields [8].

This work is methodological in nature, and it is intended to attract attention of readers to the distinctive features of the drift motion of charged particles in the crossed fields with a strong electric component, which is necessary, in particular, for the correct interpretation of the kinetic modeling results for the Hall discharges. This field of research is currently being actively developed (see, e.g., [9, 10]). The set of equations for the drift motion in the strong electric field obtained in [6] is derived in terms of the Lagrange mechanics. On the example of the model configurations of the electric and magnetic fields, a number of key effects are demonstrated, the description of which goes beyond the scope of the classical drift theory.

The paper is structured as follows. In Section 2, within the framework of the approach developed in [3], we derive the Lagrange function for the guiding center of a particle moving in the strong electric field. Using the Euler–Lagrange equations, we demonstrate the transition to the well known set of drift equations taking into account corrections to the motion of the particle guiding center in the strong electric field. In Section 3, we present the numerical simulation results for the trajectories of charged particles in some model configurations of the crossed fields, demonstrating the effects associated with the particle drift motion in the strong electric field. The main results of the work are summarized in Conclusions.

2. DRIFT THEORY IN THE CASE OF STRONG ELECTRIC FIELD IN TERMS OF THE LAGRANGE MECHANICS

We derive the equations of the drift theory extended to the case of the strong electric field using the Lagrange formalism. We set the Lagrangian as a function of the independent variables: coordinates $\mathbf{q}$, velocities $\dot{\mathbf{q}}$, and time t:

$$L(\mathbf{q},\dot{\mathbf{q}},t) = \mathbf{p}\cdot\dot{\mathbf{q}} - H(\mathbf{q},\mathbf{p},t). \tag{2}$$

Here, the dot denotes time differentiation; $H(\mathbf{q},\mathbf{p},t)$ is the Hamiltonian of the charged particle in the electric and magnetic fields given by the following expression:

$$H(\mathbf{q},\mathbf{p},t) = \frac{1}{2m}\left[\mathbf{p} - \frac{Ze}{c}\mathbf{A}(\mathbf{q},t)\right]^2 + Ze\Phi(\mathbf{q},t), \tag{3}$$

where $\mathbf{A}$ is the vector potential of the magnetic field: $\mathbf{B} = \nabla\times\mathbf{A}$; Φ is the electrostatic potential, which determines the electric field in accordance with the expression $\mathbf{E} = -\nabla\Phi - (1/c)\partial\mathbf{A}/\partial t$.

We assume $\mathbf{r} = \mathbf{q}$ and $\mathbf{v} = (1/m)(\mathbf{p} - (Ze/c)\mathbf{A})$ and then write the Lagrangian as a function of variables $(\mathbf{r},\mathbf{v})$:

$$L = \left[\frac{Ze}{c}\mathbf{A}(\mathbf{r},t) + m\mathbf{v}\right]\cdot\dot{\mathbf{r}} - \left[Ze\Phi(\mathbf{r},t) + \frac{m}{2}|\mathbf{v}|^2\right]. \tag{4}$$

Further, variables $\mathbf{v}$ and $\dot{\mathbf{r}}$ will be considered as the independent ones.

To proceed to the description of drift motion, we represent the particle radius-vector $\mathbf{r}$ in the form of a superposition of the radius-vector of the guiding center $\mathbf{R}$ and the Larmor radius $\boldsymbol{\rho}_{\text{L}}$:

$$\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}_{\text{L}}, \tag{5}$$

where

$$\boldsymbol{\rho}_{\text{L}} = \frac{[\mathbf{u}\times\boldsymbol{\Omega}]}{\Omega^2}. \tag{6}$$

Here, $\mathbf{u} = \mathbf{v}_\perp - \mathbf{V}_{\text{E}}$ is the cyclotron rotational velocity in the reference frame moving at the electric drift velocity; and $\mathbf{v}_\perp$ is the transverse particle velocity $\boldsymbol{\Omega} = -Ze\mathbf{B}/mc$.

We introduce the three orthogonal unit vectors:

$$\mathbf{b} = [\mathbf{e}_1\times\mathbf{e}_2], \quad \mathbf{e}_1 = [\mathbf{e}_2\times\mathbf{b}], \quad \mathbf{e}_2 = [\mathbf{b}\times\mathbf{e}_1], \tag{7}$$

where $\mathbf{b} = \mathbf{B}/B$ is the unit vector directed along the magnetic field. The $\mathbf{e}_1$ and $\mathbf{e}_2$ vectors can be easily determined by defining one of them in the form of $\mathbf{e}_1 = \mathbf{u}/u$; then, in accordance with expression (6), the second vector will be $\mathbf{e}_2 = (\Omega/u)\boldsymbol{\rho}_{\text{L}}$. Following this definition, one can express the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, which rotate around the $\mathbf{b}$ vector with a frequency Ω, through the phase of the Larmor rotation θ:

$$\mathbf{e}_1 = \mathbf{e}_i\cos\theta + \mathbf{e}_j\sin\theta, \quad \mathbf{e}_2 = -\mathbf{e}_i\sin\theta + \mathbf{e}_j\cos\theta,$$

where $\mathbf{e}_{i,j}$ are the orthogonal unit vectors in the plane of the particle Larmor rotation, $[\mathbf{e}_i\times\mathbf{e}_j] = \mathbf{b}$.

Using relations (5)–(7), the velocity and radius-vector of the particle can be written as follows:

$$\mathbf{v} = v_\parallel\mathbf{b} + \mathbf{V}_{\text{E}} + u\mathbf{e}_1, \quad \mathbf{r} = \mathbf{R} + \frac{u}{\Omega}\mathbf{e}_2. \tag{8}$$

Substituting expressions (8) into the Lagrangian (4), we obtain

$$L = m\left\{\frac{\Omega}{B}\mathbf{A}(\mathbf{r},t) + v_{\parallel}\mathbf{b}(\mathbf{r},t) + \mathbf{V}_{\mathrm{E}}(\mathbf{r},t) + u\mathbf{e}_1\right\} \times\left(\dot{\mathbf{R}} + \left(\frac{u}{\Omega}\mathbf{e}_2\right)^{\cdot}\right) - \left\{Ze\Phi(\mathbf{r},t) + \frac{m}{2}|v_{\parallel}\mathbf{b}(\mathbf{r},t) + \mathbf{V}_{\mathrm{E}}(\mathbf{r},t) + u\mathbf{e}_1|^2\right\}. \quad (9)$$

Next, we use the notions concerning the orders of magnitude of the quantities, which are standard for the drift theory: $(\boldsymbol{\rho}_{\mathrm{L}}\cdot\nabla)\mathbf{C} \sim \varepsilon\mathbf{C}$, $\varepsilon \ll 1$, where $\mathbf{C}$ is an arbitrary slowly varying vector. We average expression (9) over the cyclotron period, and determine all the quantities at the point of the guiding center position up to the second-order in ε:

$$\mathbf{C}(\mathbf{r}) = \mathbf{C}(\mathbf{R}) + \rho_{\mathrm{L}i}\frac{\partial\mathbf{C}(\mathbf{R})}{\partial R_i} + \frac{1}{2}\rho_{\mathrm{L}i}\rho_{\mathrm{L}j}\frac{\partial^2\mathbf{C}(\mathbf{R})}{\partial R_i\partial R_j} + O(\varepsilon^3).$$

Finally, we obtain:

$$\begin{aligned} L = {} & m\left\{\frac{\Omega}{B}\mathbf{A} + v_{\parallel}\mathbf{b} + \mathbf{V}_{\mathrm{E}}\right\}\cdot\dot{\mathbf{R}} \\ & + \dot{\left(\frac{mu}{\Omega}\right)}\left\{\frac{u}{2B}(\nabla\cdot\mathbf{A} - \mathbf{b}\cdot(\mathbf{b}\cdot\nabla)\mathbf{A}) + \frac{v_{\parallel}u}{2\Omega}\nabla\cdot\mathbf{b} + \frac{u}{2\Omega}(\nabla\cdot\mathbf{V}_{\mathrm{E}} - \mathbf{b}\cdot(\mathbf{b}\cdot\nabla)\mathbf{V}_{\mathrm{E}})\right\} \\ & - \left(\frac{mu^2}{2\Omega}\right)\left\{1 - \frac{v_{\parallel}}{\Omega}\mathbf{b}\cdot\nabla\times\mathbf{b} - \frac{1}{\Omega}\mathbf{b}\cdot\nabla\times\mathbf{V}_{\mathrm{E}}\right\}\dot{\theta} \\ & - Ze\left(\Phi + \frac{u^2}{4\Omega^2}\nabla^2\Phi\right) + \frac{m}{2}\left(v_{\parallel}^2 + V_{\mathrm{E}}^2 + u^2 \times\left\{1 - \frac{v_{\parallel}}{2\Omega}\mathbf{b}\cdot\nabla\times\mathbf{b} - \frac{1}{2\Omega}\mathbf{b}\cdot\nabla\times\mathbf{V}_{\mathrm{E}}\right\}\right). \end{aligned} \quad (10)$$

In expression (10), all the smoothed quantities are determined as functions of coordinates at the point of the guiding center position $\mathbf{R}$.

Expression (10) includes the term with $\nabla\cdot\mathbf{A} - \mathbf{b}\cdot(\mathbf{b}\cdot\nabla)\mathbf{A}$, which can be vanished by choosing the corresponding calibration of the vector potential $\mathbf{A}$, as it is conventionally done in classical electrodynamics.[1] Then the term with $d(mu/\Omega)/dt$ becomes the second-order quantity. With an accuracy up to the zero-order terms, the Lagrange function (10) can be written in the desired form

$$L = \frac{Ze}{c}\mathbf{A}^*\cdot\dot{\mathbf{R}}_{\mathrm{d}} - \frac{m^2c}{2Ze}\mu\dot{\theta} - h. \quad (11)$$

Here,

$$\mathbf{A}^* = \mathbf{A} + \frac{mc}{Ze}(v_{\parallel}\mathbf{b} + \mathbf{V}_{\mathrm{E}}) \quad (12)$$

is the modified vector potential, $\mu = u^2/B$ is the magnetic moment, and h is the smoothed Hamilton function of the charged particle, determined by the following expression:

$$h = Ze\Phi + \frac{m}{2}\left(v_{\parallel}^2 + V_{\mathrm{E}}^2 + \mu B\right),$$

which coincides with function (3) with an accuracy up to the terms of the retained order. The Lagrange function (11) generalizes the well-known Littlejohn's Lagrangian [3, 4] to the case of the strong transverse electric field $\mathbf{E}_{\perp} = O(1)$.

The first term in expression (11) coincides in form[2] with the Lagrangian (1) obtained by Morozov and Solov'ev with the additional correction to the modified vector potential associated with $\mathbf{V}_E$. As shown in [2], in the case of $\mathbf{b}\cdot\nabla\times\mathbf{b} = 0$, the equation of motion for the guiding center can be represented in the following form[3]

$$[\dot{\mathbf{R}}\times\mathbf{B}^*] = 0, \quad (13)$$

which formally coincides with the equation of motion for a particle with negligible mass moving in some fictitious magnetic field $\mathbf{B}^* = \nabla\times\mathbf{A}^*$. It follows from Eq. (13) that the guiding center trajectory lies along the $\mathbf{B}^*$ field lines, i.e.,

$$\dot{\mathbf{R}} = \frac{v_{\parallel}}{B}\mathbf{B}^*. \quad (14)$$

The generalization of Eq. (14) to the case of $\mathbf{b}\cdot\nabla\times\mathbf{b} \neq 0$ and $\mathbf{b}\cdot\nabla\times\mathbf{V}_{\mathrm{E}} \neq 0$ can be done by replacing B by $B_{\parallel}^* = \mathbf{b}\cdot\mathbf{B}^*$ in the denominator. In the framework of the classical drift theory, the strict derivation of Eq. (14), generalized to the case of $\mathbf{b}\cdot\nabla\times\mathbf{b} \neq 0$, was performed by Boozer in [11].

Using the Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad (15)$$

[1] An alternative method for deriving the expression (11) is based on the fundamental property of invariance of the equations of motion with respect to the substitution $L \to L + dF/dt$, where F is an arbitrary scalar function. The choice of such a function in the form of $F(\mathbf{R},t) = -\frac{mu^2}{2\Omega B}\mathbf{e}_2\cdot(\mathbf{e}_2\cdot\nabla)\mathbf{A}$ results in the cancellation of "unnecessary" terms in expression (9).

[2] In Eq. (12), the $v_{\parallel}$ and $\mathbf{R}$ variables are considered to be independent, and, in the subsequent derivation of the drift equations of motion, their independent variation is used; to derive the drift equations from expression (1) for the Lagrangian L^*, it is necessary to consider $v_{\parallel}$ as a function of $\mathbf{R}$: $v_{\parallel} = v_{\parallel}(\mathbf{R})$.

[3] In the case of the strong electric field, one should also set $\mathbf{b}\cdot\nabla\times\mathbf{V}_{\mathrm{E}} = 0$.

we obtain the extended set of drift equations in the strong electric field. Substituting the generalized coordinates $q_i = v_\parallel, \mu, \theta$ and the Lagrangian (11), which is a function of six independent variables $L = L(\mathbf{R}, v_\parallel, \mu, \theta)$, into Eq. (15), we obtain

$$\mathbf{b} \cdot \dot{\mathbf{R}} = v_\parallel, \quad \dot{\theta} = -\Omega, \quad \dot{\mu} = 0. \tag{16}$$

From the first equation (at $q_1 = v_\parallel$), it follows that the product $\mathbf{b} \cdot \dot{\mathbf{R}}$ is equal to the projection of the particle velocity $v_\parallel$ on the magnetic field direction at the point of the guiding center position. The second equation (at $q_2 = \mu$) shows that, with an accuracy of up to the zero-order terms, the time derivative of the Larmor rotation phase θ coincides with the cyclotron frequency: $\dot{\theta} = -\Omega + O(\varepsilon)$. And finally, the third equation (at $q_3 = \theta$) provides the conservation of the magnetic moment μ.

Next, for $q_i = R_i$, with an accuracy of up to the first-order terms, we obtain

$$Ze\mathbf{E}^* + \frac{Ze}{c}[\dot{\mathbf{R}} \times \mathbf{B}^*] = \frac{m}{2}(\nabla V_{\mathrm{E}}^2 + \mu \nabla B), \tag{17}$$

where

$$\mathbf{E}^* = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}^*}{\partial t} = \mathbf{E} - \frac{m}{Ze}\frac{\partial}{\partial t}(v_\parallel \mathbf{b} + \mathbf{V}_{\mathrm{E}}). \tag{18}$$

Taking into account the $\mathbf{b} \cdot \dot{\mathbf{R}} = v_\parallel$ relation, we perform the vector multiplication of Eq. (17) by the $\mathbf{b}$ vector and obtain the equation for $\dot{\mathbf{R}}$ in the following form:

$$\begin{aligned}\dot{\mathbf{R}} = \frac{1}{B_\parallel^*}\bigg(& v_\parallel \mathbf{B}^* + c[\mathbf{E}^* \times \mathbf{b}] \\ &+ \frac{mc}{2Ze}[\mathbf{b} \times (\nabla V_{\mathrm{E}}^2 + \mu \nabla B)]\bigg).\end{aligned} \tag{19}$$

Here, $B_\parallel^* = \mathbf{b} \cdot \mathbf{B}^*$. By expanding the $1/B_\parallel^*$ expression in powers of ε in Eq. (19), with an accuracy up to the first-order terms, the expression for $\dot{\mathbf{R}}$ can be rewritten in a more conventional form of the sum of drift terms [8]:

$$\begin{aligned}\dot{\mathbf{R}} &= v_\parallel \mathbf{b} + \mathbf{V}_{\mathrm{E}} + \frac{u^2}{2\Omega}\frac{[\mathbf{b} \times \nabla B]}{B} + \frac{v_\parallel^2}{\Omega}[\mathbf{b} \times (\mathbf{b} \cdot \nabla)\mathbf{b}] \\ &+ \frac{c}{B\Omega}\left(\frac{\partial \mathbf{E}_\perp}{\partial t} + v_\parallel (\mathbf{b} \cdot \nabla)\mathbf{E}_\perp + (\mathbf{V}_{\mathrm{E}} \cdot \nabla)\mathbf{E}_\perp\right)_\perp \\ &+ \frac{v_\parallel}{\Omega}\left([\mathbf{b} \times (\mathbf{V}_{\mathrm{E}} \cdot \nabla)\mathbf{b}] - \frac{c\mathbf{E}_\perp}{B^2}\mathbf{b} \cdot \nabla B + \left[\mathbf{b} \times \frac{\partial \mathbf{b}}{\partial t}\right]\right) \\ &- \frac{c\mathbf{E}_\perp}{\Omega B^2}\left(\mathbf{V}_{\mathrm{E}} \cdot \nabla B + \frac{\partial B}{\partial t}\right).\end{aligned} \tag{20}$$

In Eq. (20), the first line is well known from the classical drift theory [6, 12]. The rest terms are proportional to $\varepsilon V_{\mathrm{E}}/v$ and $\varepsilon (V_{\mathrm{E}}/v)^2$, and they are absent in equations of the standard drift theory due to the assumption $V_{\mathrm{E}} \ll v$. In [6], these terms were obtained in the following compact form: $[\mathbf{b} \times (\partial \mathbf{v}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{v})]/\Omega$, where $\mathbf{v} = v_\parallel \mathbf{b} + \mathbf{V}_{\mathrm{E}}$, which is formally equivalent to the component of the hydrodynamic equation of cold plasma motion transverse to the magnetic field $\mathbf{B}$. In this case, the corrections to the mass velocity $\mathbf{V}$ associated with the motion of such plasma in the strong electric field can be also obtained from the hydrodynamic equations by means of expanding the convective term $(\mathbf{V} \cdot \nabla)\mathbf{V}$ in powers of $1/\Omega$. In Eq. (20), the second line describes the drift motions associated with the changes in the electric field: the first term describes the known polarization drift in the nonstationary field $\mathbf{E}(t)$, and two additional terms are associated with the electric field inhomogeneity $\mathbf{E}(\mathbf{r})$ along the directions of the magnetic field and electric drift. The terms in the third and fourth lines describe the drift motions caused by the inhomogeneity and nonstationarity of the magnetic field. We note that the equation of motion of the guiding center taking into account the effects of the strong electric field was used to construct the quasi-hydrodynamic models describing the dynamics of the strongly rarefied plasma in the magnetic field [13, 14]. The case of particle motion in the weak electric field but the strongly inhomogeneous magnetic field was considered in [15].

To close the set of equations of motion of the guiding center, below we obtain the equation for $v_\parallel$. To do this, we perform the scalar multiplication of expression (17) by $\mathbf{B}^*$. Taking into account Eq. (18), we obtain

$$\frac{dv_\parallel}{dt} = \frac{\mathbf{B}^*}{B_\parallel^*}\left\{\frac{Ze}{m}\mathbf{E} - \left(v_\parallel \frac{\partial \mathbf{b}}{\partial t} + \frac{\partial \mathbf{V}_{\mathrm{E}}}{\partial t}\right) - \frac{1}{2}(\nabla V_{\mathrm{E}}^2 + \mu \nabla B)\right\},$$

which, by analogy with Eq. (19), can be reduced to the following standard form:

$$\frac{dv_\parallel}{dt} = \frac{Ze}{m}\mathbf{b} \cdot \mathbf{E} + \frac{\mu B}{2}\nabla \cdot \mathbf{b} + \mathbf{V}_{\mathrm{E}} \cdot \frac{d\mathbf{b}}{dt}. \tag{21}$$

Equations (16), (20), and (21), form the desired set of equations describing the particle drift in the strong electric field.

Using Eq. (21), it is easily to obtain the expression describing the change of the kinetic energy $K = m(v_\parallel^2 + \mu B + V_{\mathrm{E}}^2)/2$, which, as in [6], has the following form:

$$\frac{dK}{dt} = Ze(\dot{\mathbf{R}} \cdot \mathbf{E}) + \frac{m\mu B}{2}\frac{\partial B}{\partial t}. \tag{22}$$

In the static fields $\partial(E, B)/\partial t = 0$, the law of conservation of energy follows from Eq. (22)

$$K + Ze\Phi = \mathrm{const}.$$

3. EXAMPLES OF DRIFT MOTION IN STRONG ELECTRIC FIELD

The effect of the strong electric field on the character of the drift motion of charged particles is studied numerically. The field configuration with the homogeneous magnetic field $\mathbf{B}$ = const and the inhomogeneous electric field $\mathbf{E} = \mathbf{E}(\mathbf{r})$ and the configuration with the inhomogeneous magnetic field $\mathbf{B} = \mathbf{B}(\mathbf{r})$ and the homogeneous electric field $\mathbf{E}$ = const are considered by turns. We choose the strictly transverse directions of the fields ($\mathbf{E} \perp \mathbf{B}$). For each configuration, the exact trajectories are calculated for the positively charged ($Z > 0$) particle $\mathbf{r}$ and its guiding center. The latter trajectory is determined by subtracting the current value of the Larmor rotation vector from the particle radius-vector, $\mathbf{r} - \boldsymbol{\rho}_L$. The resulting trajectories are compared with the results of numerical calculations of the guiding center trajectories $\mathbf{R}_{cl}$ (Eq. (20) without terms describing drifts in strong electric field) and $\mathbf{R}$ (the complete Eq. (20)). The particle velocity components involved in Eq. (20) are calculated using the exact equations of motion.

As the natural normalization constants for the spatial and temporal scales of the problem, we use the Larmor radius ($\tilde{\mathbf{r}} = \mathbf{r}/\rho_{L0} = \mathbf{r}ZeB_0/(v_0 mc)$) and the cyclotron period ($\tilde{t} = t2\pi\Omega_0 = t2\pi ZeB_0/(mc)$), which are calculated at the initial time $t = 0$. The electric field is set using the dimensionless parameter $\varkappa = cE_0/v_0B_0$ (E_0 is the electric filed at the initial point of particle location), which naturally arises within the chosen normalization. In the strictly transverse fields, this parameter is equal to V_{E0}/v_0. Further, the symbol "tilde" marking the dimensionless quantities is omitted.

3.1. Configuration with the Inhomogeneous Electric Field

In the inhomogeneous electric and homogeneous magnetic fields, Eq. (20) can be reduced to

$$\dot{\mathbf{R}} = v_\parallel \mathbf{b} + \mathbf{V}_E + \frac{c}{B\Omega}\left(v_\parallel (\mathbf{b}\cdot\nabla)\mathbf{E}_\perp + (\mathbf{V}_E\cdot\nabla)\mathbf{E}_\perp\right)_\perp,$$

and contains two additional terms associated with the strong electric field $\mathbf{E}(\mathbf{r})$.

We consider a quadrupole-lens-type configuration with a potential $\Phi = -(E_0/r_0)xy$ (here, $r_0 = \sqrt{x_0^2 + y_0^2}$ is the initial particle location) in combination with the homogeneous magnetic field $\mathbf{B} = B_0\mathbf{e}_z$ (B_0 = const) (see Fig. 1a). Examples of the trajectories of the charged particle and its guiding center in the considered system are shown in Fig. 1b. Calculated without the additional drifts in the electric field, the guiding center trajectory $\mathbf{R}_{cl}$ lies along the equipotential lines Φ = const and, with time, it asymptotically approaches the x-axis. Under the effect of the additional drift $\sim(\mathbf{V}_E\cdot\nabla)\mathbf{E}_\perp$ (associated with the effect of the centrifugal force $\mathbf{F}_c = -m(\mathbf{V}_E\cdot\nabla)\mathbf{V}_E$), one of the components of which is directed along the electric field $\mathbf{E}$, the particle guiding center $\mathbf{R}$ intersects the x-axis. The other component of the additional drift motion is co-directed with the electric drift $\mathbf{V}_E$; therefore, in equal times, the particle guiding centre $\mathbf{R}$ is displaced to a longer distance along the x-axis than $\mathbf{R}_{cl}$. The performed calculations demonstrate good agreement between the trajectory of the guiding center $\mathbf{R}$ and the exact trajectory of the charged particle (see Fig. 1b).

The interesting effect appears in the considered configuration is associated with an increase in the particle energy due to its drift in the crossed fields along the electric field $\mathbf{E}$. In accordance with expression (22), at $\mathbf{B}$ = const and $v_\parallel = 0$, the change of the kinetic energy can be described by the following equation:

$$\frac{dK}{dt} = \frac{mc^2}{B^2}\mathbf{E}_\perp\cdot(\mathbf{V}_E\cdot\nabla)\mathbf{E}_\perp.$$

The mentioned effect cannot be described within the framework of the classical drift approximation, since $Ze(\mathbf{E}_\perp\cdot\dot{\mathbf{R}}_{cl}) = Ze(\mathbf{E}\cdot\mathbf{V}_E) \equiv 0$. Figure 2 shows time variation of the particle energy along the trajectory shown in Fig. 1b. As easily seen, even a slight displacement of the guiding center $\mathbf{R}$ along the electric field (displacement of the order of several Larmor radii) leads to increase of kinetic energy of the particle by an order of magnitude.

3.2. Configuration with the Inhomogeneous Magnetic Field

Now we consider the charged particle motion in the configuration of the magnetic field $\mathbf{B} = \mathbf{e}_\varphi B_0 r_0/r$, formed by the straight current, and the uniform electric field $\mathbf{E} = -E_0\mathbf{e}_z$ (E_0 = const) (see Fig. 3a). We use the cylindrical coordinate system $\{r, \varphi, z\}$. In this geometry, Eq. (20) can be reduced to the form:

$$\dot{\mathbf{R}} = v_\parallel\mathbf{b} + \mathbf{V}_E + \frac{u^2/2 + v_\parallel^2}{\Omega}\frac{[\mathbf{b}\times\nabla B]}{B} - \frac{c^2}{\Omega B^3}\mathbf{E}_\perp(\mathbf{E}_\perp\cdot[\mathbf{b}\times\nabla B]),$$

from which it follows that the motion of the charged particle guiding center across the magnetic field is composed of the electric drift, toroidal drift (the sum of the gradient and centrifugal drifts) and the additional drift in the strong electric field, which occurs in the case of $\mathbf{V}_E\cdot\nabla B \neq 0$. Since the additional "electric" drift has a quadratic dependence on the electric field vector, its direction is always opposite to the toroidal drift direction. Obviously, when the condition

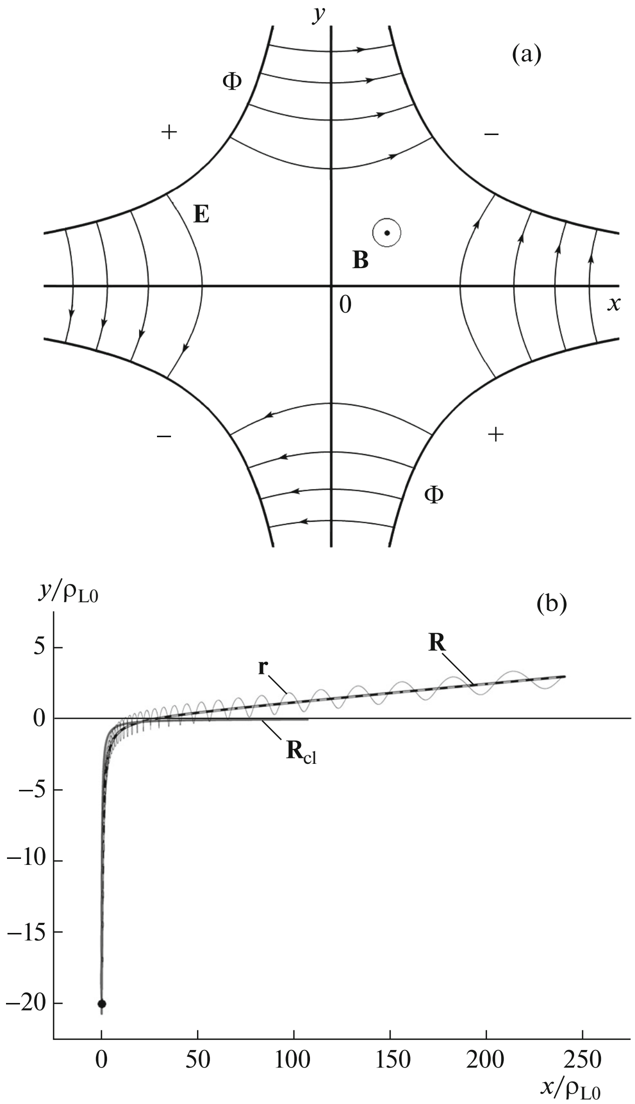

Fig. 1. (a) Field configuration of a quadrupole electrostatic lens with the axial magnetic field, and (b) trajectories in the considered configuration: **r** is the particle trajectory; **R** and $\mathbf{R}_{cl}$ are the trajectories of the guiding center calculated with and without the account of the additional drift motions in the electric field, respectively. The dashed line corresponds to the exact trajectory of the guiding center; the black circle corresponds to the initial particle location ($x = 0$, $y = -20\rho_{L0}$). Here, $v_\parallel = 0$, $v_\perp = v_0$: $v_x = v_y = v_0/\sqrt{2}$, $\varkappa = 0.5$.

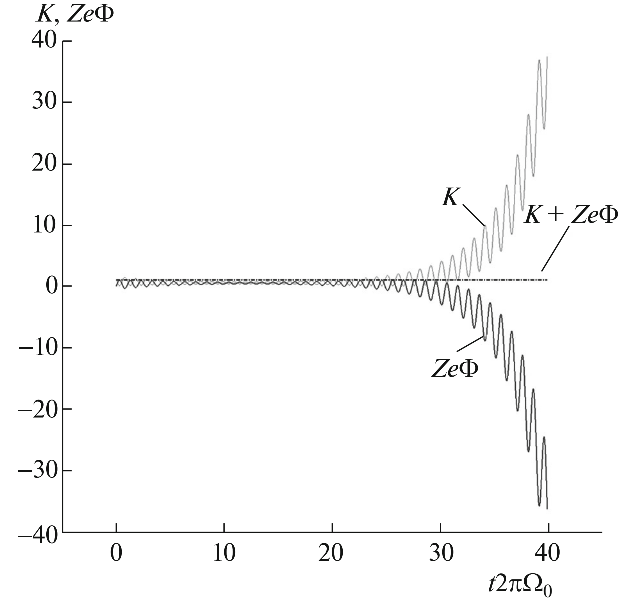

Fig. 2. Time dependences of the particle kinetic K and potential $Ze\Phi$ energies normalized to the initial energy. The dependences are calculated in the same time interval as the trajectories shown in Fig. 1b.

$u^2 + 2v_\parallel^2 = 2V_E^2/B$ is true, the toroidal drift can be fully compensated. At $v_\parallel = 0$, from the condition of conservation of the magnetic moment, it is easy to find the critical point r_{crit}, at which $\dot{R}_z = 0$

$$r_{crit} = r_0\left(\frac{\mu B_0}{2V_{E0}^2}\right)^{1/3}.$$

The example of trajectories of the charged particle and its guiding center at $v_\parallel = 0$ and $\varkappa = 0.2$ is shown in Fig. 3b. Without additional drift the particle guiding center $\mathbf{R}_{cl}$ is displaced along the r and z-axes due to the electric and toroidal drifts, respectively. In fact, after passing the critical point $r = r_{crit}$, the particle starts drifting in the direction of the electric field, which is opposite to the toroidal drift direction. This is confirmed by the results of numerical calculations (see curves for **R** and **r** in Fig. 3b). In turn, the guiding center displacement along the electric field results in the changes in the particle kinetic energy, in accordance with the following law:

$$\frac{dK}{dt} = m\left[\frac{u^2}{2} + v_\parallel^2 - V_E^2\right](\mathbf{V}_E \cdot \nabla B).$$

The results of calculations of the kinetic energy time evolution at different electric fields E_0 (different parameters $\varkappa$) are shown in Fig. 4a. It can be seen that, in the weak electric fields ($\varkappa = 0.05$ and 0.1), for a given time interval, the particle loses energy, as it moves against the field. At $\varkappa = 0.2$, the particle also slows down moving within the interval $r \in [r_0, r_{crit})$, but after passing the critical point $r = r_{crit}$, at which $dK/dt = 0$, it starts accelerating (see Fig. 3b). The trajectories of the guiding center calculated at different parameters $\varkappa$ are shown in Fig. 4b.

In [16], based on the effect of "drift acceleration" (changes in the particle kinetic energy occurring during its drift in the direction of the electric field), in

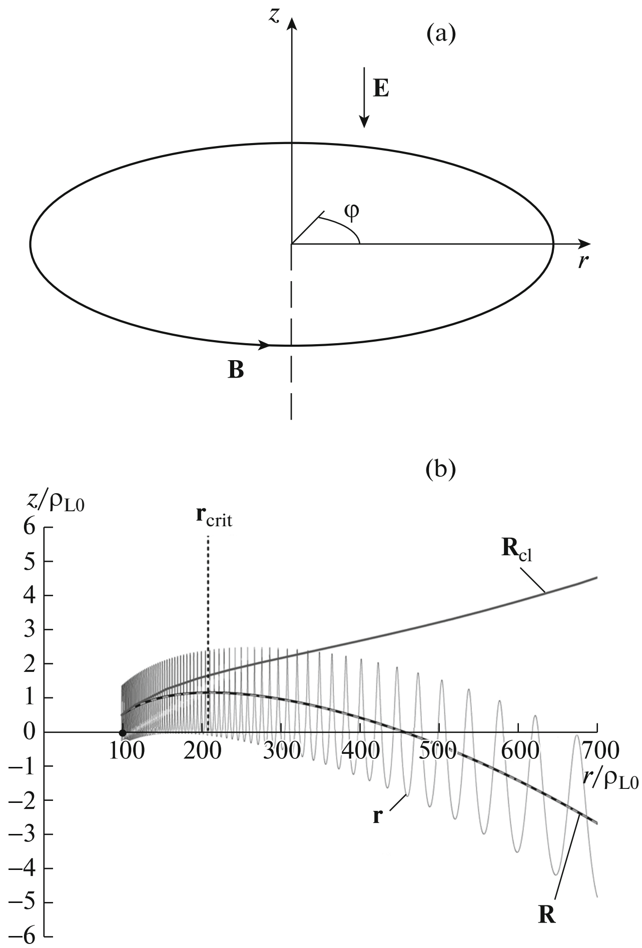

Fig. 3. (a) Field configuration formed by the magnetic field of straight current and the axial electric field; (b) trajectories in the $(r-z)$ plane: $\mathbf{r}$ is the particle trajectory; $\mathbf{R}$ and $\mathbf{R}_{cl}$ are the trajectories of the guiding center calculated with and without the account of the additional drift motions in the electric field, respectively. The dashed line corresponds to the exact trajectory of the guiding center; the black circle corresponds to the initial particle location ($r = 100\rho_{L0}$, $z = 0$). Here, $v_{\|} = 0$, $v_{\perp} = v_0$: $v_r = v_z = v_0/\sqrt{2}$, $\varkappa = 0.2$.

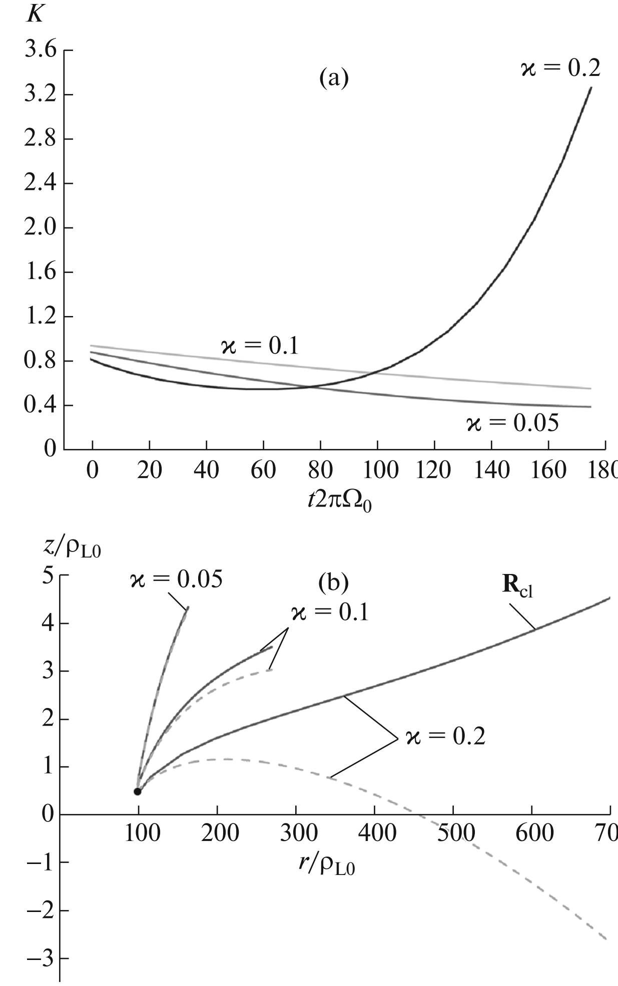

Fig. 4. (a) Time dependences of the smoothed particle kinetic energy K corresponding to the guiding center trajectories shown in Fig. 4b; (b) trajectories of the particle guiding center in the $(r-z)$ plane in the magnetic field of straight current and the axial electric field: the solid lines correspond to the classical trajectories of the drift center calculated without the account of the additional drift motions in the electric field; the dashed lines correspond to the exact trajectories of the guiding center; the black circle corresponds to the initial particle location ($r = 100\rho_{L0}$, $z = 0$). Here, $v_{\|} = 0$, $v_{\perp} = v_0$: $v_r = v_z = v_0/\sqrt{2}$, $\varkappa = 0.05$, $\varkappa = 0.1$, $\varkappa = 0.2$.

the field configuration similar to that considered above, the recuperation scheme was proposed for conversion of the thermal plasma energy into the electric field energy. The results of [16] were based on the model of the classical drift theory, in the framework of which the sign of the energy increment is predetermined by the directions of the fields, i.e., $dK/dt < 0$ at $\mathbf{V}_E \cdot \nabla B < 0$ and vice versa. When the additional drift motions are taken into account, the radial size of the spatial region, within which the particles lose their energy, is reduced to the r_{crit} value. In the case of the sufficiently strong electric field $|V_E| \sim 0.1|v_0|$, its order-of-magnitude estimate is $\sim 10^2 \rho_L$.

4. CONCLUSIONS

The additional terms (as compared to the equation of the classical drift theory) in the equation (20) of the drift motion of the particle guiding center were analyzed. It was shown that the drift motion of a nonrelativistic charged particle in the crossed slowly varying magnetic and strong electric fields differs from that predicted by the classical drift theory. This motion has the specific components, which can considerably contribute to the resultant particle motion. The additional drift motions are caused by the field inhomogeneity in the $\mathbf{B}$ and $\mathbf{V}_E$ directions, and it is necessary to take them into account, if the velocity of the $\mathbf{E}\times\mathbf{B}$ drift is comparable to the total particle velocity. The generalized Littlejohn's Lagrangian (10) was obtained, and with its help, the set of equations (16), (20), and (21) of the extended drift theory was derived.

The distinctive features of the particle drift motion in some model crossed field configurations were studied numerically. It was shown that the dynamics of the guiding center motion in the strong electric field considerably differs from that for the usual $\mathbf{E}\times\mathbf{B}$ drift. The distinctive feature of the drift motion in the strong electric field is the presence of the drift velocity component directed along the electric field $\mathbf{E}_\perp$. In turn, such a displacement results in the changes in the particle kinetic energy, since $\dot{\mathbf{R}}\cdot\mathbf{E}_\perp \neq 0$. Obviously, such a "drift" acceleration in the constant magnetic field is not described by the classical drift theory, in the framework of which $\mathbf{V}_E\cdot\mathbf{E}\equiv 0$. It is important to note that, since the additional drift motions are caused by the electric and magnetic field inhomogeneities, they depend both on the charge and mass of the charged particles and, thus, they are able to contribute to the generation of current in plasma, as well as the well known polarization current.

FUNDING

The work was supported in part by the Ministry of Education and Science of the Russian Federation (contract no. 3.2223.2017/4.6, Section 3) and the RUDN University Program 5-100 (Section 2).

REFERENCES

1. L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics, Vol. 1, Mechanics* (Nauka, Moscow, 1988; Butterworth-Heinemann, Oxford, 1988).
2. A. I. Morozov and L. S. Solov'ev, Sov. Phys.—Doklady **4**, 1031 (1959).
3. R. G. Littlejohn, J. Plasma Phys. **29**, 111 (1983). https://doi.org/10.1017/S002237780000060X
4. J. R. Cary and A. J. Brizard, Rev. Mod. Phys. **81**, 693 (2009). https://doi.org/10.1103/RevModPhys.81.693
5. A. I. Morozov, in *Plasma Accelerators,* Ed. by L. A. Artsimovich, S. D. Grishin, G. L. Grozdovskii, L. V. Leskov, A. I. Morozov, A. M. Dorodnov, V. G. Padalka, and M. I. Pergament (Mashinostroenie, Moscow, 1973), p. 85 [in Russian].
6. A. I. Morozov and L. S. Solov'ev, in *Reviews of Plasma Physics,* Ed. by M. A. Leontovich (Consultants Bureau, New York, 1966), Vol. 2, p. 201.
7. T. G. Northrop, *The Adiabatic Motion of Charged Particles* (Wiley Interscience, New York, 1963).
8. V. I. Ilgisonis, *Classical Problems in the Physics of Hot Plasmas* (Izd. dom MEI, Moscow, 2015) [in Russian].
9. T. Charoy, J. P. Boeuf, A. Bourdon, J. A. Carlsson, P. Chabert, B. Cuenot, D. Eremin, L. Garrigues, K. Hara, I. D. Kaganovich, A. T. Powis, A. Smolyakov, D. Sydorenko, A. Tavant, O. Vermorel, et al., Plasma Sources Sci. Technol. **28**, 105010 (2019). https://doi.org/10.1088/1361-6595/ab46c5
10. J. P. Boeuf and L. Garrigues, Phys. Plasmas **25**, 061204 (2018). https://doi.org/10.1063/1.5017033
11. A. H. Boozer, Phys. Fluids **23**, 904 (1980). https://doi.org/10.1063/1.863080
12. D. V. Sivukhin, in *Reviews of Plasma Physics,* Ed. by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 1.
13. L. I. Rudakov and R. Z. Sagdeev, in *Plasma Physics and the Problem of Controlled Thermonuclear Reactions,* Ed. by M. A. Leontovich (Pergamon, New York, 1959), Vol. 3, p. 321.
14. T. F. Volkov, in *Reviews of Plasma Physics,* Ed. by M. A. Leontovich (Consultants Bureau, New York, 1968), Vol. 4, p. 1.
15. K. Sh. Khodzhaev, A. G. Chirkov, and S. D. Shatalov, J. Appl. Mech. Tech. Phys. **22**, 439 (1981).
16. A. V. Timofeev, Plasma Phys. Rep. **4**, 464 (1978).

Translated by I. Grishina