Probability and Statistics: Lecture-24

Monsoon-2020

by Pawan Kumar (IIIT, Hyderabad)
on October 7, 2020

» Checklist for online class

1. Turn off your microphone, when you are listening
2. Turn on microphone only when you have question
3. Attend tutorials to practice problems or to discuss solutions or doubts
4. Chat is not always reliable, I may not look at chat

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» Bound for Φ Function...

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Bound for Φ Function

Let $Z \sim N(0,1)$. We recall that

$$\Phi(x) = P(Z \leq x).$$

For all $x \geq 0$, the $\Phi-$function satisfies the following bound

$$\underbrace{\frac{1}{\sqrt{2\pi}}\frac{x}{x^2+1}e^{-x^2/2}}_{l.b} \leq \underbrace{1-\Phi(x)}_{Q(x)} \leq \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-x^2/2}$$

$h(x) = Q(x) - l.b \geq 0 \quad \forall x \geq 0$

» Answer to previous problem...

To show lower bound, let

$$h(x) = Q(x) - \frac{1}{\sqrt{2\pi}}\,\frac{x}{x^2+1}\,e^{-x^2/2}, \quad \forall x \geq 0$$

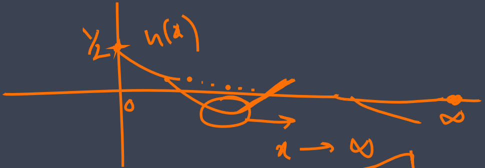
where

$$Q(x) = 1 - \Phi(x)$$

Properties of $h(x)$

① $h(0) = Q(0) - 0 = Q(0)$

$Q(0) = 1 - \phi(0) = 1 - \frac{1}{2} = \frac{1}{2}$



② $\lim_{x \to \infty} h(x)$ $\quad \boxed{2^x \quad x^2}$ $\quad 0$

$$= \underbrace{\lim_{x\to\infty} Q(x)}_{0} - \underbrace{\lim_{x\to\infty} \frac{1}{\sqrt{2\pi}}\,\frac{x}{x^2+1}\,e^{-x^2/2}}_{0}$$

$= 0$

③ We need $h'(x) < 0$

» Answer to previous problem...

$$h(x) = \underset{\sim}{Q(x)} - \frac{1}{\sqrt{2\pi}} \frac{x}{x^2+1} e^{-x^2/2}, \quad \forall x \geq 0$$

$$h'(x) = \frac{-2}{\sqrt{2\pi}} \frac{\overset{>0}{e^{-x^2/2}}}{\underset{>0}{(x^2+1)^2}} \quad \text{(Check!)} \quad \forall x \geq 0$$

$$< 0 \quad \forall x \geq 0$$

<u>Note</u>: h is strictly decreasing that starts at $h(0) = \frac{1}{2}$ and

it decr. as x incr.

It approaches 0 as $x \to \infty$

$\Rightarrow h(x) \geq 0 \quad \forall x \geq 0.$

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$Z \sim N(0,1)$

$$\underline{X} = \underbrace{\sigma Z + \mu,}_{} \quad \text{where } \sigma > 0$$

Definition of Normal Random Variables

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$$X = \sigma Z + \mu, \quad \text{where } \sigma > 0$$

* We have expectation of X, $E[X]$

Definition of Normal Random Variables

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$$X = \sigma Z + \mu, \quad \text{where } \sigma > 0$$

$\text{Var}(ax+b) = a^2\text{Var}(X)$

- We have expectation of X, $E[X]$

0 since $Z \sim N(0,1)$

$$E[X] = \sigma E[Z] + \mu = \mu,$$
$$\text{Var}(X) = \sigma^2 \text{Var}(Z) = \sigma^2$$

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- In this case, we write $X \sim N(\mu, \sigma^2)$

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check

- In this case, we write $X \sim N(\mu, \sigma^2)$
- Conversely, if $X \sim N(\mu, \sigma^2)$, then $Z = \dfrac{X - \mu}{\sigma}$ is standard RV, i.e., $Z \sim N(0, 1)$

» CDF and PDF of Normal Random Variable...

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CDF and PDf of Normal Random Variable

* To find the CDF of $X \sim N(\mu, \sigma^2)$, we have the following

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\begin{aligned}
F_X(x) &= P(X \le x) = P(\sigma Z + \mu \le x) \\
&= P\left(Z \le \frac{x - \mu}{\sigma}\right)
\end{aligned}
$$

CDF and PDf of Normal Random Variable

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$$\begin{aligned} F_X(x) &= P(X \leq x) = P(\sigma Z + \mu \leq x) \\ &= P\left(Z \leq \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right). \end{aligned}$$

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$$f_X(x) = \frac{d}{dx} F_X(x)$$

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$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx} \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$f_X(x) = \frac{d\phi}{dg} \frac{dg}{dx}$

$\phi(g(x))$

$g(x) = \frac{x-\mu}{\sigma}$

$g'(x) =$

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$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{d}{dx}\Phi\left(\frac{x-\mu}{\sigma}\right)$$
$$= \frac{1}{\sigma}\Phi'\left(\frac{x-\mu}{\sigma}\right)$$

$g'(x)$

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$X = \sigma Z + \mu$

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f_X(x) &= \frac{d}{dx} F_X(x) = \frac{d}{dx} \Phi\left(\frac{x-\mu}{\sigma}\right) \\
&= \frac{1}{\sigma} \Phi'\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma} f_Z\left(\frac{x-\mu}{\sigma}\right)
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$$

$\frac{1}{\sigma} f_Z(z)$

CDF and PDF of Normal Random Variable

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$$= \frac{1}{\sigma}\Phi'\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma}f_Z(\frac{x-\mu}{\sigma})$$
$$= \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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$$F_X(x) = P(X \leq x)$$

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then

$F_X(b) - F_X(a)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$F_X(x) = P(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$P(a < X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

» Solved Example...

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Let $X \sim N(-5, 4)$

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Let $X \sim N(-5, 4)$

- Find $P(X < 0)$
- Find $P(-7 < X < -3)$

» Solved Example...

$\boxed{\phi(x) + \phi(-x) = 1}$

$F_X(x) = \phi\left(\frac{x-\mu}{\sigma}\right)$

$N(\mu, \sigma^2)$

$\phi(-x) =$

-5 -3 0

Solved Example

Let $X \sim N(-5, 4)$ $\sigma^2 = 2$

1 * Find $P(X < 0)$ $= F_X(0) = \phi\left(\frac{0-(-5)}{2}\right) = \phi(2.5)$ (get value from table)

2 * Find $P(-7 < X < -3)$ $= F_X(b) - F(a) = \phi\left(\frac{(-3)-(-5)}{2}\right) - \phi\left(\frac{(-7)-(-5)}{2}\right)$

3 * Find $P(X > -3 \mid X > -5)$

$= \phi(1) - \phi(-1)$

$= 2\phi(1) - 1$

3) $\frac{P(X > -3, X > -5)}{P(X > -5)}$

$= \frac{P(X > -3)}{P(X > -5)} = \frac{1 - P(X \le -3)}{1 - P(X \le -5)}$

$= \phi(-3)$

$\phi(-5)$

» Answer to previous problem...

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» Linear Transformation of a Normal RV is a Normal RV...

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Theorem

If $X \sim N(\mu_X, \sigma_X^2)$, and $Y = aX + b$, where $a, b \in \mathbb{R}$, then $Y \sim N(\mu_Y, \sigma_Y^2)$ where

$$\mu_Y = a\mu_X + b, \quad \sigma_Y^2 = a^2\sigma_X^2.$$

$$Y = aX + b = a(\sigma_x Z + \mu_x) + b \quad \text{where} \quad Z = N(0,1)$$

$$= a\sigma_x Z + a\mu_x + b$$

$$E[Y] = a\sigma_x \cancel{E[Z]}^{0} + E[a\mu_x + b]$$

$$= a\mu_x + b$$

$$\mathrm{Var}(Y) = \mathrm{Var}(a\sigma_x Z + a\mu_x + b) = a^2\sigma_x^2 \cancel{\mathrm{Var}(Z)}^{1} = a^2\sigma_x^2$$

» Gamma Distribution...

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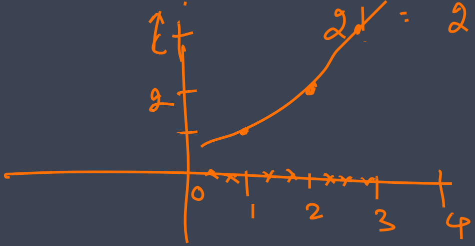
Gamma = Extension of factorial (Euler)

$n! = n \cdot n-1 \cdots 1$

$2! = 2 \cdot 1$

- Widely used distribution



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- Related to exponential and normal

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Gamma Function: Extension of Factorial Function

The Gamma function denoted by $\Gamma(x)$ is an extension of the factorial function to real numbers.

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$$\Gamma(n) = (n-1)!$$

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Gamma Function: Extension of Factorial Function

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$\boxed{\frac{1}{2}! = \frac{\sqrt{\pi}}{2}}$

Recall: If $n \in \{1, 2, 3, \ldots\}$, then

$$\Gamma(n) = (n-1)!$$

Real

Generally, for any positive number α, $\Gamma(\alpha)$ is defined as

Euler

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \quad \text{for } \alpha > 0.$$

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Gamma Function: Extension of Factorial Function

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$0! = 1$

$$\Gamma(n) = (n-1)!$$

$\Gamma(1) = 0! = 1$

Generally, for any positive number $\alpha, \Gamma(\alpha)$ is defined as

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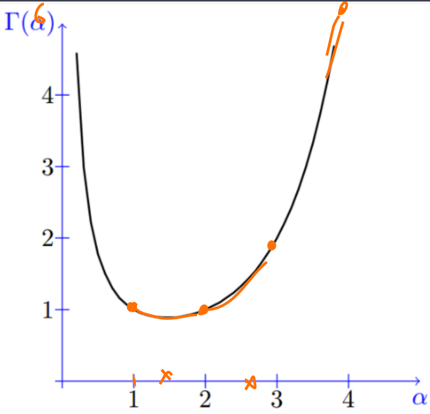

Gamma function for positive real values