Probability and Statistics: Lecture-20

Monsoon-2020

by Pawan Kumar (IIIT, Hyderabad)
on September 25, 2020

» Online Quiz

Class starts at : 10:42

1. Please login to gradescope
2. Attempt the online quiz-~~2~~ 3
3. You may use calculator if necessary
4. Time for the quiz is mentioned in the quiz

» Checklist

1. Turn off your microphone
2. Turn on microphone only when you have question

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» Expected Value of a Function of Continuous Random Variable...

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PMF of X

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$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

PDF

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We recall that the linearity of $E[\cdot]$ holds:

1. $E[aX + b] = aE[X] + b$ for all $a, b \in \mathbb{R}$
2. $E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$

» Example of Expected Value for Continuous Random Variable...

Example

Let the PDF of a continuous R.V. be given by

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find $E[\underbrace{X^n}_{g(x)}]$, $n \in \mathbb{N}$.

$g(x) = x^n, \ n \in \mathbb{N}$

$\frac{1}{n+2} + \frac{1}{2(n+1)} - 0 - 0$

$$E[g(x)] = E[x^n] = \int_{-\infty}^{\infty} x^n \underbrace{f_x(x)}_{\text{given}} dx$$

$$= \int_0^1 x^n \left(x + \frac{1}{2}\right) dx = \left[\frac{x^{n+2}}{n+2} + \frac{1}{2} \frac{x^{n+1}}{n+1} \right]_0^1 = \frac{3n+4}{2(n+1)(n+2)}$$

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So, for a continuous random variable, we have

g(x) g(x) PDF f_X

$$\mathrm{Var}(X) = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\,dx$$

$$= E[X^2] - E[X]^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx - \mu_X^2$$

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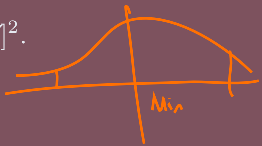

So, for a continuous random variable, we have

$$\mathrm{Var}(X) = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\, dx$$
$$= E[X^2] - E[X]^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx - \mu_X^2$$

$\mathrm{Var}(X + 100) = \mathrm{Var}(X)$

Recall that for $a, b \in \mathbb{R}$, we have

$$\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X).$$

» Example: Expected Value and Variance

Example

Consider the following PDF of the continuous random variable X

Example

Consider the following PDF of the continuous random variable X

$$f_X(x) = \begin{cases} \frac{3}{x^4} & x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$R_X = [1, \infty)$

Find Mean and Variance of X.

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_1^{\infty} x \cdot \frac{3}{x^4}\, dx = 3\int_1^{\infty} \frac{1}{x^3}\, dx$$

$$= \left[-\frac{3}{2} x^{-2}\right]_1^{\infty} = \frac{3}{2}.$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_1^{\infty} \frac{3}{x^2}\, dx = \left[-3x^{-1}\right]_1^{\infty} = 3$$

$$\mathrm{Var}(X) = E[x^2] - (E(x))^2 = 3 - 9/4 = 3/4$$

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Example of a function of continuous random variable

Let $X \sim$ Uniform$(0,1)$, and let $Y = e^X$.

- Find the CDF of Y
- Find the PDF of Y
- Find $E[Y]$

Find the mean and variance of X.

$R_X = (-\infty, \infty)$

$\boxed{R_X} = \{x \mid f_X(x) > 0\}$

Corresp. PDF

Given $X \sim$ Uniform$(0,1)$

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

CDF

$R_X = [0,1]$

$R_Y = (1, e]$

» Answer to previous problem...

CDF of Y

$F_Y(y) = P(Y \leq y) = 0$ for $y < 1$

$F_Y(y) = 1$ for $y \geq e$

$F_Y(y) = P(Y \leq y) = P(e^X \leq y)$

$= P(X \leq \ln y) = F_X(\ln y)$

$= \ln y$

Since $0 \leq \ln y \leq 1$

for $y \in [1, e]$

Summarize

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 1 \\ \ln y & 1 \leq y < e \\ 1 & \text{for } y \geq e \end{cases}$$

PDF of Y

$$f_Y(y) = F_Y'(y) = \begin{cases} 1/y & 1 \leq y \leq e \\ 0 & \text{otherwise} \end{cases}$$

» Answer to previous problem...

$$E[Y] = E[e^X]$$

$$= \int_{-\infty}^{\infty} e^x f_X(x)\, dx$$

$$= \int_0^1 e^x \cdot 1\, dx$$

$$= \left[e^x\right]_0^1 = e^1 - e^0$$

$$= e - 1$$

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= \int_1^e y \cdot \frac{1}{y}\, dy = e - 1$$

» Example: Function of Continuous Random Variable...

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$X \sim U(a,b)$ $\quad P(c \le X \le d) = \frac{d-c}{a-b}$

Example

Let $X \sim \text{Uniform}(-1, 1)$ and $Y = X^2$. Find the CDF and PDF of Y.

$R_X = [-1, 1]$, $\quad R_Y = [0, 1]$

$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$

$= \frac{\sqrt{y} - (-\sqrt{y})}{1 - (-1)} = \sqrt{y}$

CDF is cont. & diff.

Summarize

$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ \sqrt{y} & \text{for } 0 \le y \le 1 \\ 1 & \text{for } y > 1 \end{cases}$

$f_Y(y) = F_Y'(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

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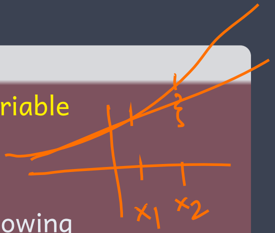

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We can directly find the PDF of Y using the following formula

$$f_Y(x) = \begin{cases} \dfrac{f_X(x_1)}{g'(x_1)} = f_X(x_1) \cdot \dfrac{dx_1}{dy} & \text{where } g(x_1) = y \\ 0 & \text{if } g(x) = y \text{ does not have a solution} \end{cases}$$

Since g is strictly $\uparrow$ g^{-1} is well defined. For each $y \in R_Y$ $\exists$ a unique x_1 s.t. $g(x_1) = y$

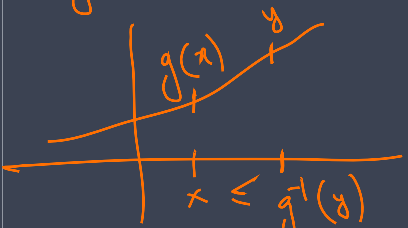
$x_1 = g^{-1}(y)$

$$\underbrace{F_Y(y)} = P(Y \le y) = P(g(X) \le y)$$

$$= P(X \le g^{-1}(y)) = \underbrace{F_X(g^{-1}(y))}$$

$g : \overset{x_1}{R_X} \to \overset{y \in}{R_Y}$

$g^{-1} : R_Y \to R_X$



$y \geq g(x)$

$g^{-1}(y) \geq x$