Shallow water equations & Poincaré waves

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Shallow water equations & Poincaré waves

- fundamental in fluid dynamics
- thin fluid layer compared to its horizontal extension
- Poincaré waves: frictionless and Coriolis dependent nature
- Subsets of solutions : Rossby waves, Kelvin waves, Inertia-gravity waves

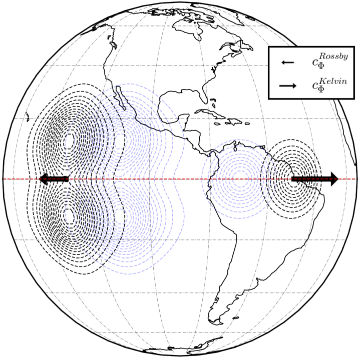


Figure: Earth view of the equatorial domain.

Differential equation system

From first order perturbation we got :

$$\begin{cases} \partial_t u - fv = -g\partial_x h \\ \partial_t v + fu = -g\partial_y h \\ \partial_t h + a_0(\partial_x u + \partial_y v) = 0 \end{cases} \tag{1}$$

- equatorial study to simply the coriolis parameter dependence
- Beta plane approximation : $f = \beta y$
- scale dependence of solutions behavior : study of the zonal wave number

Equatorial solutions

The equatorial study leads at first order to the following solutions, using slow variable : ξ, τ :

$$\begin{cases} v^0(y,\xi,\tau) = \partial_\xi \eta(\xi,\tau) e^{-(1/2)y^2} H_n(y) \\ u^0(y,\xi,\tau) = \eta(\xi,\tau)[\frac{H_{n+1}(y)}{2(1-c)} - \frac{nH_{n-1}(y)}{1+c}] e^{-(1/2)y^2} \\ h^0(y,\xi,\tau) = \eta(\xi,\tau)[\frac{H_{n+1}(y)}{2(1-c)} + \frac{nH_{n-1}(y)}{1+c}] e^{-(1/2)y^2} \end{cases} \tag{2}$$

With H_n the Hermite polynomials and $c = -\frac{1}{2n+1}$ the phase velocity of the n-th mode of propagation. And $\eta(\xi,\tau)$ the envelope function, defined by KDV equation solving.

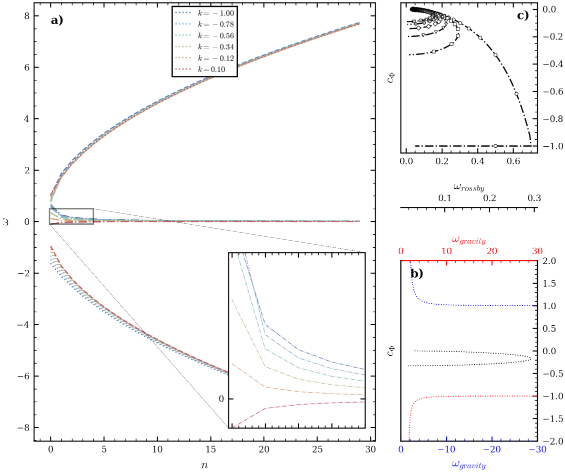
DISPERSION RELATION

The dispersion relation can be expressed by the following :

$$\sigma^3 = \sigma[k^2\epsilon^{-1} + \epsilon^{-1/2}(2n+1)] + k\epsilon^{-1} \tag{3}$$

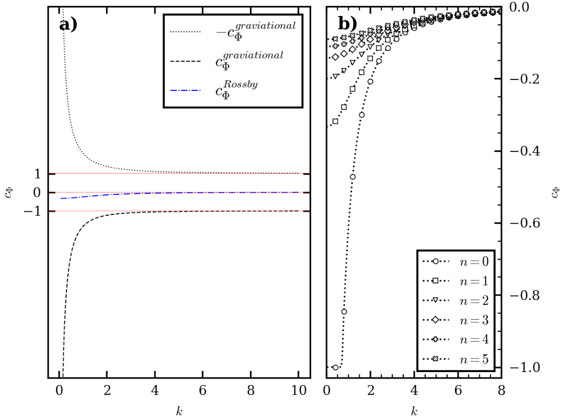
- σ is the dimensionless frequency, k the zonal wave number, ϵ is a constant depending on the system parameter.
- Third order : 3 solutions

Rossby wave frequency study



- low mode weakly dispersive
- 2 types of wave
- $n = 0$ strongly dispersive

Rossby wave spatial study

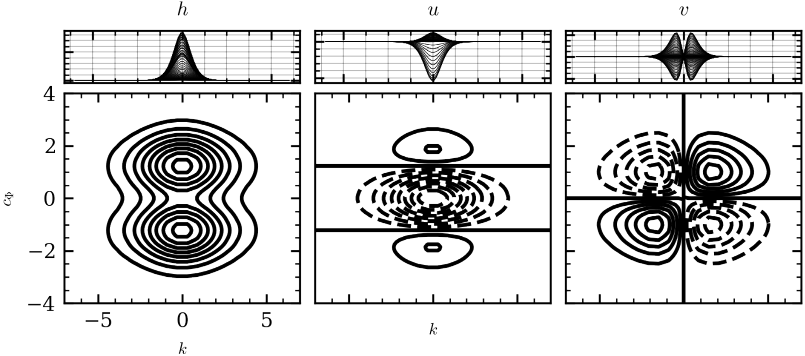


- large scale to have weakly dispersive Rossby waves
- always westward

First mode of rossby wave

For the first mode we got $c_\Phi = -1/3$ and :

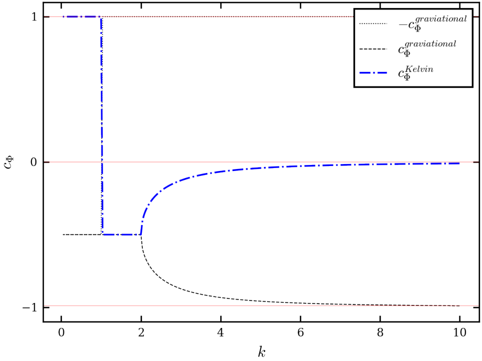
$$\eta(\xi,\tau) = A\mathrm{sech}^2[B(\xi - 0.395B^2\tau)] \quad (4)$$



Kelvin wave

An other type of solution is the Kelvin wave, with $n = -1$ defined by :

$$\begin{cases} u(\xi) = U_{-1} e^{-(1/2)\xi^2} \\ v(\xi) = 0 \\ h(\xi) = U_{-1} \frac{\sigma}{k} e^{-(1/2)\xi^2} \end{cases} \tag{5}$$



Numerical implementation

1. No damping effect
2. Numerical errors
3. soliton is a good way to test the scheme

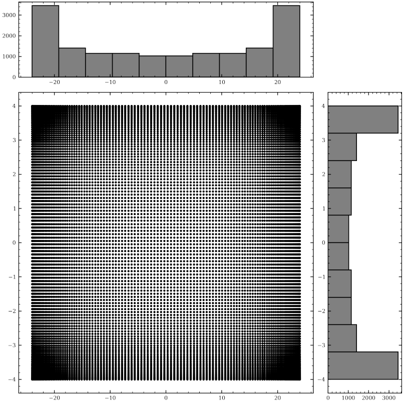
Integration scheme

- Chebyshev spectral method
- domain $[-24, 24] \times [-4, 4]$
- To be in the scope of Boyd study

Spatial discretization

$$x'_{i,j} = (\alpha \cos(\frac{i\pi}{N}), \beta \cos(\frac{j\pi}{N}))$$

To be in the following range : $[-\alpha, \alpha] \times [-\beta, \beta]$. Here we will choose $\alpha = 24$, $\beta = 4$ This gives the following mesh :



Time discretization

leap-frog like method

$$\partial_t u(t) \approx \frac{u(t+\Delta t) - u(t-\Delta t)}{2\Delta t}$$

$$\begin{cases} u^{n+1} = u^{n-1} + 2\Delta t - g\partial_x h^n + f v^n \\ v^{n+1} = v^{n-1} + 2\Delta t - g\partial_y h^n - f u^n \\ h^{n+1} = h^{n-1} - 2\Delta t a_0(\partial_x u^n + \partial_y v^n) = 0 \end{cases} \tag{6}$$

CFL conditions

CFL conditions is strongly impacted by the spectral mesh with irregular spacing

$$C = \Delta t \left(\sum_{i=1}^{n} \frac{u_i}{\Delta x_i} \right) \leq C_{\max}.$$

$$C = \Delta t \left(\frac{u_1}{\Delta x_1} \right) \leq C_{\max}.$$

$$\Delta_x = 1 - \cos(\frac{1}{N}) \approx \frac{1}{N^2}.$$

Hence we have the following condition : $\Delta_t \leq 3 C_{\max} N^{-2}$, we determined $C_{\max} = 5.532$ using many, time and space discretization

Boundary conditions

- Chebyshev spectra methods : cannot use periodic boundary conditions
- Simple Dirichlet conditions
- $h = u = v = 0$

Conservation of mass and energy

- Mass :

$$M = \sum_{i,j} (a_0 + h_{i,j}) \Delta x_i \Delta y_j$$

- Energy :

$$E = \frac{1}{2} \sum_{i,j} (u_{ij}^2 + v_{ij}^2 + ((a_0 + h_{ij}))g)(a_0 + h_{ij}) \Delta x_i \Delta y_j$$

- Conservation of mass and energy for the simulation

Enstrophy Conservation

- Enstrophy conservation : strength of potential vorticity
- $q = \frac{f+(\partial_x v - \partial_y u)}{h}$
- $\Omega = h\frac{q^2}{2}$
- $\Omega = \frac{1}{2}\sum_{i,j} h_{ij} q_{ij}^2 \Delta x_i \Delta y_j$

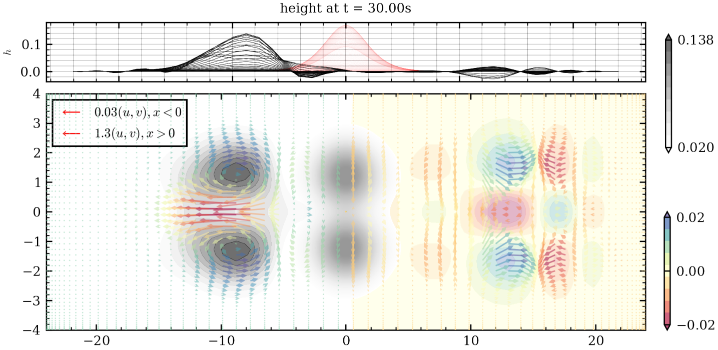
Rossby wave

1. Rossby wave simulation
2. Kelvin wave simulation
3. unstable wave simulation

Simulation parammeters

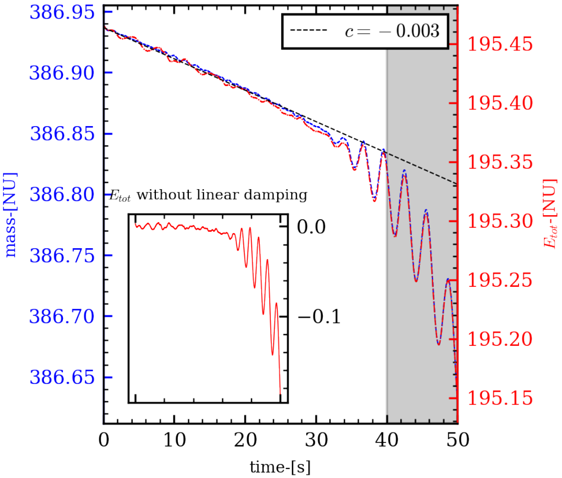
a_0	β	g	N	Δt
1	1	1	64	$2e^{-3}$

Rossby wave



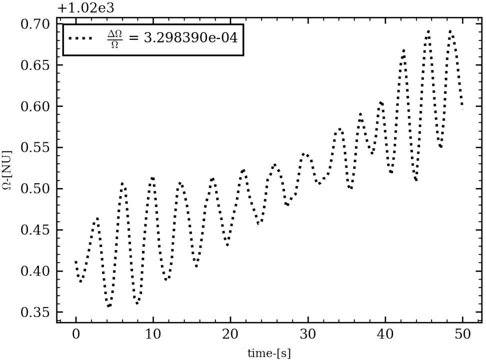
- westward propagation of the soliton
- strongly dispersive instabilities
- periodic shedding
- one can show that these instabilities are barotropic modes

Mass and energy conservation



- quite good conservation ($\sim e^{-4}$)
- linear numerical damping
- sponge layer to avoid effect of Boundaries

Enstrophy Conservation



- Conservative value
- radiative instabilities have great impact in the v-field
- radiative shed are antisymmetric parts of the wave
- oscillations are caused by successive radiations
- constant increasing die to shedding

Phase speed evaluation

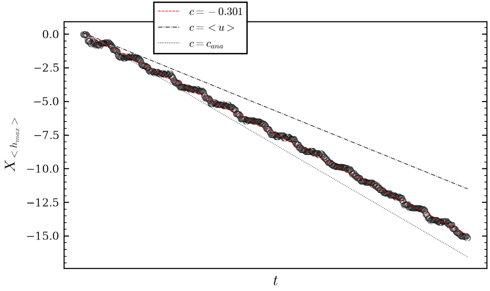


Figure: Phase speeds evaluation

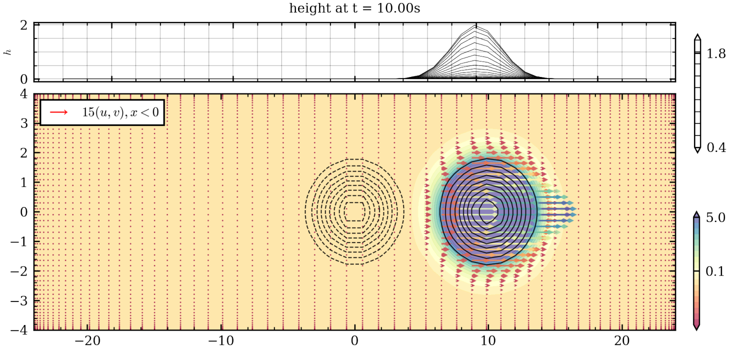
- Position of a maximum window during time
- $c_\Phi = -0.301$
- close to the $\frac{1}{3}$ theoretical one

Kelvin wave

Following wave satisfying the Kelvin equation

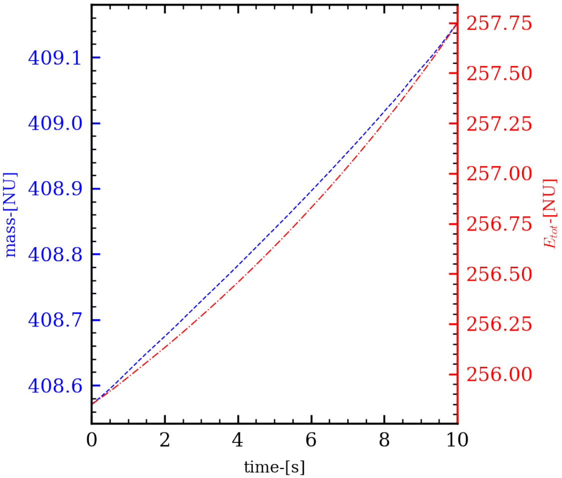
$$\begin{cases} h(x,y,t=0) = H_0 \frac{\sigma}{k} \exp(-((kx)^2 + (ly)^2)) \\ u(x.y,t=0) = H_0 \exp(-((kx)^2 + (ly)^2)) \\ v(x,y,t=0) = 0 \end{cases} \tag{7}$$

Kelvin wave



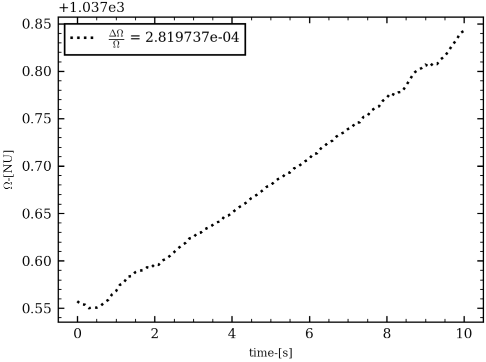
- eastward propagation of the wave
- faster than Rossby waves
- no dispersion
- no north-south velocity field

Energy and mass conservation



- quite conservation : relative conservation of $\sim e^{-3}$
- relative error greater than for Rossby waves (CFL condition)

Enstrophy Conservation



- quite constant $(2e^{-4})$
- no oscillations : no instabilities

Phase velocity & dispersion

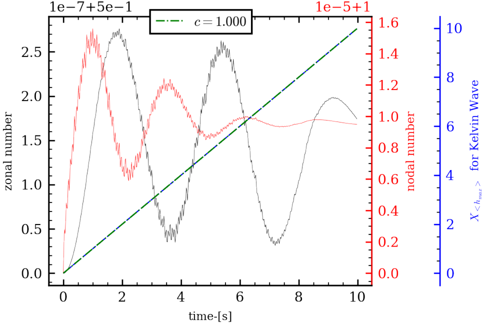


Figure: Phase speed and dispersion evaluation

- Fit of the kelvin wave by an original Kelvin wave
- $c_\Phi = 1.000 = c_{ana}$
- relative dispersion of $1e^{-5}$

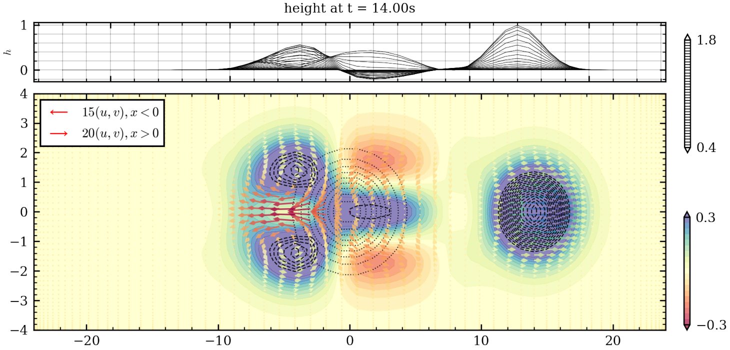
Unstable wave study

Kelvin like wave shape, with velocity, gravity spreading.

$$\begin{cases} h(x,y,t=0) = H_0 \exp(-((0.5x)^2 + (y)^2)) \\ u(x.y,t=0) = 0 \\ v(x,y,t=0) = 0 \end{cases} \tag{8}$$

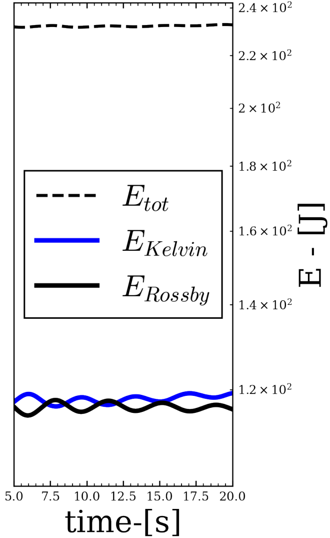
Note that as the gravity is the only force applied at the beginning, the wave should spread equally in every direction, hence the **mass of fluid should be equally distributed** east-west.

Unstable wave study



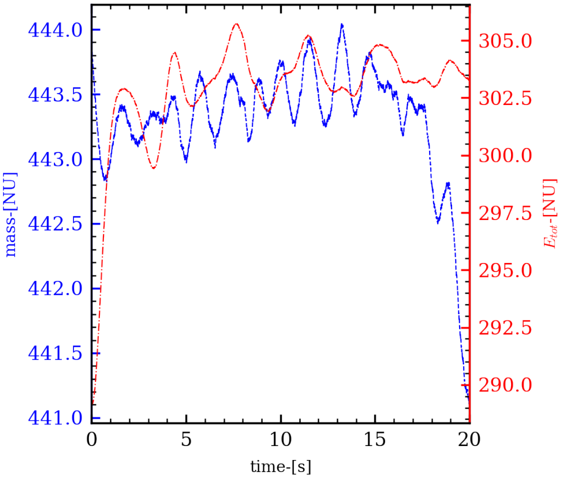
- Rossby soliton $n = 1$ westard
- Kelvin wave eastward
- Radiative instabilities
- energy repartition between modes

Energy repartition between mode



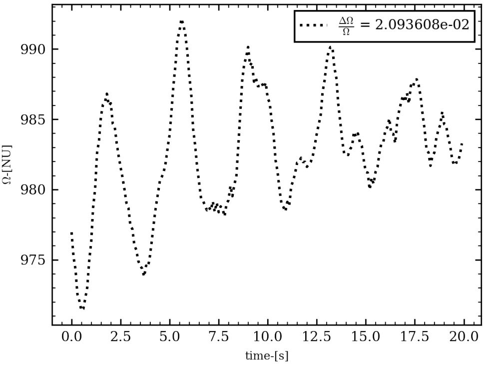
- $V_{\text{Kelvin}} = 2\pi H_0 kl$
- 2 times smaller than the initial unstable wave
- equally distributed Mass, but Kelvin wave is 3 times faster
- $T_{\text{Kelvin}} = 9T_{\text{Rossby}}$.
- $\pi/2$ phase shifted

Mass and Energy conservation



- quite conservation : relative conservation of $6 \sim e^{-3}$
- no dissipation (Rossby decrease, Kelvin increase)
- oscillations correlated with instabilities shed

Enstrophy Conservation



- high relative value $2e^{-}2$
- CLF and instabilities

Phase speed & dispersion

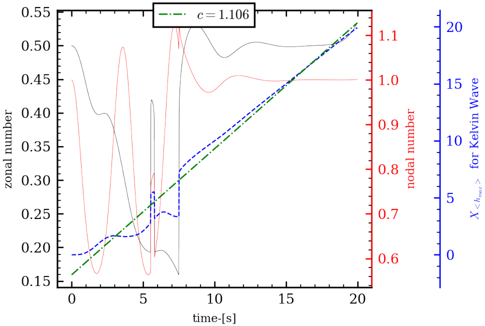


Figure: Phase speed and dispersion evaluation of Kelvin Wave

- Only the height of the wave has changed
- $c_\Phi = 1.106$
- no dispersion at high t (zonal and nodal wave number constant)

Phase speed & dispersion

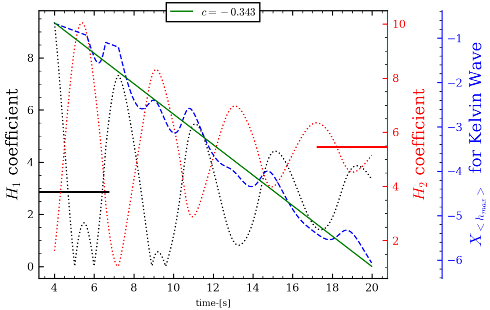


Figure: Phase speed and dispersion evaluation of Rossby Wave

- $c_\Phi = -0.304$
- Hermite coefficient oscillating around a steady state
- stands for variation in the ratio : $\frac{k}{\sigma}$, when instabilities are shed

Conclusion

- Chebyshev and 2 order integration scheme
- Analyses on 3 different types of wave
- Study of modes of propagation
- Using mass, energy and enstrophy conservation
- Study of dissipation and propagation speed of the wave

Perspective

- Study of sponge layer to avoid influence of boundary
- Study of higher modes, and their repartition in energy