Probability and Statistics: Lecture-19

Monsoon-2020

by Pawan Kumar (IIIT, Hyderabad)
on September 23, 2020

» Online Quiz

1. Please login to gradescope
2. Attempt the online quiz-2
3. You may use calculator if necessary
4. Time for the quiz is mentioned in the quiz

» Checklist

1. Turn off your microphone
2. Turn on microphone only when you have question
3. The grades will be uploaded by Today

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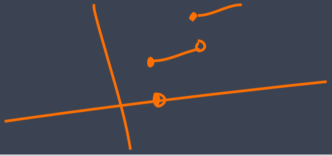
» Continuous Random Variable

Definition: Continuous Random Variable

A random variable X with CDF $F_X(x)$ is said to be continuous if $F_X(x)$ is a continuous function for all $x \in \mathbb{R}$.

» Continuous Random Variable



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» Continuous Random Variable

PMF $\rightarrow P(X = x) = 0$

Definition: Continuous Random Variable

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* CDF is always a continuous function whereas PMF may not be continuous
* We will usually assume that the CDF of a continuous R.V. is differentiable
* Although PMF does not makes sense for continuous random variable, we define probability density function

» Probability Density Function...

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» Probability Density Function...

- The probability density function (PDF) would be defined as probability per unit length
- Consider a continuous random variable X and define $\underbrace{f_X(x)}_{\text{PDF}}$ as follows

$$f_X(x) = \lim_{\Delta \to 0^+} \frac{P(x < X \leq x + \Delta)}{\Delta}$$

$X \in (x, x+\Delta)$

$\Delta \to 0$

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- We recall that $P(x < X \leq x + \Delta) = F_X(x + \Delta) - F_X(x)$.

$P(a < X \leq b) = F_X(b) - F_X(a)$

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Recall Calculus

$$\lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon} = f'(x) = \frac{df}{dx}$$

- Here, $f_X(x)$ gives the probability density at point x
- We recall that $P(x < X \leq x + \Delta) = F_X(x + \Delta) - F_X(x)$.
- We then have

CDF

$$f_X(x) = \lim_{\Delta \to 0} \frac{F_X(x + \Delta) - F_X(x)}{\Delta} = \frac{dF_X(x)}{dx} = F_X'(x)$$

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$$f_X(x) = \lim_{\Delta \to 0} \frac{F_X(x + \Delta) - F_X(x)}{\Delta} = \frac{dF_X(x)}{dx} = F'_X(x)$$

Definition of Probability Density Function

Let X be a continuous R.V. with continuous CDF $F_X(x)$. The function $f_X(x)$ defined by

$$f_X(x) = \frac{dF_X(x)}{dx} = F'_X(x),$$

is called the probability density function of X. We assume that $F_X(x)$ is differentiable.

» Example of CDF and PDF for Continuous Random Variable...

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Example: Derive PDF from CDF for Continuous R.V

Let X be a continuous random variable.

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Let X be a continuous random variable. Let X denote a real number chosen uniformly at random. Here uniformly means that all intervals in $[a, b]$ that have same length must have same probability.

Example: Derive PDF from CDF for Continuous R.V

Let X be a continuous random variable. Let X denote a real number chosen uniformly at random. Here uniformly means that all intervals in $[a, b]$ that have same length must have same probability. The CDF is given by:

$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

$\boxed{X \sim \text{Uniform}(a, b)}$

$X \sim \text{Uniform}[0, 1]$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

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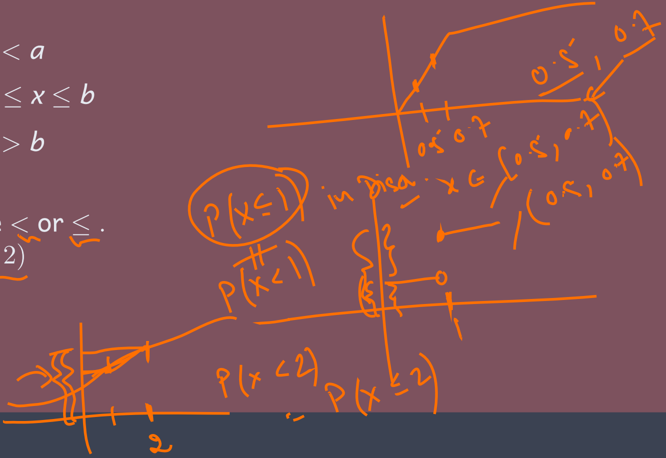
  That is, $P(X < 2) = P(X \le 2)$



» Example of CDF and PDF for Continuous Random Variable...

Example: Derive PDF from CDF for Continuous R.V

Let X be a continuous random variable. Let X denote a real number chosen uniformly at random. Here uniformly means that all intervals in $[a, b]$ that have same length must have same probability. The CDF is given by:

$a = 0, b = 1$ $1 \neq 0$ $f(x)$

$$F_X(x) = \begin{cases} 0 & \text{for } x < a \\ \dfrac{x-a}{b-a} & \text{for } a \le x \le b \\ 1 & \text{for } x > b \end{cases}$$

$\lim_{\epsilon \to 0}$

does not exist

$0 \neq$ Derivative $= 0$

$x = 0, \; x = 1$

Why?

$\frac{f(x+\epsilon) - f(x)}{\epsilon}$ at

PDF

* Does not matter if we use $<$ or $\le$.
  That is, $P(X < 2) = P(X \le 2)$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$$

$\lim_{\epsilon \to 0} \; \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon) - f(x)}{\epsilon}$

Example: Derive PDF from CDF for Continuous R.V

Let X be a continuous random variable. Let X denote a real number chosen uniformly at random. Here uniformly means that all intervals in $[a, b]$ that have same length must have same probability. The CDF is given by:

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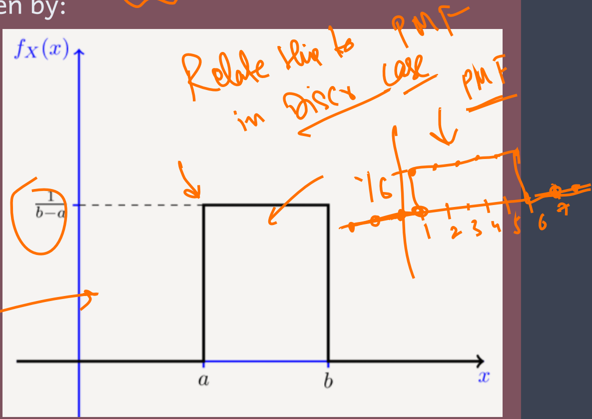

» Properties of PDF...

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* Since the PDF is the derivative of CDF, we have

» Properties of PDF...

derivative $\leftrightarrow$ anti derivative

- Since the PDF is the derivative of CDF, we have

$$F_X(x) = \underbrace{\int_{-\infty}^{x} f_X(u)\, du}$$

$$\frac{d}{dx} F_X(x) = f_X(x)$$

$f_x \; d \; x$

» Properties of PDF...

- Since the PDF is the derivative of CDF, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

- $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\, du$

$a < b$

$$\int_{-\infty}^{b} f_X(u)\, du - \int_{-\infty}^{a} f_X(u)\, du$$

$$\int_{a}^{b} f_X(u)\, du.$$

» Properties of PDF...

* Since the PDF is the derivative of CDF, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

$$\boxed{\sum_x p_X(x) = 1}$$

* $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(u)\, du$
* $\int_{-\infty}^{\infty} f_X(u)\, du = 1$

$P(-\infty < X \leq -\infty) = \text{Sample space}$

$P(S) = 1$ (By axiom)

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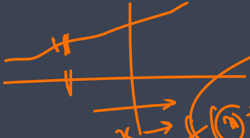


$\lim_{\epsilon \to 0}$ $f(x+\epsilon) - f(x) \geq 0$ ≥ 0

$\epsilon \geq 0$

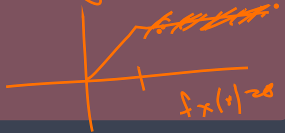
$\frac{df}{dx} =$

Since f is $\uparrow$ $f(x+\epsilon) - f(x) \geq 0$

Properties of PDF

* $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

$f_X(x)$ is the derivative of a non-decreasing fn, CDF



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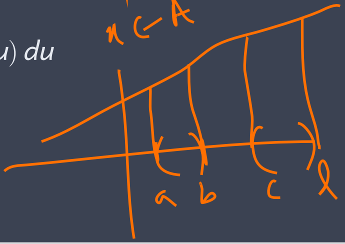
» Properties of PDF...

* Since the PDF is the derivative of CDF, we have

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du$$

$A = (a, b) \cup (c, d)$

$x \in A$



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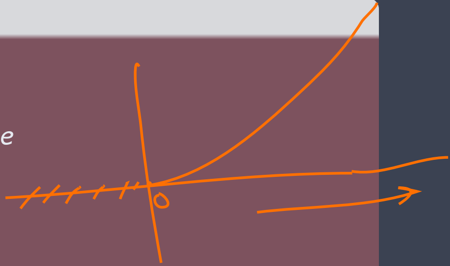
* More generally, for a set A, $\quad P(x \in A) = \int_A f_X(u)\, du$
* If $A = [0, 1] \cup [3, 4]$: $P(X \in A) = \int_0^1 f_X(u)\, du + \int_3^4 f_X(u)\, du$

» Example: PDF and CDF of Continuous Random Variable

Example: PDF and CDF of Continuous R.V.

PDF

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & otherwise \end{cases}$$



where c is a positive constant.

» Example: PDF and CDF of Continuous Random Variable

Example: PDF and CDF of Continuous R.V.

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

where c is a positive constant.

1. Find c

» Example: PDF and CDF of Continuous Random Variable

Example: PDF and CDF of Continuous R.V.

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$

where c is a positive constant.

1. Find c
2. Find the CDf of X, $F_X(x)$

» Example: PDF and CDF of Continuous Random Variable

Example: PDF and CDF of Continuous R.V.

$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & \textit{otherwise} \end{cases}$$

where c is a positive constant.

1. Find c
2. Find the CDf of X, $F_X(x)$
3. Find $P(1 < X < 3)$

$P(1 < x < 3) = \int_1^3 f_x(x)\,dx$

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$$f_X(x) = \begin{cases} ce^{-x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

where c is a positive constant.

1. Find c
2. Find the CDf of X, $F_X(x)$
3. Find $P(1 < X < 3)$

» Answer to previous problem...

a) Find c

We know that

$$\int_{-\infty}^{\infty} \underbrace{f_X(u)\,du}_{} = 1 \qquad (\text{Prop. } 2)$$

$$= \int_0^{\infty} c e^{-u}\,du = c\left[-e^{-u}\right]_0^{\infty}$$

$$= c$$

$$\Rightarrow \boxed{c = 1}$$

b) CDF of X

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\,du$$

$\underline{\text{For } x < 0} \Rightarrow F_X(x) = 0$

$\underline{\text{For } x \geq 0}$

$$F_X(x) = \int_0^{x} e^{-u}\,du$$

$$= 1 - e^{-x}$$

» Answer to previous problem...

$$\Rightarrow F_X(x) = \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

c) $P(1 < X < 3)$

We have both CDF & PDF.

Using CDF

$$P(1 < X < 3) = F_X(3) - F_X(1)$$
$$= (1 - e^{-3}) - (1 - e^{-1})$$

Equivalently using PDF

$$P(1 < X < 3) = \int_1^3 f_X(x)\,dx$$
$$= \int_1^3 c \cdot e^{-x}\,dx, \quad c = 1$$
$$= \int_1^3 e^{-x}\,dx = \left[-e^{-x}\right]_1^3$$
$$= e^{-1} - e^{-3}$$

» Range, Expectation, Variance of Continuous Random Variable...

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Definition: Range of Continuous Random Variable

The range of a random variable X is the set of possible values of the random variable. If X is a continuous random variable, we can define the range of X as the set of real numbers x for which the PDF is larger than zero, i.e,

$$R_X = \{x \mid f_X(x) > 0\}$$

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Recall that the expected value of discrete R.V. is

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Definition: Expected Value of Continuous R.V

Recall that the expected value of discrete R.V. is

$$EX = \sum_{x_k \in R_x} x_k P_X(x_k)$$

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Definition: Expected Value of Continuous R.V

Recall that the expected value of discrete R.V. is

$$EX = \sum_{x_k \in R_x} x_k P_X(x_k)$$

Replacing sum by integral, and PMF by PDF we have

$$EX = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

» Example of Expected Value of Continuous Random Variable...

» Example of Expected Value of Continuous Random Variable...

Example

Let $X \sim \text{Uniform}(a, b)$. Find EX.

PDF of X

$$f_X(x) : \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & x < a \text{ or } x > b \end{cases}$$

$(-\infty, a] = 0 \qquad (b, \infty] = 0$

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx = \int_a^b x \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b = \frac{a+b}{2}$$