Probability and Statistics: Lecture-5

Monsoon-2020

by Pawan Kumar (IIIT, Hyderabad)
on August 19, 2020

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» Why do we need Probability?

What is probability?

Probability is a measure that we use to measure how likely an event or outcome is. For example, today there is 10% chance of rain.

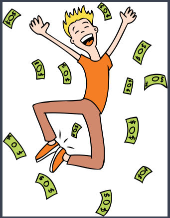
» Why do we need Probability?

What is probability?

Probability is a measure that we use to measure how likely an event or outcome is. For example, today there is 10% chance of rain.

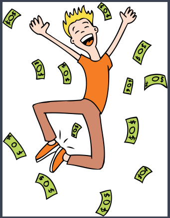
How likely is that one player or team will win the game?

How likely is that one will win the lottery?

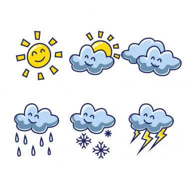


» Motivation for Probability...

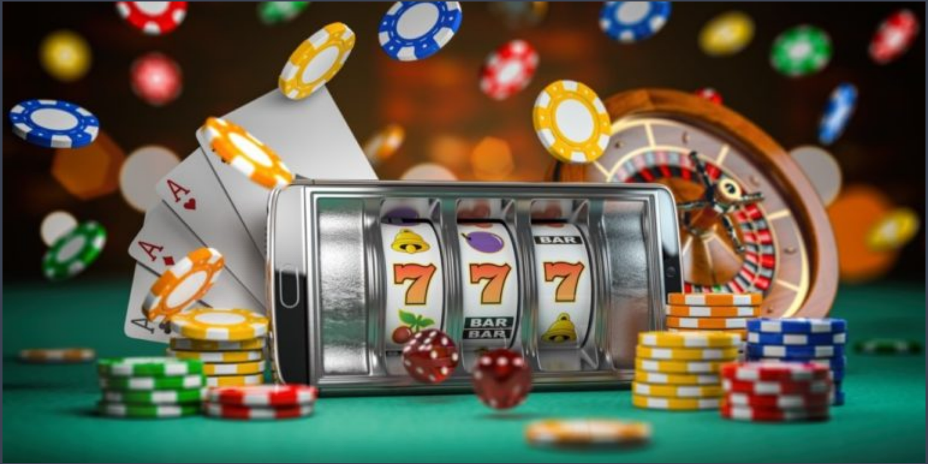
How likely is that one will win the lottery?



How likely it is to rain or snow?



» Study of Probability started with Games of Chance...



» Game of Chance, Probability, and War

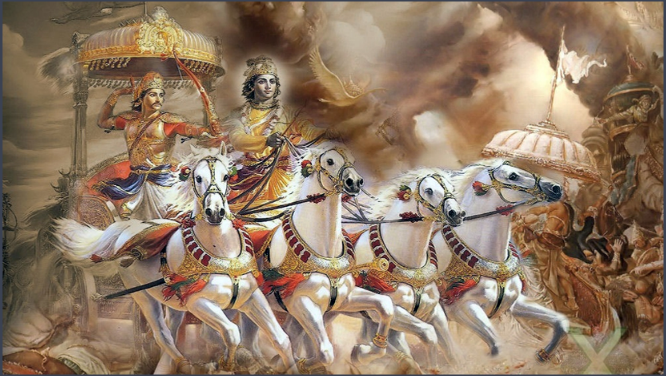


illustration of great war

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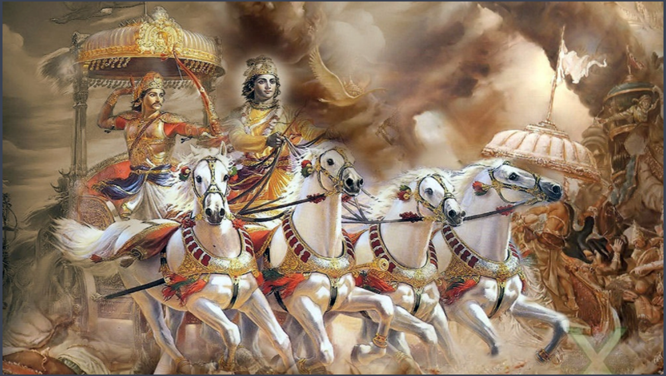

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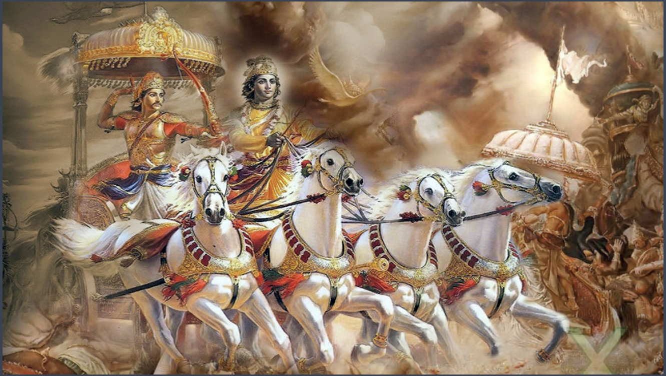

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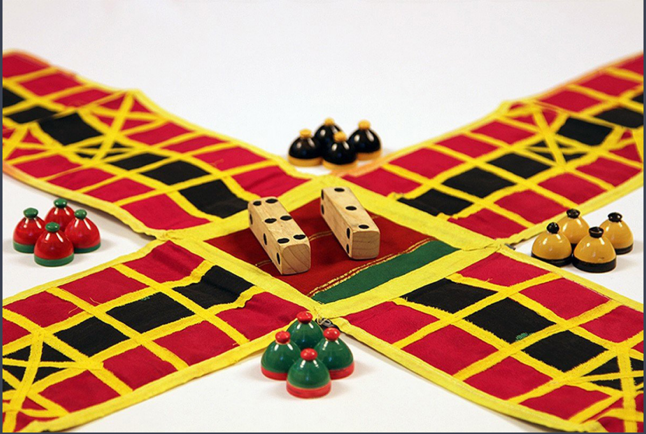
* Mahabharat, happened around 900 BCE (Proof dates to 400BCE)
* Struggle between two major group of cousins: Adventure, Drama, Suspense, Thriller, etc
* ... and a game of chance!

» What Does Game of Chance has to do with Mahabharat in India?



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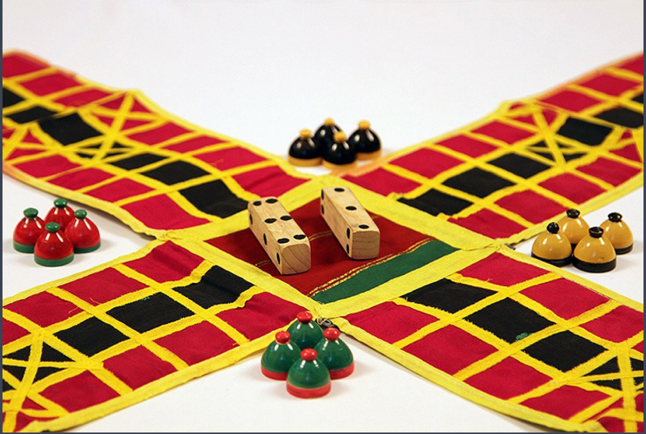

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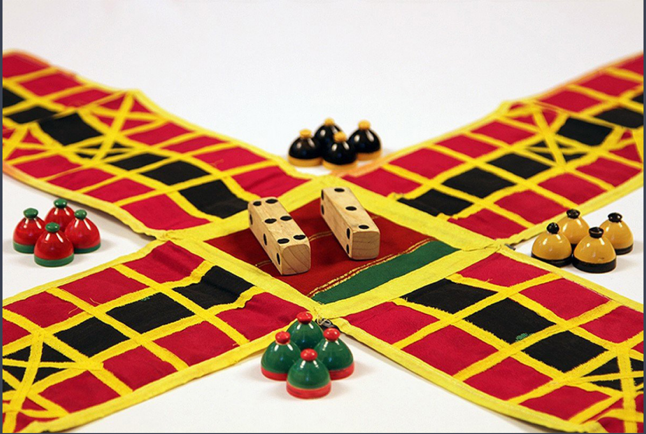

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* In ludo, there is one dice with 6 sides
* There are two cuboidal dices with number of dots on them
* The move is determined by a randomly throwing the two dices

» Game of Chance and Early History of Probability

Cardino, Italy

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- On the left is Cardano, an Italian mathematician, who studied game of chance
- He gambled for about 25 years!
- His work on probability were published in famous 15 page "Liber de Ludo Aleae"

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- Oystein Ore's biography of Cardano, titled Cardano: The Gambling Scholar:

> ...I have gained the conviction that this pioneer work on probability is so extensive and in certain questions so successful that it would seem much more just to date the beginnings of probability theory from Cardano's treatise rather than the customary reckoning from Pascal's discussions with his gambling friend de Méré and the ensuing correspondence with Fermat, all of which took place at least a century after Cardano began composing his *De Ludo Aleae*.

Random Experiment

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Trial

When we repeat a random experiment several times, we call each one of them a trial.

» Examples of Experiments and Samples Spaces

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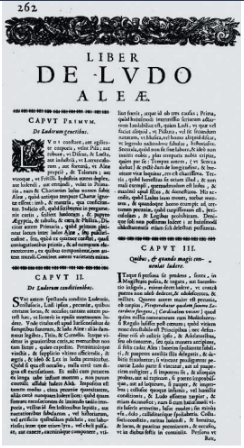
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LIBER

DE LVDO

ALEÆ

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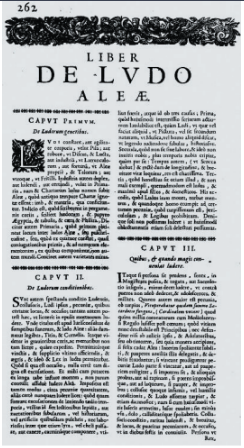

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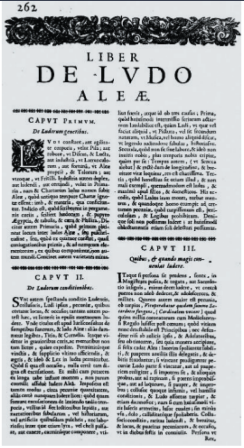
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> So there is one general rule, namely, that we should consider the whole circuit, and the number of those casts which represents in how many ways the favorable result can occur, and compare that number to the rest of the circuit, and according to that proportion should the mutual wagers be laid so that one may contend on equal terms.

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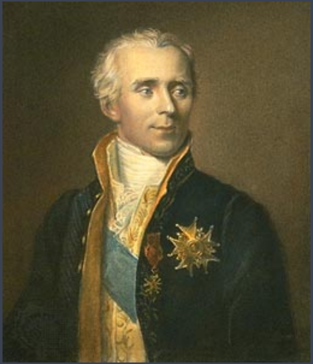

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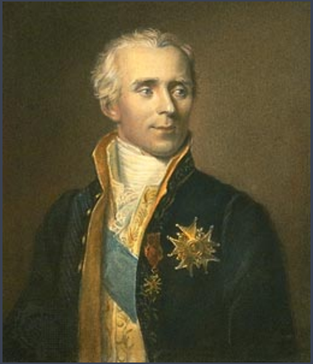

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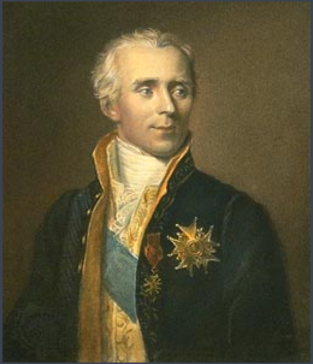

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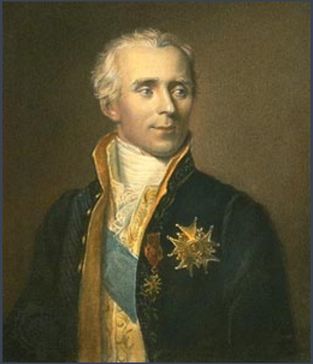

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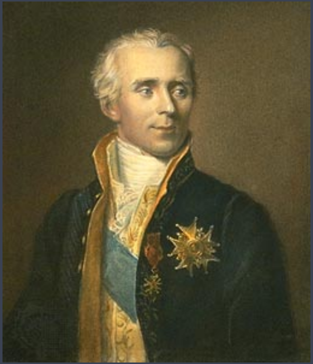

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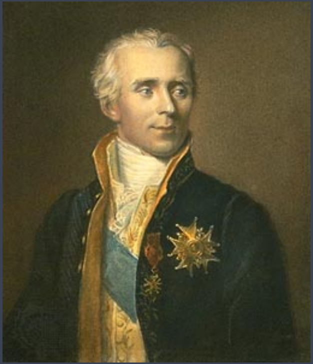

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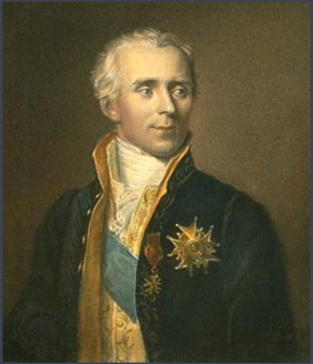

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  - Physics: Existence of black holes, Gravitational collapse, stability of solar, Speed of sound, Surface tension, etc

» Definition of Classical Probability...the way it evolved

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La théorie des hasards consiste à réduire tous les événemens du même genre, à un certain nombre de cas également possibles, c'est-à-dire, tels que nous soyons également indécis sur leur existence; et à déterminer le nombre des cas favorables à l'événement dont on cherche la probabilité. Le rapport de ce nombre à celui de tous les cas possibles, est la mesure de cette probabilité qui n'est ainsi qu'une fraction dont le numérateur est le nombre des cas favorables, et dont le dénominateur est le nombre de tous les cas possibles.

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Translation of last line: The probability of an event is a fraction whose numerator is the number of favourable cases and whose denominator is total number of cases. In the first line, he also mentions that all the events are equally possible (or equally likely).

» Finite Probability Space

Definition

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* Event: a set of outcomes
* Probability of the event: sum of outcome probabilities
* Is $p_i = 0$ possible?

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A finite sample space where each outcome is equally likely

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* Let $S = \{s_1, s_2, \cdots, s_N\}, \quad P(s_i) = P(s_j), \quad \forall i, j$
* Since all outcomes are equally likely,

$$P(s_i) = \frac{1}{N}, \quad \forall i \in \{1, 2, \cdots, N\}$$

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$$P(s_i) = \frac{1}{N}, \quad \forall i \in \{1, 2, \cdots, N\}$$

* If A is any event with cardinality $|A| = M$, then

$$P(A) = \frac{|A|}{|S|} = \frac{M}{N}$$

» Probability and Axioms of Probability

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Probability

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* Axiom 1: For any event A, $\quad P(A) \geq 0$

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Probability Axioms...

* Axiom 1: For any event A, $\quad P(A) \geq 0$
* Axiom 2: Probability of the sample space S is $P(S) = 1$

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Probability Axioms...

* Axiom 1: For any event A, $\quad P(A) \geq 0$
* Axiom 2: Probability of the sample space S is $P(S) = 1$
* Axiom 3: If $A1, A2, A3, \cdots$ are disjoint events, then $P(A_1 \cup A_2 \cup A_3 \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$

» Quiz

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Prove the following:

* For any event A, $P(A^c) = 1 - P(A)$

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* If $A \subseteq B$, then $P(A) \leq P(B)$
* $P(A - B) = P(A) - P(A \cap B)$

» Calculate Probabilities...

Coin Toss example

A coin is considered fair if the likelihood of getting heads or tails is same. Calculate the probability of obtaining head, when the coin is tossed once.

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Coin Toss example

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Answer

There are two favourable cases, the set of all possible cases is $\{H, T\}$. The number of head possible in one toss of the coin is 1. Hence, the probability of obtaining head is the fraction: $\frac{1}{2}$

» Quiz

Quiz

We roll *a* fair dice. What is the probability of the event $E = \{1, 6\}$?

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Answer

* Total number possibilities are $\{H, H\}, \{H, T\}, \{T, T\}, \{T, H\}$
* Favourable cases are $\{H, H\}, \{H, T\}, \{T, H\}$
* Hence the probability of atleast one head is = $\frac{3}{4}$

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Ten fair coins are tossed simultaneously. What is the probability of getting atleast one head?

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Ten fair coins are tossed simultaneously. What is the probability of getting atleast one head?

Answer

* Total number possibilities are 2^{10}
* Only case with no head: $\{T, T, T, T, T, T, T, T, T, T\}$. Favourable cases are: $2^{10} - 1$

» Calculate Probabilities...

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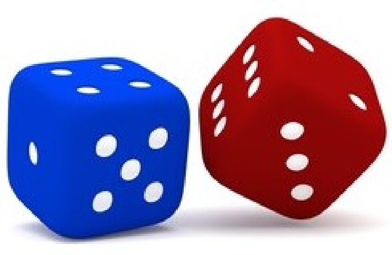
Ten fair coins are tossed simultaneously. What is the probability of getting atleast one head?

Answer

* Total number possibilities are 2^{10}
* Only case with no head: $\{T, T, T, T, T, T, T, T, T, T\}$. Favourable cases are: $2^{10} - 1$
* Hence the probability of atleast one head is = $\dfrac{2^{10} - 1}{2^{10}}$

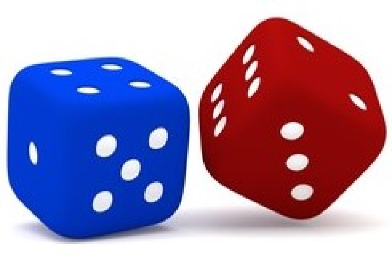
» Computing probabilities...

Consider an experiment of throwing two dice: blue and red one.



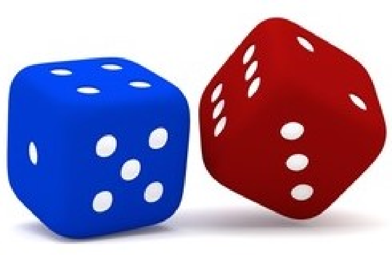
» Computing probabilities...

Consider an experiment of throwing two dice: blue and red one.



What did we assume?

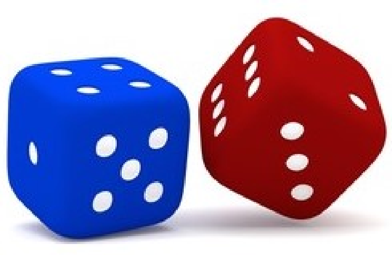
Consider an experiment of throwing two dice: blue and red one.



What did we assume?

* The outcome is $(x, y), \quad x, y, \in \{1, 2, \ldots, 6\}$

Consider an experiment of throwing two dice: blue and red one.

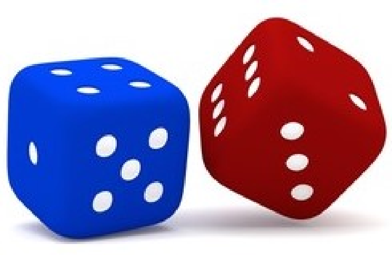


What did we assume?

* The outcome is $(x, y), \quad x, y, \in \{1, 2, \ldots, 6\}$
* The number of outcomes = 36

» Computing probabilities...

Consider an experiment of throwing two dice: blue and red one.

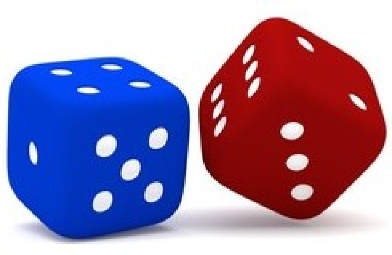


What did we assume?

- The outcome is (x, y), $\quad x, y, \in \{1, 2, \ldots, 6\}$
- The number of outcomes = 36
- We assume that all 36 are equiprobable

» Computing probabilities...

Consider an experiment of throwing two dice: blue and red one.



What did we assume?

- The outcome is $(x, y), \quad x, y, \in \{1, 2, \ldots, 6\}$
- The number of outcomes = 36
- We assume that all 36 are equiprobable
- Probability space: all outcomes

» Computing probabilities...

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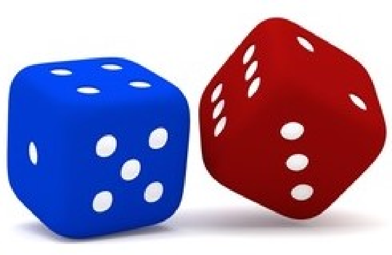

What did we assume?

* The outcome is $(x, y), \quad x, y, \in \{1, 2, \ldots, 6\}$
* The number of outcomes = 36
* We assume that all 36 are equiprobable
* Probability space: all outcomes
* Event: some favourable outcomes that we need to define precisely

» Attendance Quiz-2

https://tinyurl.com/y3ct7ucj