Probability and Statistics: Lecture-6

Monsoon-2020

by Pawan Kumar (IIIT, Hyderabad)
on August 21, 2020

» Table of contents

» Computing probabilities...

Experiment: Throw two dice.

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* Probability space: all outcomes
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* Probability: (#favorable)/36

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Hence, probability, $p = \frac{15}{36}$

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* Consider an event of "first bit" = "last bit"
* What is the **probability** of the above event?

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- **Observation:** In **half** of the cases, first bit does not equal last bit
- Hence, probability of the event is $1/2$

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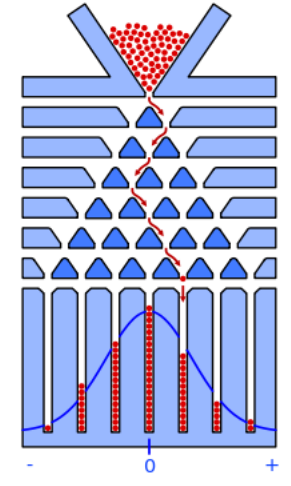
* We also study continuous distributions
* In the continuous case, we need to consider some other measure
* What is the probability that a number 1.5 is picked in $[1, 2]$? Does this make sense?

» Probability, Galton Board, and Pascal Triangle

Movie of Galton board here!

» Probability, Galton Board, and Pascal Triangle



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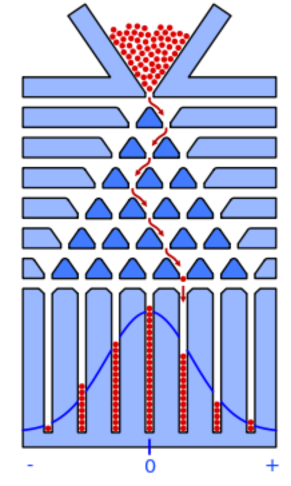

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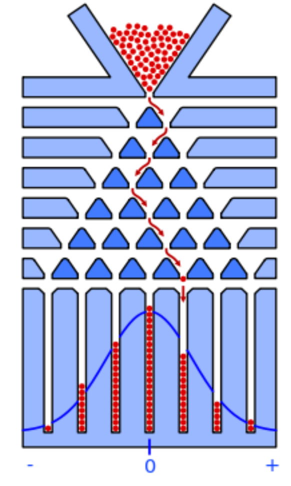

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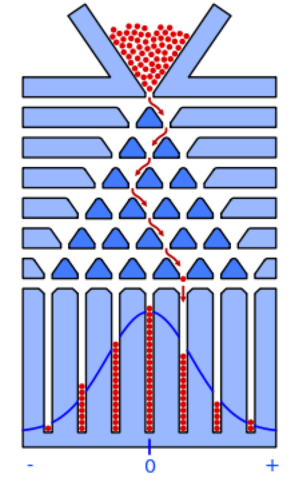

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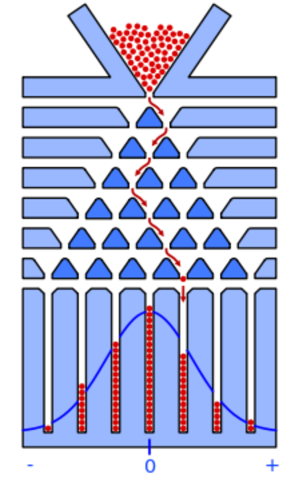

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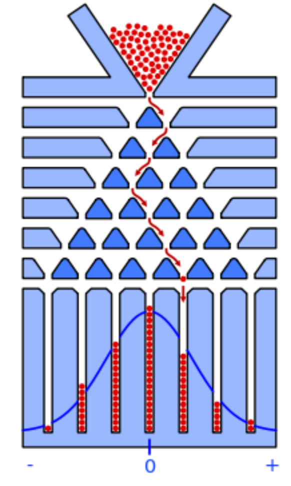

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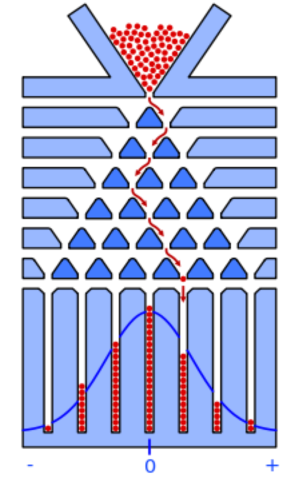

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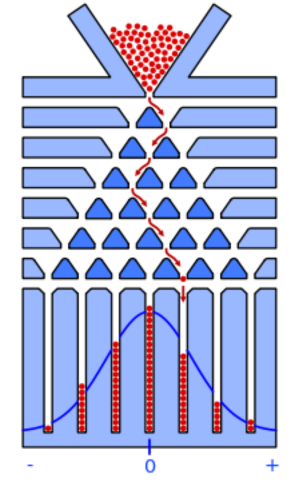

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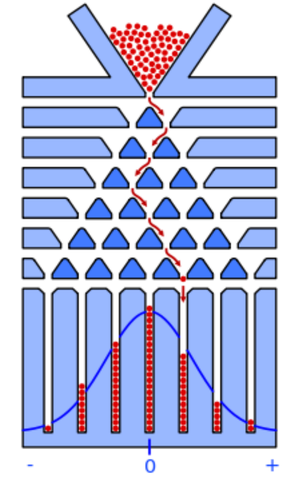

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- Let us analyze this in detail ...

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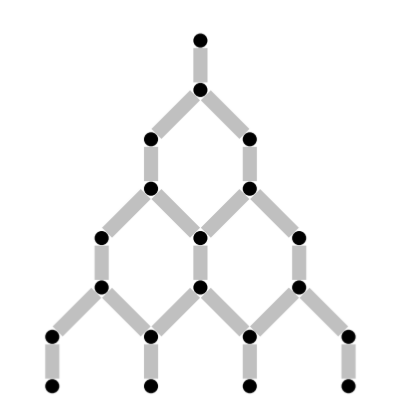

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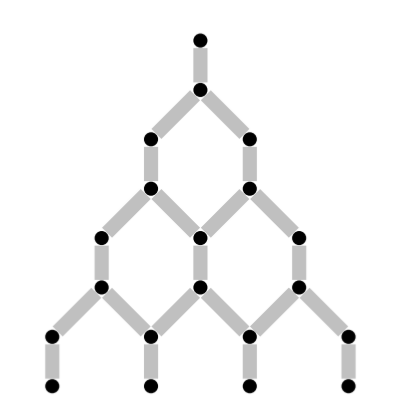

* Assume that at each level the beams are splitted evenly

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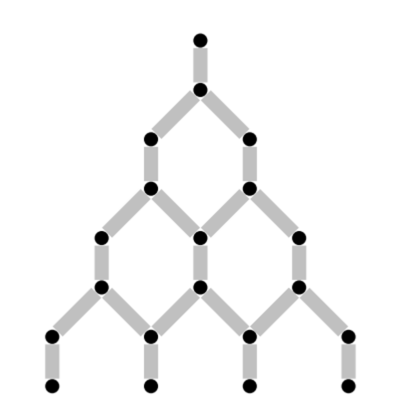

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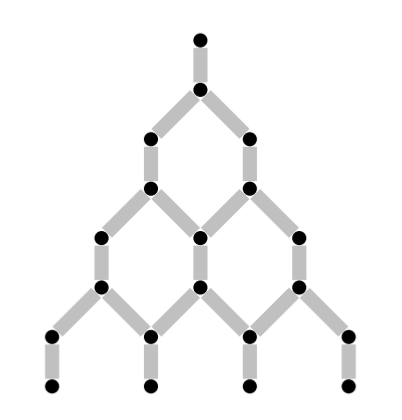

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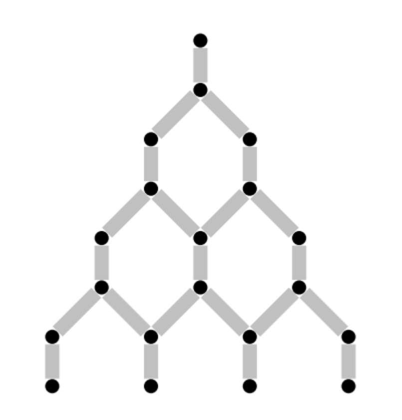

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1

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1/4 2/4 1/4

1/8 3/8 3/8 1/8

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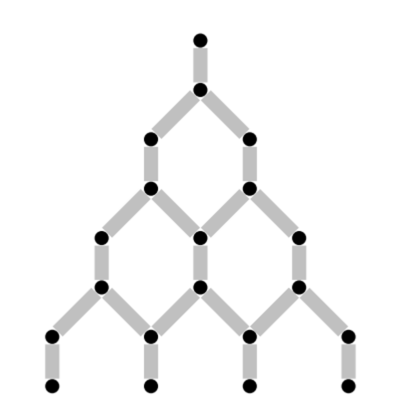

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			1			
		1/2		1/2		
	1/4		2/4		1/4	
1/8		3/8		3/8		1/8

* The probability of current bin = $\binom{n}{k}/2^n$

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What is wrong with these dice?

* So far we assumed equiprobable outcomes
* On the left, we have skewed dice, i.e., unfair dice
* Will our previous definition of calculating probability work?
* Unfortunately not, how to deal with skewed cases?

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- Since the dice is skewed, frequency of outcomes are different
  - That is, for example, number 4 may appear more frequently
- Let p_i denote the probability of ith number
- If we throw the dice long enough, the frequencies stabilise
- For example, the probability of getting an even number = $p_2 + p_4 + p_6$
- Also, we know that the sum of all the probabilities sum to 1, even though the dice is skewed!
  - That is: $p_1 + p_2 \cdots + p_6 = 1$

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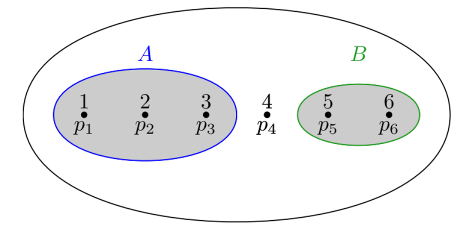

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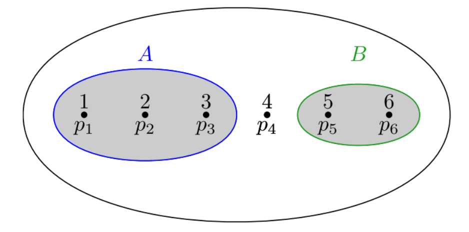

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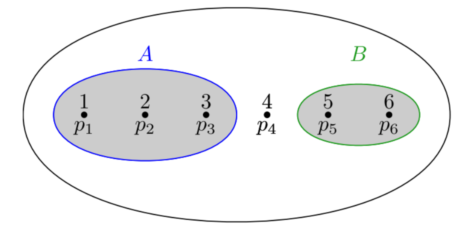

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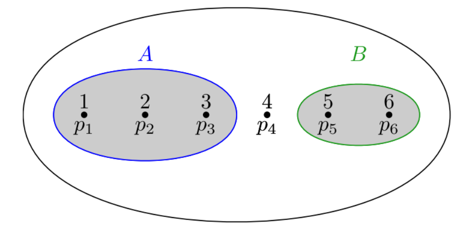

- Consider event $A = \{1, 2, 3\}$
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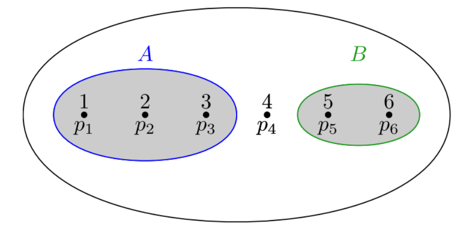

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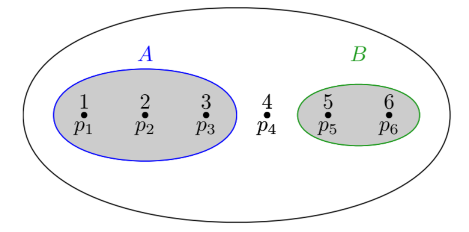

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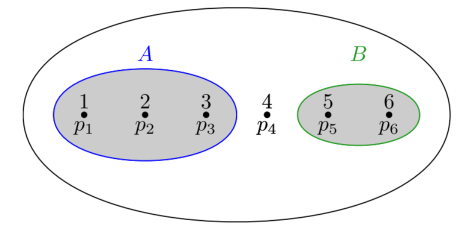

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* Consider the event A or $B = \{1,2,3,5,6\}$
* $Pr(A \text{ or } B) =$

$$Pr(A) + Pr(B) = p_1 + p_2 + p_3 + p_5 + p_6$$

Why $Pr(A \text{ or } B) = Pr(A) + Pr(B)$?

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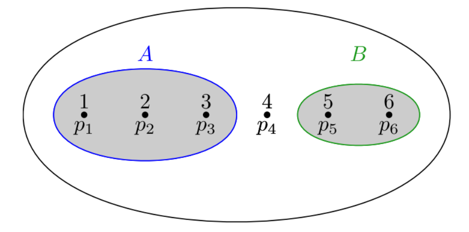

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Why $Pr(A \text{ or } B) = Pr(A) + Pr(B)$?
When are we allowed to add probabilities?

» Mutually Exclusive Events Events

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$$Pr(A) + Pr(A^c) = 1 \quad \implies \quad Pr(A^c) = 1 - Pr(A)$$

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A set of n events denoted by $A_1, A_2, \cdots, A_n$ are called mutually disjoint events if the following holds:

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From Axiom-3, it holds for countably infinite unions.

» Quiz: Use concept of disjoint events

Quiz

In a cricket tournament with four teams denoted by $\{A, B, C, D\}$, team A has 20% chance of winning, while team B has a 40% chance of winning. What is the probability that A or B win the tournament?

» Quiz-Use probability axioms

Quiz

Suppose we know the following:

* there is a 50% chance that it will be hot Today
* there is a 30% chance that it will be hot Tomorrow
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* probability that it will be hot today and tomorrow
* probability that it will be hot today but not tomorrow
* probability that it either will be hot today or tomorrow, but not both

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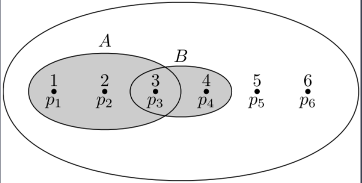

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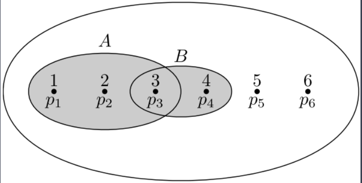

- For the events A and B, which relation (1) holds?

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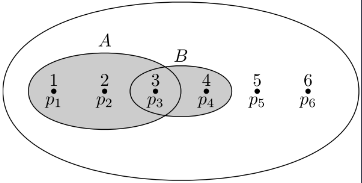

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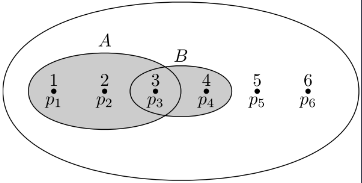

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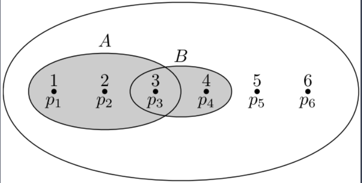

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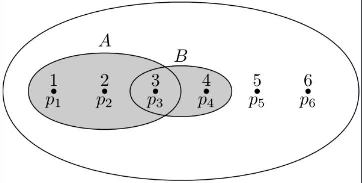

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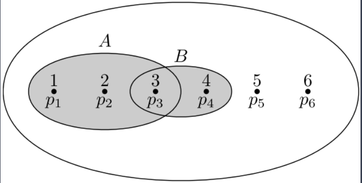

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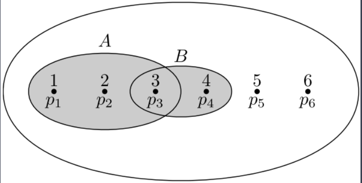

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* Recall discrete mathematics: inclusion-exclusion principle. (Proof?)

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$$P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots (-1)^{n-1} P(\cap_{i=1}^{n} A_i)$$

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* For an event $A \subset S$, by 3rd axiom

$$P(A) = \sum_{s_j \in A} P(s_j)$$

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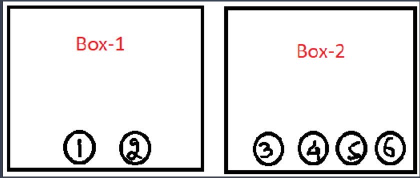

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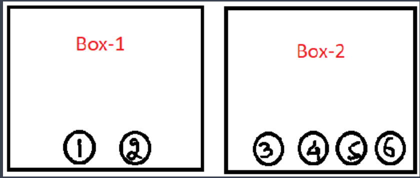

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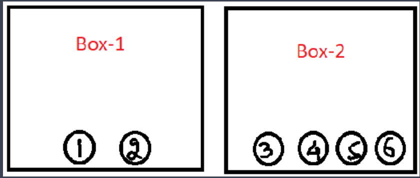

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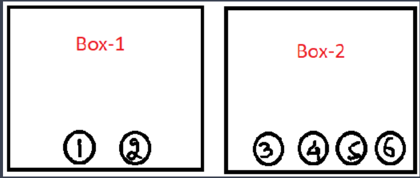

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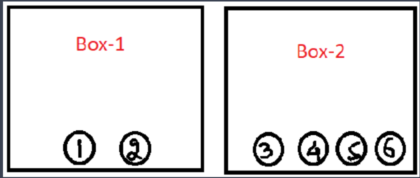

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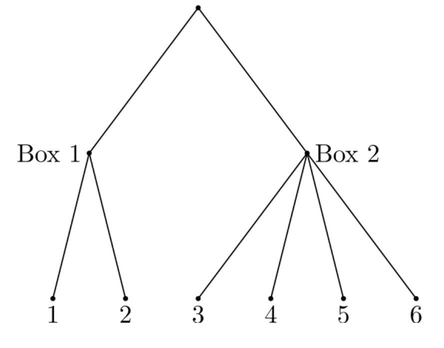
* Question: What are the probabilities $p_1, p_2, \cdots, p_6$?



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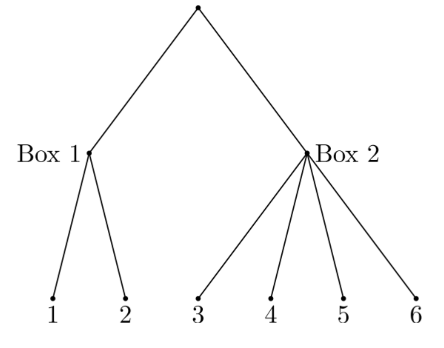

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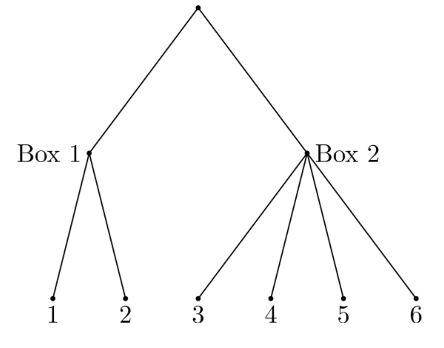

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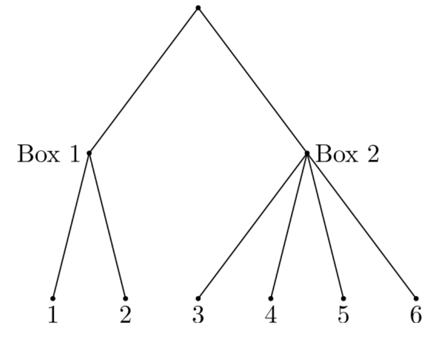

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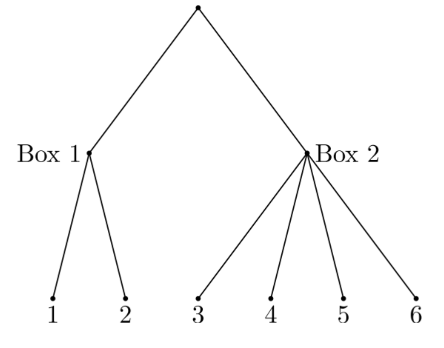

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* From above, we have

$$p_1 = p_2 = 1/4, \quad p_3 = p_4 = p_5 = p_6 = 1/8$$

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Question: How should the prisoner put the balls such that the probability of his release is maximized?