

Probability and Statistics: Lecture-8

Monsoon-2020

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» Motivation for Conditional Probability with an Example ...

- * **Experiment:** Throw two dice A and B simultaneously

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Roll two 6-sided dice, yielding values D_1 and D_2 . Let E be the event $D_1 + D_2 = 4$. What is $P(E)$?

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$$P(E|F) =$$

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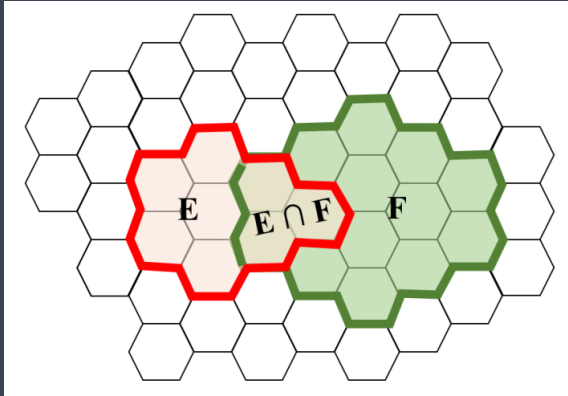
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With equally likely outcomes:

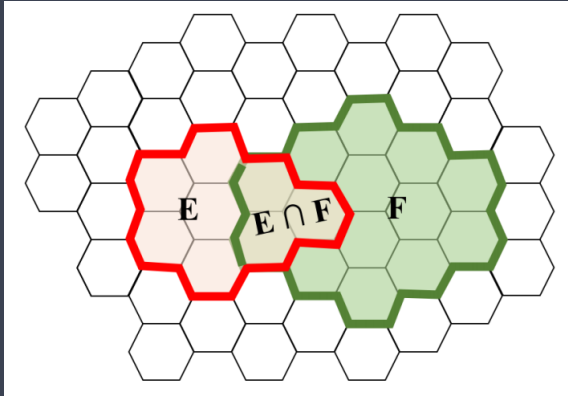
$$P(E|F) = \frac{|E \cap F|}{|S \cap F|} = \frac{|E \cap F|}{|F|}$$

» Graphical Illustration/Example of Conditional Probability...



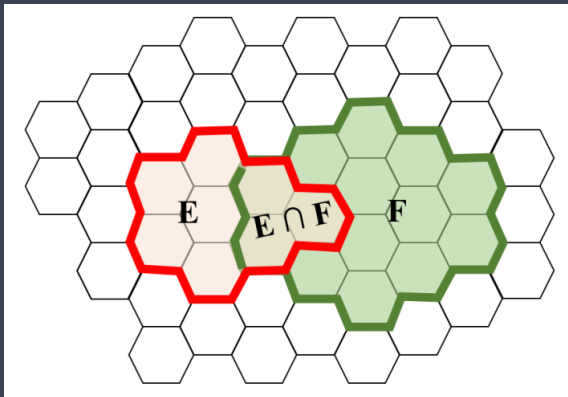
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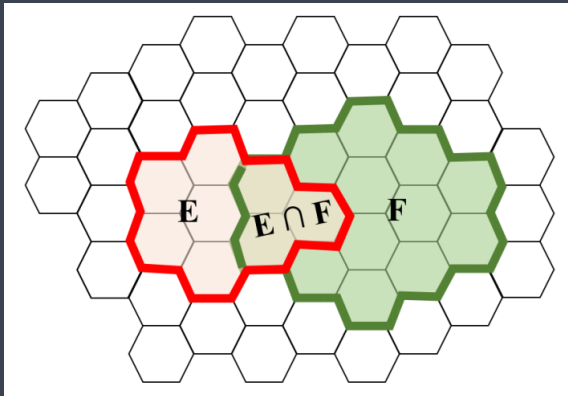
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* **Question:** What is $P(E)$? Here $P(E) = \frac{8}{50} \approx 0.16$

* **Question:** What is $P(E|F)$? Here $P(E|F) = \frac{3}{14} \approx 0.21$

» Probability of Receiving Spam Emails...

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Email Spam Conditional Probability Problem

24 emails are sent, 6 each to 4 users.

- * 10 of the 24 emails are spam.
- * All possible outcomes are equally likely.

Question-1

Let event E = user 1 receives 3 spam emails. What is $P(E)$?

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24 emails are sent, 6 each to 4 users.

- * 10 of the 24 emails are spam.
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Question-1

Let event E = user 1 receives 3 spam emails. What is $P(E)$?

Question-2

Let event F = user 2 receives 6 spam emails. What is $P(E|F)$?

» Law of Total Probability...

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Conditional Probability Implies Chain Rule...

$$P(E|F) = \frac{P(E \cap F)}{P(F)} \Rightarrow P(E \cap F) = P(F)P(E|F)$$

These hold even when outcomes are not equally likely!

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Law of Total Probability (Theorem)

$$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$$

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Proof:

» Compute $P(E)$ from $P(E|F)$ Using Probability Tree...

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Problem

Flips a fair coin.

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Flips a fair coin.

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You win if you roll a 6. What is $P(\text{winning})$?

Solution using probability tree:

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Solution using **total probability**: