Adaptive Debiased SGD in High-dimensional GLMs with Streaming Data

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Big Data

Big data is about

> *"datasets whose size is beyond the ability of typical database software tools to capture, store, manage, and analyze" [Manyika et al., 2011]*
>
> *McKinsey Global Institute*

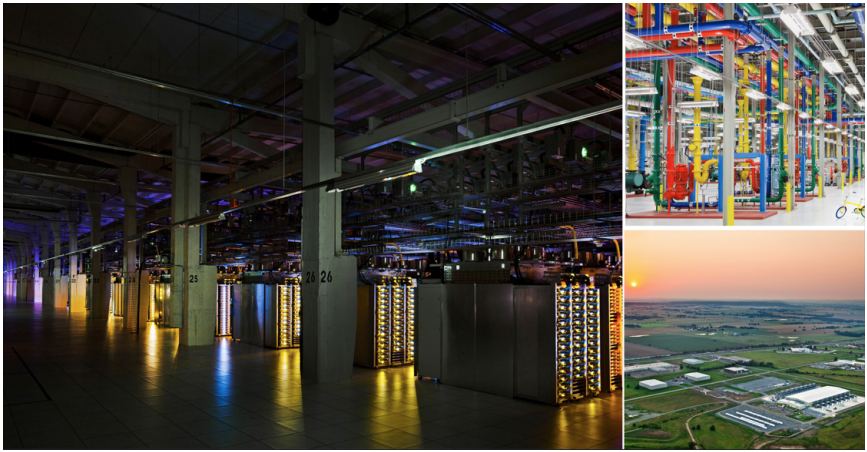


Figure: Google Data Center

Streaming Data

Many applications must process large streams of live data and provide results in near-real-time

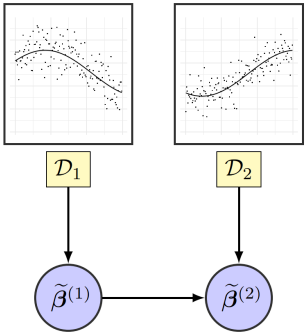
- Algorithmic trading
- Real-time operations management
- Online fraud detection
- Proximity/location tracking
- Intrusion detection systems
- Traffic management
- Real-time recommendations
- Social media/data analytics
- Precision agriculture
- Internet of things
- Gaming data feed
- ...

Some of the major challenges of handling streaming data.

- Historical data is no longer available (needs one pass process).
- Each data has large number of features (high-dimensional).
- Critical memory requirements.
- Limiting computing power.

Streaming Data

Proposed solution: Online learning



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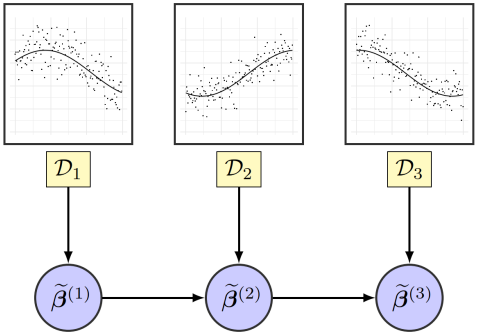

Streaming Data

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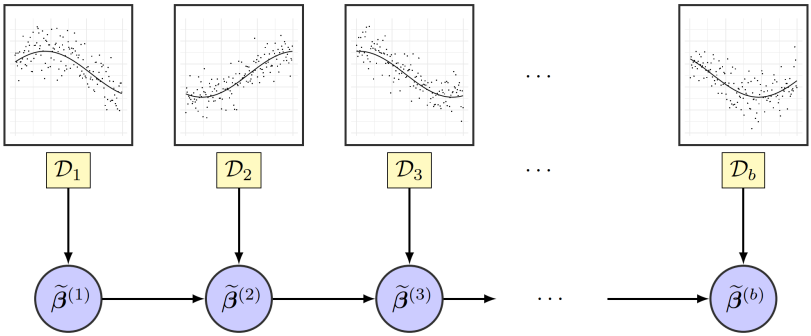


Objective: Develop an "online" estimator that

- can be updated sequentially using raw data in current batch only
- requires little time and space complexities
- still achieves desired statistical property

High-dimensional GLM

A sequence of n data points $\mathcal{D} = \{(\mathbf{x}_i, y_i)\}_{i=1}^n$ are independently and identically sampled from a generalized linear model (GLM):

$$\mathbb{P}(y \mid \mathbf{x}; \boldsymbol{\beta}^*) \propto \exp\left\{ \frac{y\left(\mathbf{x}^\top \boldsymbol{\beta}^*\right) - \Phi\left(\mathbf{x}^\top \boldsymbol{\beta}^*\right)}{c(\sigma)} \right\}, \tag{1}$$

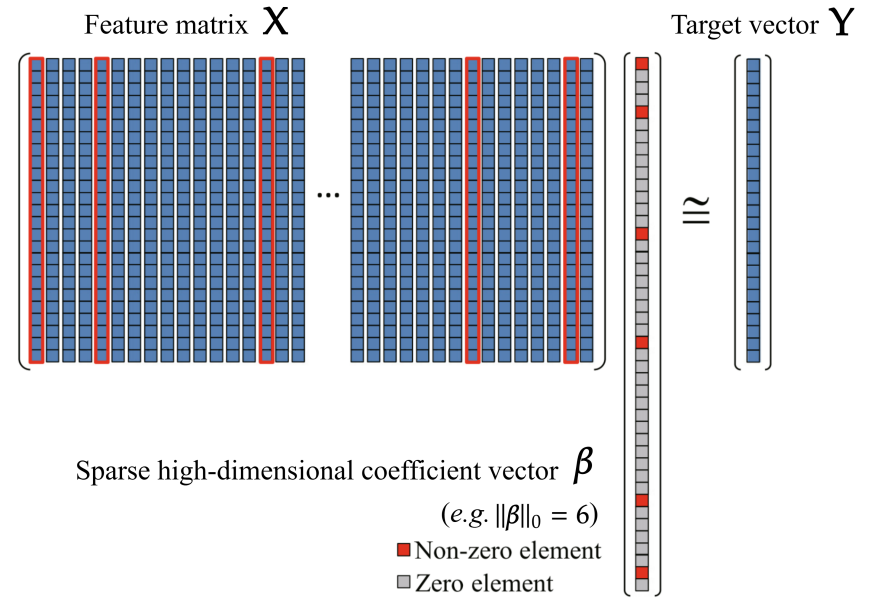
- $\mathbf{x} \in \mathbb{R}^p$. High-dimensional case: $n \ll p$.
- Sparsity: $\|\boldsymbol{\beta}^*\|_0 = s_0$.

The ground truth $\boldsymbol{\beta}^*$ is the minimum of the population loss function

$$\boldsymbol{\beta}^* = \arg\min_{\boldsymbol{\beta}} \mathbb{E}\left[F_n(\boldsymbol{\beta})\right], \tag{2}$$

where $F_n(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^n \left\{-y_i(\mathbf{x}_i^\top \boldsymbol{\beta}) + \Phi(\mathbf{x}_i^\top \boldsymbol{\beta})\right\}$, the negative log-likelihood of model $\mathbb{P}$.

Example: High-dimensional Linear Regression



Traditionally, we estimate $\boldsymbol{\beta}^*$ using data $\mathcal{D} = \{(\mathbf{x}_i, y_i)\}_{i=1}^n$ via two steps:

$$\widetilde{\boldsymbol{\beta}} = \underset{\boldsymbol{\beta}\in\mathbb{R}^p}{\arg\min} \left\{ F_n(\boldsymbol{\beta}) + \lambda_n \|\boldsymbol{\beta}\|_1 \right\}, \tag{3}$$

and

$$\widehat{\boldsymbol{\beta}}_{j,\mathrm{de}} = \widetilde{\boldsymbol{\beta}}_j - \boldsymbol{\Theta}_j^\top \nabla F_n(\widetilde{\boldsymbol{\beta}}), \tag{4}$$

where $\boldsymbol{\Theta}$ is the inverse of the covariance matrix (information matrix). $\boldsymbol{\Theta}_j$ can be estimated by [Zhang and Zhang, 2014, van de Geer et al., 2014]...

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By doing so, we have asymptotic normality

$$(\widehat{\boldsymbol{\beta}}_{j,\mathrm{de}} - \boldsymbol{\beta}_j^*)/\widehat{\sigma}_j \to \mathcal{N}(0,1) \text{ in distribution as } n \to \infty. \tag{5}$$

And we are able to make all kinds of statistical inference based on

$$\mathbb{P}\{\boldsymbol{\beta}_j^* \in (\widehat{\boldsymbol{\beta}}_{j,\mathrm{de}} - z_{\alpha/2}\widehat{\sigma}_j, \widehat{\boldsymbol{\beta}}_{j,\mathrm{de}} + z_{\alpha/2}\widehat{\sigma}_j)\} \to 1-\alpha \text{ as } n \to \infty. \tag{6}$$

To do

$$\widetilde{\boldsymbol{\beta}} = \underset{\boldsymbol{\beta}\in\mathbb{R}^p}{\arg\min} \left\{ \frac{1}{n}\sum_{i=1}^{n}\{-y_i(\mathbf{x}_i^\top\boldsymbol{\beta}) + \Phi(\mathbf{x}_i^\top\boldsymbol{\beta})\} + \lambda_n\|\boldsymbol{\beta}\|_1 \right\},$$

and

$$\widehat{\boldsymbol{\beta}}_{j,\mathrm{de}} = \widetilde{\boldsymbol{\beta}}_j - \boldsymbol{\Theta}_j^\top \nabla F_n(\widetilde{\boldsymbol{\beta}}),$$

Remark

we need to store data of size $\mathcal{O}(n \times p)$, and do calculations involving the covariance matrix of size $\mathcal{O}(p \times p)$. In large datasets (*e.g.* text data, web corpus, gene expression data...) p can be extensive. Let's say $p = 3,000,000$, then,

$$p \times p = 9,000,000,000,000. \tag{7}$$

To operate this $p \times p$ matrix of double-precision floating-point (float64) numbers, you would need at least approximately 65.54 terabytes (TB) of storage.

Our Solution

Question

Can we bring it "online"?

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Table: A comparison in time and space complexities for statistical inference of one single coefficient. LM: linear model; GLM: generalized linear model.

Works	Model	One-pass?	Space complexity
[Chen et al., 2020]	LM	Yes	$\mathcal{O}(np)$
[Deshpande et al., 2023]	LM	No	$\mathcal{O}\left(p^2\right)$
[Han et al., 2023]	LM	Yes	$\mathcal{O}(p)$
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Proposed Method	GLM	Yes	$\mathcal{O}(p)$

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Main Idea

Optimization + Statistics

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Overview of the Algorithm

Upon the arrival of the k-th data, update $\widehat{\boldsymbol{\beta}}^{(i)}$ and $\widehat{\boldsymbol{\gamma}}_{\cdot,j}^{(i)}$ with a variant of SGD, then construct the following summary statistics:

$$\boldsymbol{A}_1^{(k)} = \sum_{i=1}^{k} \boldsymbol{x}_i\{-y_i + \dot{\Phi}(\boldsymbol{x}_i^\top \widehat{\boldsymbol{\beta}}^{(i)})\}, \quad \boldsymbol{A}_2^{(k)} = \sum_{i=1}^{k} \boldsymbol{x}_i^\top \widehat{\boldsymbol{\gamma}}_{\cdot,j}^{(i)} \ddot{\Phi}(\boldsymbol{x}_i^\top \widehat{\boldsymbol{\beta}}^{(i)}) \boldsymbol{x}_i,$$

$$a_3^{(k)} = \sum_{i=1}^{k} \boldsymbol{x}_i^\top \widehat{\boldsymbol{\gamma}}_{\cdot,j}^{(i)} \ddot{\Phi}(\boldsymbol{x}_i^\top \widehat{\boldsymbol{\beta}}^{(i)}) \boldsymbol{x}_i^\top \widehat{\boldsymbol{\beta}}^{(i)}, \quad a_4^{(k)} = -\sum_{i=1}^{k} \boldsymbol{x}_i^\top \widehat{\boldsymbol{\gamma}}_{\cdot,j}^{(i)} (\boldsymbol{x}_i)_j \{\ddot{\Phi}(\boldsymbol{x}_i^\top \widehat{\boldsymbol{\beta}}^{(i)})\}.$$

All are p-dimensional vectors and scalars. The approximated debiased lasso (ADL) estimator

$$\widetilde{\boldsymbol{\beta}}_{j,\mathrm{de}}^{(k)} := \widehat{\boldsymbol{\beta}}_j^{(k)} - \frac{\{\boldsymbol{A}_1^{(k)}\}^\top \widehat{\boldsymbol{\gamma}}_{\cdot,j}^{(k)} + \{\boldsymbol{A}_2^{(k)}\}^\top \widehat{\boldsymbol{\beta}}^{(k)} - a_3^{(k)}}{a_4^{(k)}}.$$

We update summary statistics online:

$$\boldsymbol{A}_1^{(k)} \longleftarrow \boldsymbol{A}_1^{(k-1)} + \boldsymbol{x}_k\{-y_k + \dot{\Phi}(\boldsymbol{x}_k^\top \widehat{\boldsymbol{\beta}}^{(k)})\}.$$

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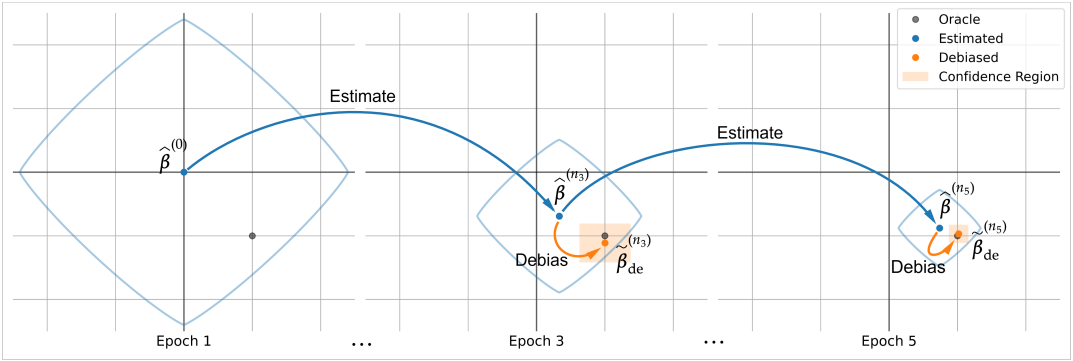
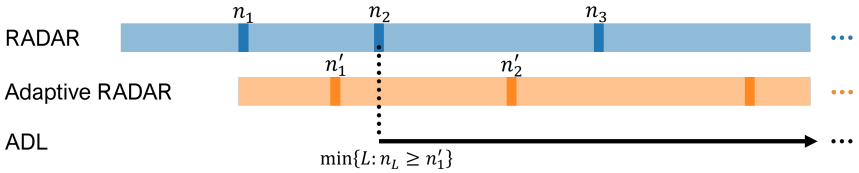
Remark

Now we can recurse $\{\widehat{\boldsymbol{\beta}}^{(k)}, \widehat{\boldsymbol{\gamma}}_{\cdot,j}^{(k)}, \boldsymbol{A}_1^{(k)}, \boldsymbol{A}_2^{(k)} a_3^{(k)}, a_4^{(k)}\}$ and clear all other variables in memory.

Overview of the Algorithm





Complexities

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Theorem (Convergence Rate)

With some regularity conditions and proper choice of hyperparameters, and initial error $d_0 = \|\widehat{\boldsymbol{\beta}}^{(0)} - \boldsymbol{\beta}^*\|_1$ *the following events*

$$\|\widehat{\boldsymbol{\beta}}^{(i)} - \boldsymbol{\beta}^*\|_1 \leq C_1 s_0 d_0 \sqrt{\frac{(\log p)^3}{i}}, \quad \|\widehat{\boldsymbol{\beta}}^{(i)} - \boldsymbol{\beta}^*\|_2 \leq C_1 d_0 \sqrt{\frac{s_0 (\log p)^3}{i}} \tag{8}$$

for $i \geq n_1$ *hold uniformly for a universal constant* C_1 *with high probability.*

Theoretical Results

Theorem (Convergence Rate)

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for $i \geq n_1$ *hold uniformly for a universal constant* C_1 *with high probability.*

Theorem (Asymptotic Normality)

Further with

$$\left(s_0 s_1 d_0 d_1 + s_0^2 d_0^2 + s_0 s_1 d_0^2\right) (\log p)^4 \log_2(n) = o_{\mathbb{P}}(\sqrt{n}),$$

then, for any fixed $j \in [p]$,

$$\left(\widetilde{\boldsymbol{\beta}}_{j,de}^{(n)} - \boldsymbol{\beta}_j^*\right) / \widetilde{\nu}_j^{(n)} \to \mathcal{N}(0,1) \text{ as } n \to \infty \text{ in distribution.}$$

Remark

Most existing methods rely on the bounded individual probability condition that

$$P(y_i = 1 \mid X_i) \in (c, 1-c)$$

for all $1 \leq i \leq n$ and some $c \in (0, 1/2)$. To the best of our knowledge, only [Guo et al., 2021, Cai et al., 2023] and ours remove this.

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Simulation with Synthetic Data (1)

True coefficients: $\beta^*_{D_1} = 1, \beta^*_{D_2} = -1$ and others are set to 0.

Table: Case 2: $n = 200$, $p = 500$, $s_0 = 6$, $\mathbf{\Sigma} = \{0.5^{|i-j|}\}_{i,j=1,\ldots,p}$. Simulation results are averaged over 200 replications. Other methods: deLasso [van de Geer et al., 2014], LSW [Cai et al., 2023], ODL [Luo et al., 2023].

	β^*_k	deLasso	LSW	ODL			ADL		
Sample size n		200	200	40	120	200	40	120	200
Coverage probability	0	0.983	0.996	1.000	0.999	0.998	0.951	0.951	0.961
	1	0.598	0.936	1.000	0.880	0.705	0.891	0.955	0.943
	-1	0.610	0.946	1.000	0.861	0.670	0.881	0.961	0.945
Absolute bias	0	0.114	0.113	0.022	0.019	0.017	0.901	0.425	0.309
	1	0.325	0.260	0.657	0.590	0.539	1.008	0.423	0.328
	-1	0.326	0.259	0.660	0.598	0.548	1.034	0.426	0.324
Coverage length	0	0.753	1.207	3.423	1.691	1.287	4.419	2.053	1.532
	1	0.769	1.261	3.434	1.689	1.286	4.555	2.086	1.536
	-1	0.767	1.271	3.416	1.648	1.282	4.431	2.078	1.541
Time (s)		6.550	20.04		0.628			0.365	

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Remark

deLasso and ODL fail because they rely on the bounded individual probability condition.

Simulation with Synthetic Data (2)

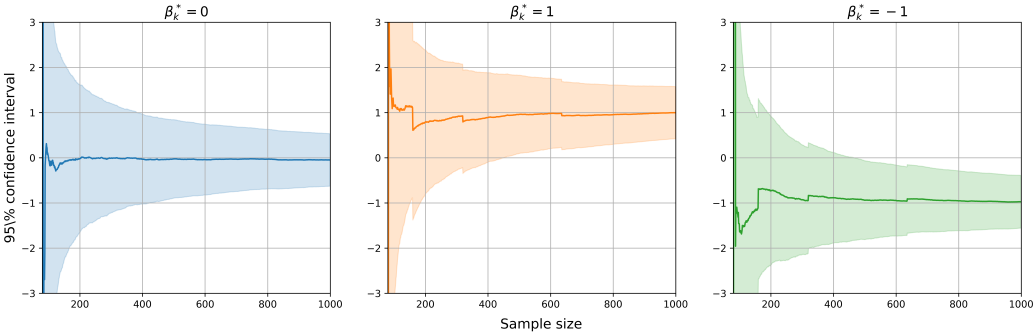


Figure: $n = 1000$, $p = 20000$, $s_0 = 20$, $\mathbf{\Sigma} = \{0.5^{|i-j|}\}_{i,j=1,\ldots,p}$. Simulation results are averaged over 200 replications (around 110.4s for each replication).

Large-scale URLs Data

Real data application: Detection of malicious websites [Ma et al., 2010].

- x: 3,231,961 features containing lexical, host-based information...
- y: Binary, is this URL a phishing site or not?
- Logistics regression.

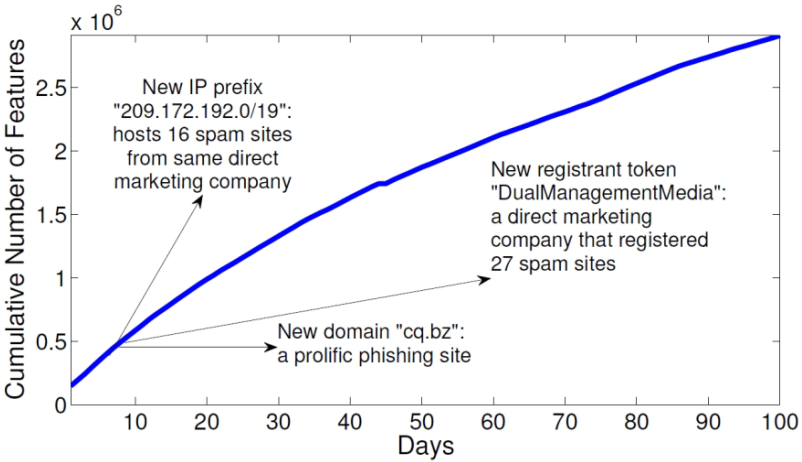
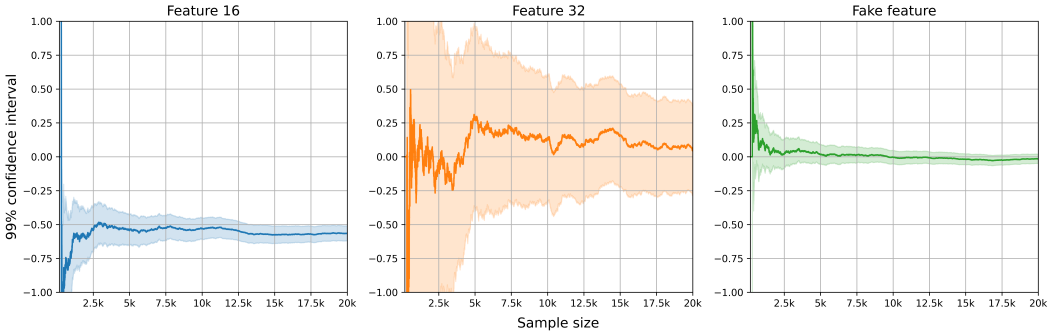


Figure: The number of features also grows with incoming data.

Large-scale URLs Data

Real data application on logistic regression:

- x: 3,231,961 features containing lexical, host-based information...
- y: Binary, is this URL a phishing site or not?



- An incoming data needs around 2s to arrive on average.
- ADL consumes around 0.3s on each data.

10-K Financial Report Data

Data source: [Kogan et al., 2009]. We used more than 10,000 10-K forms from 2002 to 2004 to predict future market volatility.

- x: 4,500,000 features containing unigram and bigram frequency.
- y: Increment of market volatility (log scale).
- Linear regression

Figure: Form 10-K is an annual report required by the U.S. Securities and Exchange Commission (SEC), that gives a comprehensive summary of a company's financial performance including, organizational structure, executive compensation, equity, subsidiaries, and audited financial statements....

10K Financial Report Data

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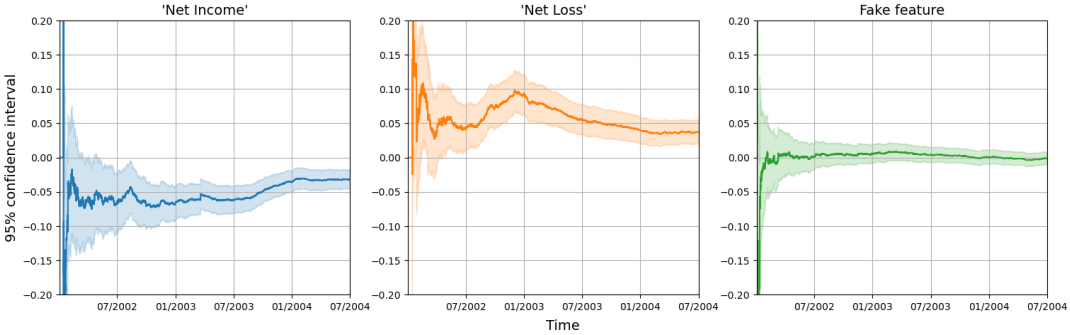
- y: Increment of market volatility (log scale).



- ADL consumes around 0.45s on each data.

Remark (Leverage/Announcement Effect)

"Bad news" has a larger impact on market volatility than "good news".

Conclusion & Remarks

In this research, we develop online statistical inference approach in high-dimensional GLM that

- sequentially updates without retrieving historical data.
- has a space complexity $\mathcal{O}(p)$, instead of $\mathcal{O}(p^2)$ nor $\mathcal{O}(np)$.
- provides theoretical guarantees for convergence and asymptotic normality.
- achieves similar performance as offline counterparts.

Future works:

- non i.i.d. data, covariate shift
- non-convex objectives
- ...

Suggested reference:

- Han, R., Luo, L., Luo, Y., Lin, Y. and Huang, J. (2024). "Adaptive debiased SGD in high-dimensional GLMs with streaming data". Manuscript submitted for publication.

Thanks for watching!

$\mathcal{Q}\&\mathcal{A}$

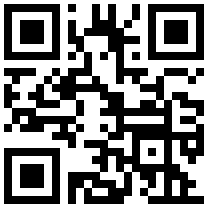


Figure: Contact info: chattelion.luo@connect.polyu.hk

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