Probability and Statistics: Lecture-7

Monsoon-2020

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on August 24, 2020

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Shrewd Prisoner Problem

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Question: How should the prisoner put the balls such that the probability of his release is maximized?

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* Put 1 white and 0 black in urn-1, and put 49 white and 100 black in urn-2
* The probability of success is $\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{49}{99} \approx 3/4$

» Extension of the problem

Quiz-1

What is the chance of success of the prisoner if he had been allowed to distribute 100 balls among 4 urns?

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What is the chance of success of the prisoner if he had been allowed to distribute 100 balls among 4 urns?

Quiz-2

What if the number of balls is increased?

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- Given that Wanda and Fred have no reason to trust, what is the good option?


Wanda
Fred
DON'T CONFESS
CONFESS
DON'T CONFESS
2/2
10/0
CONFESS
0/10
5/5

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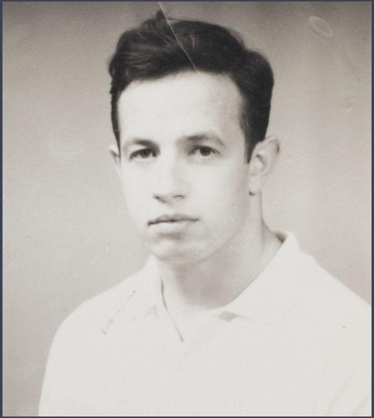
- Indeed, if both don't confess, then it is best for them; 2 years each!
- But another best option is when they both confess. Because in other cases, one of then has to spend 10 years in jail
- This is part of co-operative games, and $5/5$ is called Nash equilibrium
- These topics are part of topic known as game theory (John Nash!)

» About John Nash



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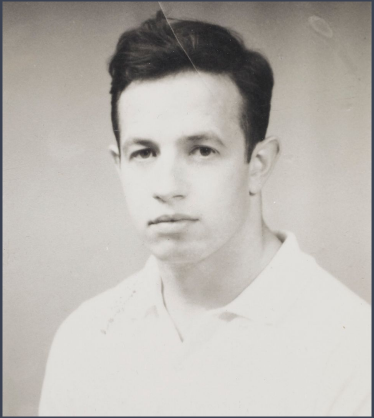

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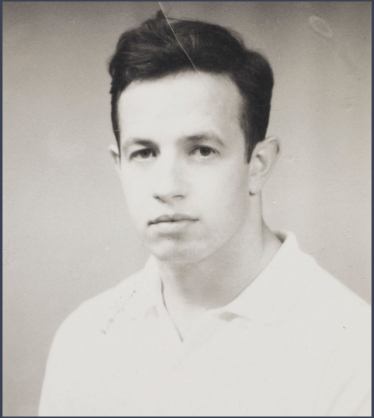

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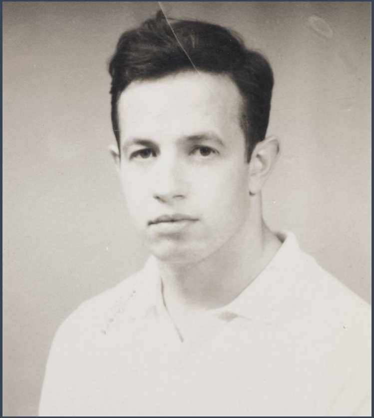

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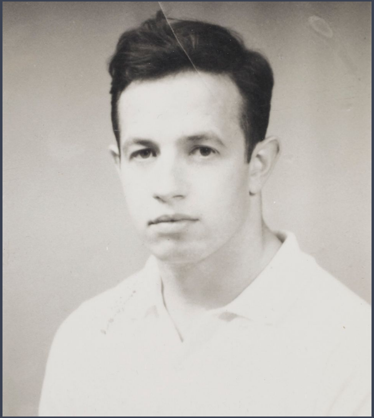

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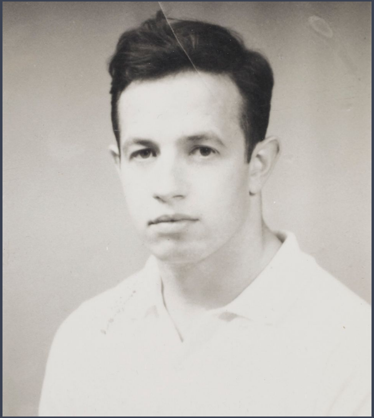

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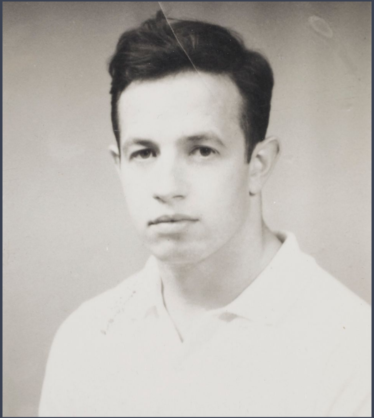

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More on this in Topics in Applied Optimization Elective!

» Watch Bar Scene in “A Beautiful Mind” Movie

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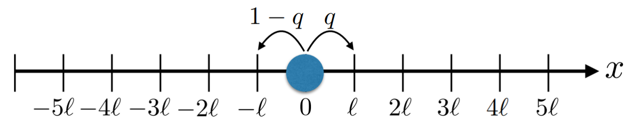
Movie of Bar Scene Here!

Movie of 1st Brownian Motion

Movie of 2nd Brownian Motion

» Random Walk, Probability



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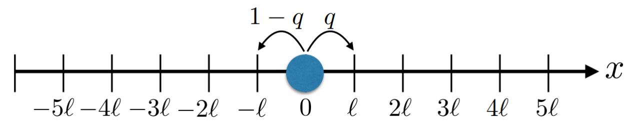

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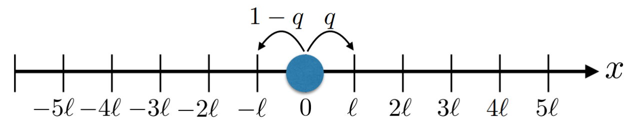

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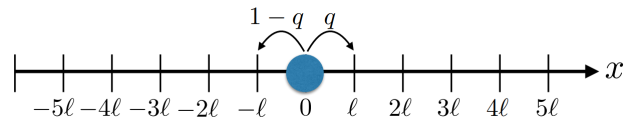

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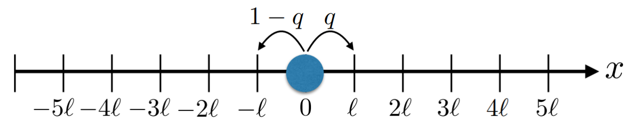

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Simple Random Walk in 1D

A walk is called simple random walk in 1D if there is a equal probability of either going to right or going to the left. Above, we set $p = q = 1/2$

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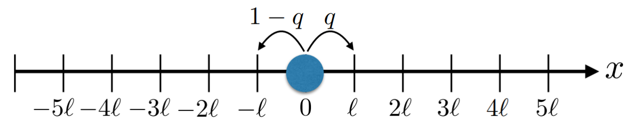

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Question

What is the probability that the person after ith step is at $x = 0$?

» Analysis of Simple Random Walk in 1D

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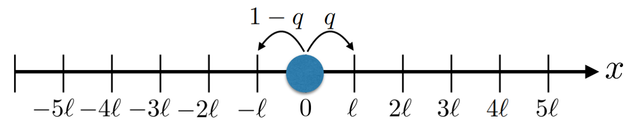

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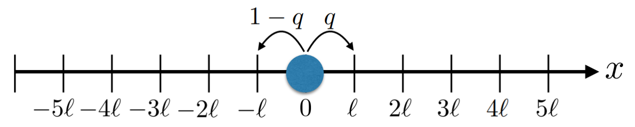

Draw the choice tree (Hint: Galton Board!):

» Analysis of Biased Random Walk in 1D, Binomial Distribution

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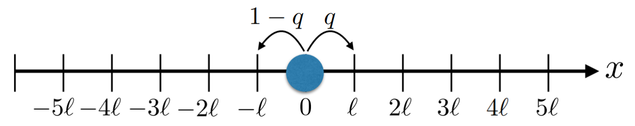

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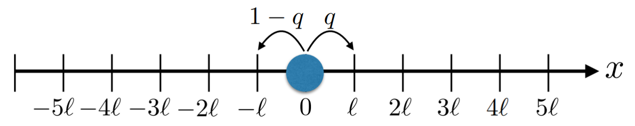

Draw the choice tree for unbiased random walk, derive binomial distribution:

» Random Walkers on 2D Grid...

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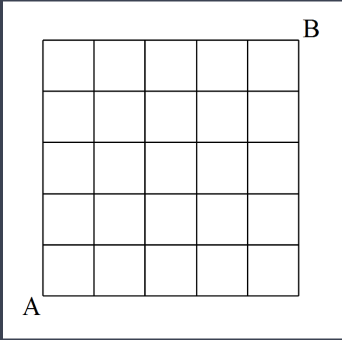

* Alice starts at point A, and each second, walks one edge right or up (if a point has two options, each direction has a 50

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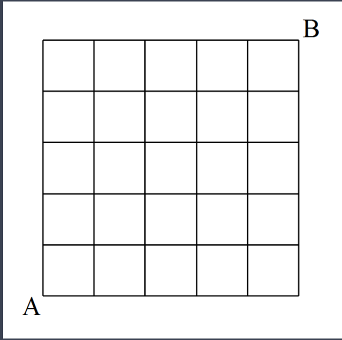

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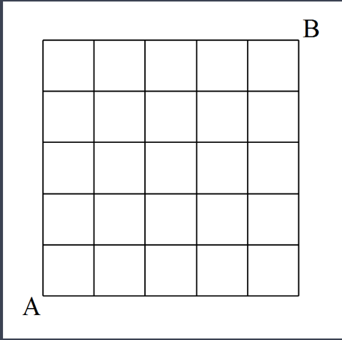

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* What is the probability Alice and Bob meet during their random walks?

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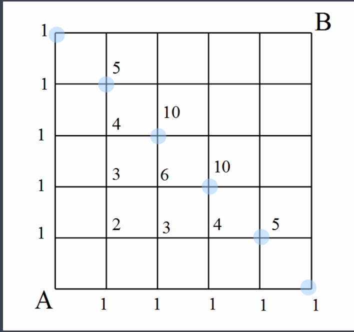

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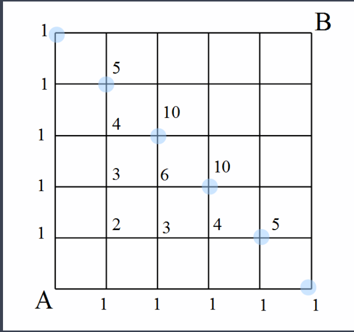

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* Number of ways they could have taken the 5 steps is 2^5 each, so combined, by product rule they can take $2^5 2^5 = 4^5 = 1024$ total paths!

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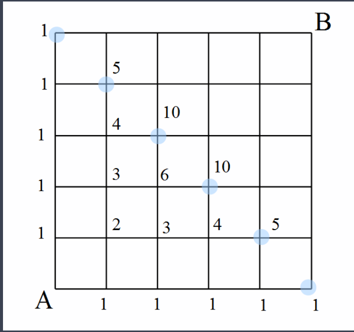

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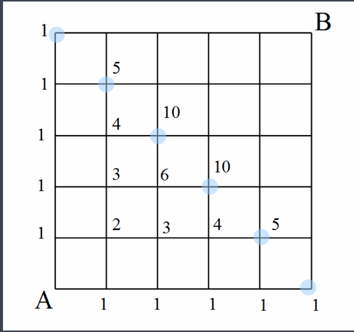

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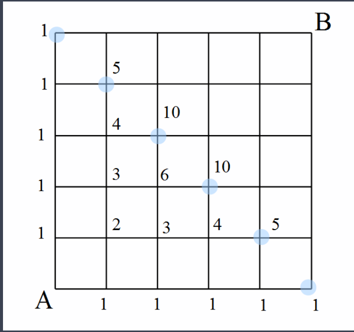

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- Hence, the total number of ways Bob and Alice could meet

$$1^2 + 5^2 + 10^2 + 5^2 + 1^2 = 252$$

- The probability that they meet is

$$252/1024 = 24.6\%$$

» Conditional Probability...

* **Experiment:** Throw two dice A and B simultaneously
* **Event:** Odd number on first die
* **Question:** What is the probability of the event E?

* **Event:** Odd number on first die A, and die B always shows even
* **Question:** What is the probability of the event E?