Probability and Statistics: Lecture-14

Monsoon-2020

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on September 9, 2020

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Independent Random Variables

We consider two random variables independent if

» Independent Random Variables...

$P(A \cap B) = P(A, B)$

Independent Random Variables

We consider two random variables independent if

All events

$$P(X = x, Y = y) = P(X = x)P(Y = y), \quad \text{for all } x, y.$$

Independent Random Variables

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In general, if two random variables are independent, then

$$P(X \in A, X \in B) = P(X \in A)P(Y \in B), \quad \text{for all sets } A \text{ and } B$$

Also, we have

» Independent Random Variables...

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In general, if two random variables are independent, then

$$P(X \in A, \cancel{X} \in B) = P(X \in A)P(Y \in B), \quad \text{for all sets } A \text{ and } B$$

Also, we have

$$P(A|B) = P(A)$$

$$P(Y = y \mid X = x) = P(Y = y), \quad \text{for all } x, y$$

» Example: independent random variables, PMF

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Problem

We toss a coin twice, and let X denote the number of heads we observe.

» Example: independent random variables, PMF

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We toss a coin twice, and let X denote the number of heads we observe. After this, we then toss the coin two more times and define Y to be the number of heads we observe.

X Y

» Example: independent random variables, PMF

H H $\{HH, HT, TH, TT\}$

X Y Y (same)

Problem

We toss a coin twice, and let X denote the number of heads we observe. After this, we then toss the coin two more times and define Y to be the number of heads we observe. What is $P((X<2)$ and $(Y>1))$?

Sol:

$P((X<2)$ and $(Y>1)) = P(X<2)\,P(Y>1)$

because X and Y are independent

$1 < Y \leq 2$

$\Rightarrow Y = 2$

$$= \left(P_X(0) + P_X(1)\right) P_Y(2)$$

$$= \left(\frac{1}{4} + \frac{1}{2}\right) \cdot \frac{1}{4} = \frac{3}{16}$$

» Definition of Expectation..

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Definition of Expectation

The expectation of a discrete random variable X is defined as

» Definition of Expectation..

$X = \{x_1, x_2, \ldots, x_k\}$

① PMF of r.v. X ② X = values.

Definition of Expectation

The expectation of a discrete random variable X is defined as

$$E[X] = \sum_{x:p(x)>0} \underbrace{p(x)}_{\text{PMF}} \cdot x \qquad E[X] = \sum_{i=1}^{k} x_i\, p(x_i)$$

$p(x) \geq 0$

$p(x) = 0$ ✗

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The expectation of a discrete random variable X is defined as

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* Note we sum over all values of X that have non-zero probability

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Definition of Expectation

The expectation of a discrete random variable X is defined as

$$E[X] = \sum_{x:p(x)>0} p(x) \cdot x$$

* Note we sum over all values of X that have non-zero probability
* The other names of expectations are: mean, expected value, weighted average, center of mass, first moment

» Example of Expectation...

$X = \{1, 2, 3, 4, 5, 6\}$

Example of Expectation...

What is the expected value of 6-sided die roll?

Step ① What is r.v.?

② PMF of X

$$p(x) = \begin{cases} 1/6, & x \in \{1, 2, 3, \cdots 6\} \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \sum_{i=1}^{k=6} x_k \, p(x_k) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6}$$

$$= \frac{1+2+3+4+5+6}{6} = \frac{\cancel{21}^{7}}{\cancel{6}_{2}} = 3.5$$

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What is the expected value of 6-sided die roll?

Answer to the question

* Define a random variable: $X =$ RV for value of a roll

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- Define a random variable: $X =$ RV for value of a roll
- Find PMF:

$$P(X = x) = \begin{cases} 1/6, & x \in \{1, \ldots, 6\} \\ 0, & \text{otherwise} \end{cases}$$

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* Solve

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* Find PMF:

$$P(X = x) = \begin{cases} 1/6, & x \in \{1, \ldots, 6\} \\ 0, & \text{otherwise} \end{cases}$$

* Solve

$$E[X] = 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = \left(\frac{7}{2}\right) = 3.5$$

» Recall: Linear Functions...

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Definition of linear function

Let A, B be two vector spaces.

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Definition of linear function

Let A, B be two vector spaces. A function $f : A \to B$ is called linear function if the following holds:

Definition of linear function

Let A, B be two vector spaces. A function $f : A \to B$ is called linear function if the following holds:

- $f(u+v) = f(u) + f(v)$, $u, v \in A$

» Recall: Linear Functions...

Definition of linear function

Let A, B be two vector spaces. A function $f : A \to B$ is called linear function if the following holds:

* $f(u+v) = f(u) + f(v)$
* $f(cu) = cf(u)$, c is scalar.

» Recall: Linear Functions...

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Let A, B be two vector spaces. A function $f : A \to B$ is called linear function if the following holds:

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Equivalent definition is:

» Recall: Linear Functions...

Definition of linear function

Let A, B be two vector spaces. A function $f : A \to B$ is called linear function if the following holds:

* $f(u+v) = f(u) + f(v)$
* $f(cu) = cf(u)$, c is scalar.

combine

$u \in A$
$u+v \in A$

Equivalent definition is:

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v), \quad \forall u, v \in A, \quad \alpha, \beta \text{ scalars}$$

» Quiz on linear functions...

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Problem on linear function

Which of the following functions are linear?

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Problem on linear function

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1. $f(x) = x$

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Problem on linear function

Which of the following functions are linear?

1. $f(x) = x$ ✓ Lin
2. $f(x) = ax + b$ ✗

» Quiz on linear functions...

Problem on linear function

Which of the following functions are linear?

1. $f(x) = x$
2. $f(x) = ax + b$
3. $f(x) = 10$

» Quiz on linear functions...

Problem on linear function

Which of the following functions are linear?

1. $f(x) = x$
2. $f(x) = ax + b$
3. $f(x) = 10$
4. $f(x) = Ax + b$

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6. $f(x, y) = 3x + 4y$
7. $f(x, y, z) = 3 + x + y + z$ ✗

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Problem on linear function

Which of the following functions are linear?

1. $f(x) = x$
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6. $f(x, y) = 3x + 4y$
7. $f(x, y, z) = 3 + x + y + z$
8. $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y \\ y - z \\ z \end{pmatrix}$

» Properties of Expectation...

» Properties of Expectation...

$E : \text{Sp.R.V} \to \mathbb{R}$

Linearity of Expectation

$$E[aX + b] = aE[X] + b$$

$$E[c] = \sum_{i=1} p(x_k) c = c[1]$$

Linearity of Expectation

$$E[aX + b] = aE[X] + b$$

Let $X = 6$ sided dice roll. Let $Y = 2X - 1$. If $E[X] = 3.5$, then what is $E[Y] = ?$

$$E[Y] = E[2X - 1]$$
$$= 2E[X] - E[1]$$
$$= 2E[X] - 1$$
$$= 2 \cdot 3.5 - 1$$
$$= 6$$

» Properties of Expectation...

Linearity of Expectation

$$E[aX + b] = aE[X] + b$$

Expectation of a sum

$$E[X + Y] = E[X] + E[Y]$$

Let $X = 6$ sided dice roll. Let $Y = 2X - 1$. If $E[X] = 3.5$, then what is $E[Y] = ?$

» Properties of Expectation...

$E[X] = \sum x_i\, p(x_i)$

Linearity of Expectation

$$E[aX + b] = aE[X] + b$$

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Let $X =$ 6 sided dice roll. Let $Y = 2X - 1$. If $E[X] = 3.5$, then what is $E[Y] =?$

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Linearity of Expectation

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Expectation of a sum

$$E[X + Y] = E[X] + E[Y]$$

Let $X = 6$ sided dice roll. Let $Y = 2X - 1$. If $E[X] = 3.5$, then what is $E[Y] = ?$

Consider two dice rolls. Let X = roll of dice 1, $Y =$ roll of die 2, then $E[X + Y] = ?$

$= E[X] + E[Y]$

$= 3.5 + 3.5$

$= 7$ Ans

» Properties of Expectation...

Linearity of Expectation

$$E[aX + b] = aE[X] + b$$

Expectation of a sum

$$E[X + Y] = E[X] + E[Y]$$

Expectation of a function of random variable

$$E[g(X)] = \sum_x g(x)p(x)$$

Let $X =$ 6 sided dice roll. Let $Y = 2X - 1$. If $E[X] = 3.5$, then what is $E[Y] =?$

Consider two dice rolls. Let X = roll of dice 1, $Y =$ roll of die 2, then $E[X + Y] =?$

» Properties of Expectation...

$g(x) = ax + b$

special case

Linearity of Expectation

$$E[aX + b] = aE[X] + b$$

Expectation of a sum

$$E[X + Y] = E[X] + E[Y]$$

Expectation of a function of random variable

PMF of X

$$E[g(X)] = \sum_x g(x)p(x)$$

Let $X = 6$ sided dice roll. Let $Y = 2X - 1$. If $E[X] = 3.5$, then what is $E[Y] =?$

Consider two dice rolls. Let X = roll of dice 1, $Y =$ roll of die 2, then $E[X + Y] =?$

$1 - x$

Let $X = 6$ sided dice roll. Let $g(x) = x^2$. If $E[X] = 3.5$, then what is $E[g(X)] =?$

PMF

$E[g(x)] = \sum_{i=1}^{6} g(x_i)\, p(x_i)$

$= \sum_{i=1}^{6} x_i^2\, p(x_i)$

$E[g(x)] = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + \ldots + 6^2 \cdot \frac{1}{6}$

» Proof of Linearity of Expectation...

$$\boxed{E[aX + b] = a\,E[X] + b}$$

$$E[aX+b] = \sum_x \underbrace{(ax+b)\,p(x)} = \sum_x ax\,p(x) + b\,p(x$$

$$= a\sum_x x\,p(x) + b\,\cancel{\sum p(x)} \rightarrow 1$$

$$= a\,E[x] + b$$

» Proof of Expectation of Function of a Random Variable...

$$\boxed{E[g(X)] = \sum_x g(x)\, p(x)}$$

Assignment

» Quiz on Expectation...

» Quiz on Expectation...

Question on Expectation

Let X be a discrete random variable, whose PMF is given as follows

$$P_X(x_k) = \begin{cases} \frac{1}{3}, & \text{for } x \in \{-1, 0, 1\}, \\ 0, & \text{otherwise.} \end{cases}$$

$g(x)$

Let $Y = |X|$. What is $E[Y]$?

$$E[Y] = \sum_x g(x)\, p(x) = g(-1)\,p(-1) + g(0)\,p(0) + g(1)\,p(1)$$

$$= 1 \cdot \frac{1}{3} + 0 + 1 \cdot \frac{1}{3} = \frac{2}{3}$$

» Expectation and Game of Chance...

» Expectation and Game of Chance...

Expectation and Game of Chance Problem

Consider a game where a die is rolled. After the roll, the total money you gain is the output of the die. How much money would you like to pay to play this game?

$E[X] = 3.5$

» Saint Petersberg Paradox...

Saint Petersberg Paradox

Consider a fair coin.

Saint Petersberg Paradox

Consider *a* fair coin. We toss the coin, if it is heads,the game ends, and you win Rs 2.

» Saint Petersberg Paradox...

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Consider *a* fair coin. We toss the coin, if it is heads,the *game* ends, *and* you win Rs 2. If it turns out to be tails, I toss it *again*.

» Saint Petersberg Paradox...

Saint Petersberg Paradox

Consider *a* fair coin. We toss the coin, if it is heads,the *game* ends, *and* you win Rs 2. If it turns out to be tails, I toss it *again*. If it is heads on the second toss, you win Rs 4.

» Saint Petersberg Paradox...

Saint Petersberg Paradox

Consider *a* fair coin. We toss the coin, if it is heads,the *game* ends, *and* you win Rs 2. If it turns out to be tails, I toss it *again*. If it is heads on the second toss, you win Rs 4. If not, I toss it *again*, *and* if it is heads on the third toss,you win Rs 8.

Saint Petersberg Paradox

Consider *a* fair coin. We toss the coin, if it is heads,the *game* ends, *and* you win Rs 2. If it turns out to be tails, I toss it *again*. If it is heads on the second toss, you win Rs 4. If not, I toss it *again*, *and* if it is heads on the third toss,you win Rs 8. In general, if the first heads *appears at* the nth toss, then you win Rs 2^n.

T T T . . . . . T H

n-1th 2^n

Saint Petersberg Paradox

Consider *a* fair coin. We toss the coin, if it is heads,the *game* ends, *and* you win Rs 2. If it turns out to be tails, I toss it *again*. If it is heads on the second toss, you win Rs 4. If not, I toss it *again*, *and* if it is heads on the third toss,you win Rs 8. In *general*, if the first heads *appears at* the nth toss, then you win Rs 2^n.
There is *an* entrance fee to play this *game*. What is the maximum *amount* of money you would pay to play this *game*?

» Analyzing Solution to Saint Petersberg Paradox Problem...

» Analyzing Solution to Saint Petersberg Paradox Problem...

n	Prize	Prob	Utility
1	Rs 2	1/2	Rs 2
2	Rs 4	1/4	Rs 4
3	Rs 8	1/8	Rs 8
4	Rs 16	1/16	Rs 16
5	Rs 32	1/32	Rs 32
6	Rs 64	1/64	Rs 64
7	Rs 128	1/128	Rs 128
8	Rs 252	1/252	Rs 252
⋮	⋮	⋮	⋮

100

Rs 5

1/10

» Analyzing Solution to Saint Petersberg Paradox Problem...

n	Prize	Prob	Utility
1	Rs 2	1/2	Rs 2
2	Rs 4	1/4	Rs 4
3	Rs 8	1/8	Rs 8
4	Rs 16	1/16	Rs 16
5	Rs 32	1/32	Rs 32
6	Rs 64	1/64	Rs 64
7	Rs 128	1/128	Rs 128
8	Rs 252	1/252	Rs 252
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Let us make some observations.

» Analyzing Solution to Saint Petersberg Paradox Problem...

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Let us make some observations.

* If we pay Rs 1, then if the game ends in 1st toss, then we get Rs 2. That is we gain Rs 1.

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Let us make some observations.

* If we pay Rs 1, then if the game ends in 1st toss, then we get Rs 2. That is we gain Rs 1.
* If we pay say Rs 10, and the game ends in 3rd toss, then we get Rs 8, and we loose Rs 2.
* Naturally, we wonder what is the expected value?

Expected Value for Saint Petersberg Paradox...

$$E[X] = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \cdots = 1 + 1 + 1 + \cdots = \infty$$

2

» Analyzing Solution to Saint Petersberg Paradox Problem...

n	Prize	Prob	Utility
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Let us make some observations.

- If we pay Rs 1, then if the game ends in 1st toss, then we get Rs 2. That is we gain Rs 1.
- If we pay say Rs 10, and the game ends in 3rd toss, then we get Rs 8, and we loose Rs 2.
- Naturally, we wonder what is the expected value?

Expected Value for Saint Petersberg Paradox...

Rs

$$E[X] = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \cdots = 1 + 1 + 1 + \cdots = \infty$$

So, are you willing to pay any amount to play this game given that in theory you can win upto Rs ∞?

» Analyzing Solution to Saint Petersberg Paradox Problem...

n	Prize	Prob	Utility
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⋮	⋮	⋮	⋮

Let us make some observations.

* If we pay Rs 1, then if the game ends in 1st toss, then we get Rs 2. That is we gain Rs 1.
* If we pay say Rs 10, and the game ends in 3rd toss, then we get Rs 8, and we loose Rs 2.
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Expected Value for Saint Petersberg Paradox...

$$E[X] = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \cdots = 1 + 1 + 1 + \cdots = \infty$$

So, are you willing to pay any amount to play this game given that in theory you can win upto Rs ∞? Can you see why this seems to be a paradox?

» Analyzing Solution to Saint Petersberg Paradox Problem...

Expected Value for Saint Petersberg Paradox...

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» Analyzing Solution to Saint Petersberg Paradox Problem...

Expected Value for Saint Petersberg Paradox...

$$E[X] = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \cdots = 1 + 1 + 1 + \cdots = \infty$$

So, are you willing to pay any amount to play this game given that in theory (on average) you can win upto Rs ∞?

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Expected Value for Saint Petersberg Paradox...

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It is paradoxical because of the following reasons:

» Analyzing Solution to Saint Petersberg Paradox Problem...

Expected Value for Saint Petersberg Paradox...

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So, are you willing to pay any amount to play this game given that in theory (on average) you can win upto Rs ∞? Can you see why this seems to be a paradox?

It is paradoxical because of the following reasons:

* In reality, very few would pay even Rs 15 to play this game.

» Analyzing Solution to Saint Petersberg Paradox Problem...

Expected Value for Saint Petersberg Paradox...

$$E[X] = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \cdots = 1 + 1 + 1 + \cdots = \infty$$

So, are you willing to pay any amount to play this game given that in theory (on average) you can win upto Rs ∞? Can you see why this seems to be a paradox?

It is paradoxical because of the following reasons:

* In reality, very few would pay even Rs 15 to play this game. Why?

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Expected Value for Saint Petersberg Paradox...

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So, are you willing to pay any amount to play this game given that in theory (on average) you can win upto Rs ∞? Can you see why this seems to be a paradox?

It is paradoxical because of the following reasons:

* In reality, very few would pay even Rs 15 to play this game. Why?
* Paradox: you should be willing to pay any amount,

» Analyzing Solution to Saint Petersberg Paradox Problem...

Expected Value for Saint Petersberg Paradox...

$$E[X] = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \cdots = 1 + 1 + 1 + \cdots = \infty$$

So, are you willing to pay any amount to play this game given that in theory (on average) you can win upto Rs ∞? Can you see why this seems to be a paradox?

It is paradoxical because of the following reasons:

* In reality, very few would pay even Rs 15 to play this game. Why?
* Paradox: you should be willing to pay any amount, but you won't be willing to do so!

» Analyzing Solution to Saint Petersberg Paradox Problem...

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n	Prize	Prob	Utility	Exp.
1	Rs 2	1/2	Rs 2	1
2	Rs 4	1/4	Rs 4	1
3	Rs 8	1/8	Rs 8	1
4	Rs 16	1/16	Rs 16	1
5	Rs 32	1/32	Rs 32	1
6	Rs 64	1/64	Rs 32	0.5
7	Rs 128	1/128	Rs 32	0.25
8	Rs 252	1/252	Rs 32	0.125
⋮	⋮	⋮	⋮	⋮

$2 \cdot \frac{1}{2} = 1$

$4 \cdot \frac{1}{4} = 1 + 1$

$2 \cdot \frac{1}{2} +$

$32 \cdot \frac{1}{64} = \frac{1}{2}$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

$\boxed{10 \ldots}$

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» Analyzing Solution to Saint Petersberg Paradox Problem...

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4	Rs 16	1/16	Rs 16	1
5	Rs 32	1/32	Rs 32	1
6	Rs 64	1/64	Rs 32	0.5
7	Rs 128	1/128	Rs 32	0.25
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Let us make some observations.

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* If we pay say Rs 10, and the game ends in 32nd toss, then on average, we get Rs 6, and we loose Rs 2.

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Expected Value for Saint Petersberg Paradox...

$$E[X] = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} \cdots = 1 + 1 + 1 + 1 + 1 + 0.5 + 0.25 + \cdots \approx 6$$

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So, we should not pay more than Rs 6 or much lower than this!