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Differential Topology

Class Notes

1. Definition of a Topological Space, Open Sets, Closed Sets

A topological space (X, O) is a set X and a collection O of subsets of X where $o \in O$ is called an "open set" such that

- The union of any number of elements of O is also an element of O
- The intersection of any finite number of elements of O is also an element of O
- $\emptyset, X \in O$

The condition that only a finite number of intersections can be allowed is illistrated by $\cap_{n=\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$

A closed set is a set $S \subset X$ such that $X - S$ is open.

1.1. Examples of Topological Spaces.

- $\mathbb{R}$ with standard topology $\rightarrow$ standard notion of open subsets are open
- X is any set, $O = \{\emptyset, X\}$. This is called the "trivial topology"
- X is any set and O is the set of all subsets of X. This is called the "discrete topology"
- $X = \{1, 2\}$ and $O = \{\emptyset, \{1\}, \{1, 2\}\}$ is a valid topological space

2. Definition of continuity

A function $f : X \rightarrow Y$ is continuous if for each open set O in Y, $f^{-1}(O) = \{x \in X | f(x) \in O\}$ is also open in X.

2.1. Examples involving the Continuity of Maps.

- Suppose (X, O_x) is a space with discrete topology and (Y, O_y) is any topological space. Then any map $f : X \rightarrow Y$ is continuous
- Suppose (X, O_x) is a trivial topology. Then a map $f : X \rightarrow \mathbb{R}$ is only continuous if it maps each $x \in X$ to a single point in R
- Let $X = \{x_1, x_2\}$ with discrete topology. $f : \mathbb{R} \rightarrow X$ is continuous iff f maps $\mathbb{R}$ to one point in X. (The only sets both open and closed in $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$). Question: why is this equivalent to the intermediate value theorem?

3. Definition of a Neighborhood of x

A Neighborhood of an element $x \in X$ is a subset $N \subseteq X$ such that there exists an open $O \subseteq X$ where $x \in O \subseteq N$.

4. Interiors and Closures

Let S be a subset of the topological space (X, O_x).

4.1. Definition of an Interior. $\text{Int}(S) = \cup_{O \subseteq S | O \in O_x} O$ $\text{Int}(S)$ (which is open as it is the union of open subsets) is the largest open subset in S since if there is a hypothetical larger open subset in S we know that it is actually contained in the union which constructs $\text{Int}(S)$.

4.2. Proof: S is open iff $\text{Int}(S) = S$. $\text{Int}(S) \subseteq S$. Additionally, if S is open then since $S \subseteq S$, $S \subseteq \text{Int}(S)$. So if S is open then $\text{Int}(S) = S$. Going the other way, if $S = \text{Int}(S)$ then S is open as the union of open subsets of S.

4.3. Definition of a Closure. $\overline{S} = \cap_{S \subseteq C | C \text{ is closed in X}}$