# 1 Approximating a summation

It is not always possible to find a closed-form expression for a sum. For example, no closed form is known for

$$S = \sum_{i=1}^{n} \sqrt{i}$$

In such cases, we have to resort to approximating sums.
Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a *non decreasing function.* Define

$$S ::= \sum_{i=1}^{n} f)i$$

and

$$I ::= \int_1^n f(x).dx$$

Then

$$I + f(1) \leq S \leq I + f(n)$$

*Similarily*, if $f$ is *non increasing*, then

$$I + f(n) \leq S \leq I + f(1)$$

**Proof** Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is non decreasing. The value of $S$ is the sum of the areas of $n$ unit-width rectangles of height $f(1), f(2), \ldots, f(n)$. The areas are shown in the following figure:

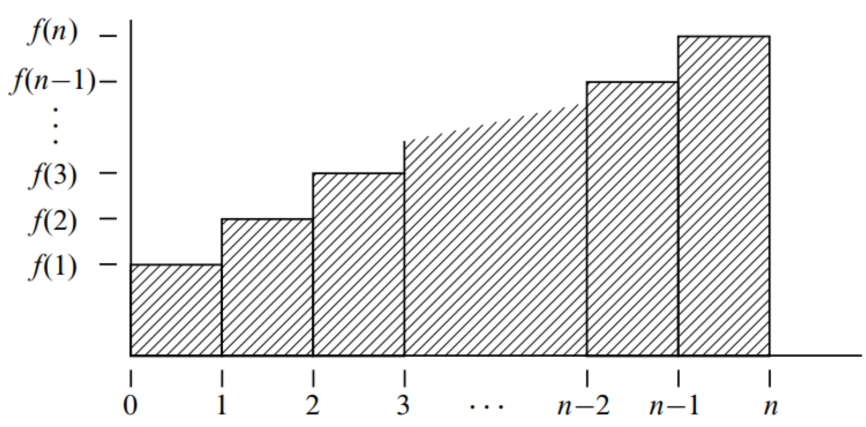


Figure 1: Figure 1

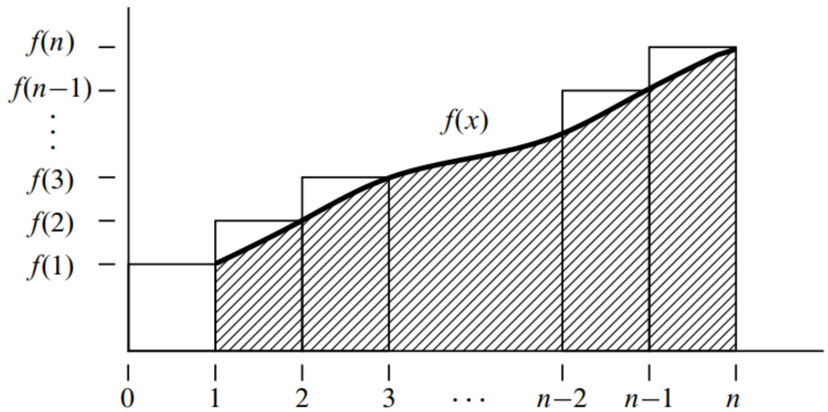


Figure 2: Figure 2

The value of $I$ is shown in the figure below:

Comparing the above two figures, the lower bound of the area can be seen as $f(1) + I$. Therefore, $S \geq f(1) + I$.

To derive the area for the upperbound, we shift the curve to one unit by left as shown below.

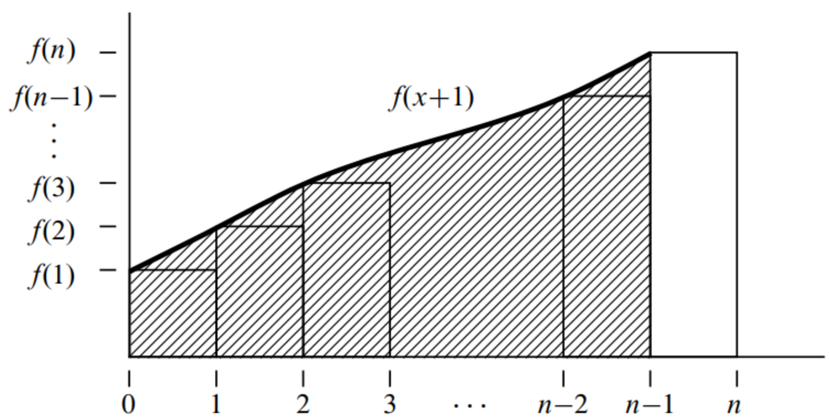


Figure 3: Figure 3

Now, the upperbound can be easily seen as $I + f(n)$, Therefore $S \leq I + f(n)$.

A similar analysis can be done for *non increasing functions*, or you can just see that the mirror image of the graph of a *non decreasing function* is a *non increasing function* and replace $x$ with $-x$ to get the mirror image and invert the equality signs.

## 2 Asymptotic Notion

Note: All of these relations are on functions.

1. **Asymptotic Equivalence ($\tilde{}$)**

   Def: $f(n)$ $g(n)$

   $$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$

   For example, $n^2$ $n^2 + n$ because

   $$\lim_{n\to\infty} \frac{n^2 + n}{n^2} = 1$$

   $\tilde{}$is an equivalence relation.

2. **Asymptotically smaller (Little Oh: o(.))**

   Def: $f(n) = o(g(n))$ iff

   $$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

   o(.) is a strict partial order relation

3. **Asymptotic Order of Growth (Big Oh: O(.))**

   Def: $f = O(g)$

   $$\limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

   O(.) is a strict partial order relation

4. **Same order of growth ($\theta$)**

   Def: $f = \theta(g)$

   $$f = O(g)$$

   and

   $$g = O(f)$$

   $\theta(.)$ is an equivalance realtion