Appendix A: Technical appendix

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1 Functional form of h

Let h be a concave function of $\hat{h}$ and I, i.e. $h = f(\hat{h}, I), f_i \geq 0, f_{ii} \leq 0$, where $i \in \{\hat{h}, I\}$. In case $I = 0, h = f(\hat{h}, 0) = f(\hat{h})$. Absent any uncertainty in the "schooling" process, the level of human capital of an agent is defined by f. On the contrary, when uncertainty arises, to be consistent with our setting, we assume h is uniformly distributed over the interval $[f(\hat{h}), f(\hat{h}, I)]$, that is $h \sim U\big(f(\hat{h}), f(\hat{h}, I)\big)$. We now prove that given f, our results remain robust under an appropriate set of conditions.

Proof. The expected utility of an agent is,

$$\begin{aligned}
U &= u(L-I) + p(h)\Big(u(H-I) - u(L-I)\Big) \\
&= -e^{\lambda(L-I)} + \Big[e^{-\lambda(L-I)} - e^{-\lambda(H-I)}\Big] \int_{f(\hat{h})}^{f(\hat{h},I)} \gamma h \big(f(\hat{h}, I) - f(\hat{h})\big)^{-1} \, \mathrm{d}h \\
&= -e^{\lambda(L-I)} + \frac{\gamma}{2}\big(f(\hat{h}, I) + f(\hat{h})\big)\Big[e^{-\lambda(L-I)} - e^{-\lambda(H-I)}\Big]
\end{aligned} \tag{1.1}$$

The corresponding first and second derivative with respect to I are,

$$U' = e^{\lambda I}\Big\{ -\lambda e^{-\lambda L} + \frac{\gamma}{2}\big(e^{-\lambda L} - e^{-\lambda H}\big) \underbrace{\big[\lambda f(\hat{h}, I) + \lambda f(\hat{h}) + f_I\big]}_{A} \Big\} \tag{1.2}$$

$$U'' = \lambda U' + \frac{\gamma}{2} e^{\lambda I}\big(e^{-\lambda L} - e^{-\lambda H}\big) \underbrace{(\lambda f_I + f_{II})}_{A'} \tag{1.3}$$

For our standard results to hold, we require that $U'' > 0$ if $U' > 0$, which is equivalent to $A' \geq 0$ or A is non-decreasing with I and $A > \frac{2\lambda e^{-\lambda L}}{\gamma(e^{-\lambda L} - e^{-\lambda H})}$. Insofar that these two conditions are satisfied within the feasible range of I given λ, the expected utility is U-shaped in I and the discontinuity of investment occurs. In the basic model, the functional form used implies $f_{II} = 0$, which effectively translates into U'' being positive when U' is positive. $\square$

2 Concavity of the expected utility function in equation 2.5

$$U = pE\Big[u\big(f(h) - I\big)\Big] + (1-p)u(L-I)$$

To prove that U is concave in I, we prove that $E\Big[u\big(f(h) - I\big)\Big]$ is concave in I.

Proof. Following the concavity of u, by definition,

$$\begin{aligned}
u\big(f(h) - tI_1 - (1-t)I_2\big) &= u\big(t(f(h) - I_1) + (t-1)(f(h) - I_2)\big) && (2.1)\\
&\geq tu\big(f(h) - I_1\big) + (t-1)u\big(f(h) - I_2\big) && (2.2)\\
\Rightarrow E\Big[u\big(f(h) - tI_1 - (1-t)I_2\big)\Big] &\geq E\Big[tu\big(f(h) - I_1\big) + (t-1)u\big(f(h) - I_2\big)\Big] && (2.3)\\
&\geq tE\Big[u\big(f(h) - I_1\big)\Big] + (t-1)E\Big[u\big(f(h) - I_2\big)\Big] && (2.4)
\end{aligned}$$

where $t \in [0,1]$ and the last line appeals to Jensen's inequality. As $E\Big[u\big(f(h) - I\big)\Big]$ is concave and u is also concave, their convex combination is concave, i.e. U is concave. □

3 Convergence of a non-linear recurrence relation

The sequence $\{h_{1,k}\}$ is of the general form

$$x_{t+1} = x_t^a + b \tag{3.1}$$

where $a, b \in (0,1)$ and $x_0 > 1$. This is known as a non-linear recurrence relation, which is notorious for its lack of a closed form solution. We therefore do not attempt to find a closed form solution but instead find the limit towards which the sequence converges. To prove the existence of the limit of $\{x_t\}$, we need to prove that the sequence is monotonic and bounded.

Lemma 3.1. *The sequence $\{x_t\}$ is monotonic. Namely, for all $t \in \mathbb{N}$, the sequence $\{x_t\}$ is either non-increasing or non-decreasing.*

Proof. Without loss of generality, assume $x_{k+1} \geq x_k$, we now prove that $x_{k+2} \geq x_{k+1}$.

$$x_{k+1} \geq x_k \Rightarrow\ x_{k+1}^a + b \geq x_k^a + b \Rightarrow\ x_{k+2} \geq x_{k+1}$$

Through a similar procedure, it can be shown that the opposite is also true, that is if $x_{k+1} \leq x_k$, $x_{k+2} \leq x_{k+1}$. □

Lemma 3.2. *Sequence $\{x_t\}$ is bounded below if $\{x_t\}$ is non-increasing and bounded above if $\{x_t\}$ is non-decreasing.*

Proof. As the values of a and b are arbitrary, we consider two cases that arise, the non-increasing case and the non-decreasing case. In the former case, $x_{t+1} \leq x_t$ for all t while the opposite is true for the latter case.

Case 1: Sequence $\{x_t\}$ is non-increasing
Since $\{x_t\}$ is non-increasing by supposition, $x_{t+1} \leq x_t$. However, by definition, $x_t > b$ for all t. Consequently, $\{x_t\}$ is bounded below when $\{x_t\}$ is non-increasing.

Case 2: Sequence $\{x_t\}$ is non-decreasing
Let $F(y) = y^a + b - y$, defined for $y > 1$. $F(y)$ is increasing for $y < a^{\frac{1}{1-a}}$ and decreasing for $y > a^{\frac{1}{1-a}}$. By definition, $y > 1 > a^{\frac{1}{1-a}}$ meaning $F(y)$ is decreasing over its entire range. As $F(1) = b$ is positive and $F(.)$ is decreasing, $F(y)$ eventually becomes negative as $y > y^*$, where y^* is the point at which $F(y^*) = 0$.
Now notice that $F(x_t) = x_t^a + b - x_t = x_{t+1} - x_t$ while $F(x_t) \geq 0$ for all t by supposition. As $F(x_t)$ is always non-negative, $x_t < y^*$ $\forall t$, i.e. $\{x_t\}$ is bounded above. □

Theorem 3.3. *The sequence $\{x_t\}$ defined by a recurrence relation $x_{t+1} = x_t^a + b$ converges to a unique limit M if $a \in (0, 1)$ and $x_0 > 1$.*

Proof. Following from Lemma 3.1 & 3.2, $\{x_t\}$ converges as it is monotonic and bounded. Let M be the limit of $\{x_t\}$,

$$\lim_{t\to\infty} x_t = M = \lim_{t\to\infty} x_{t+1} \Rightarrow M = M^a + b \tag{3.2}$$

The solution of (3.2) is then the limit to which $\{x_t\}$ converges. Due to the inverted U-shaped of F and $F(1) = b > 0$, $F = 0$ only has a unique positive root. In other words, M is unique. □