

Probability and Statistics: Lecture-3

Monsoon-2020

by Pawan Kumar (IIIT, Hyderabad)

on August 14, 2020

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» Combinations with or without repetitions...

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- * Let us try to find out...

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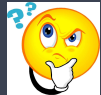
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Question

So, what could be your answer?



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We have an unlimited supply of tomatoes, cucumbers, and onions. We want to make a salad out of 4 units among these three ingredients (we do not have to use all ingredients). How many different salads we can make?

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- * But we want to do it wisely!



» **Solution continued...**

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Our goal: To pick 4 items out of 3 salads (Onions, Bell Peppers, Cucumber)

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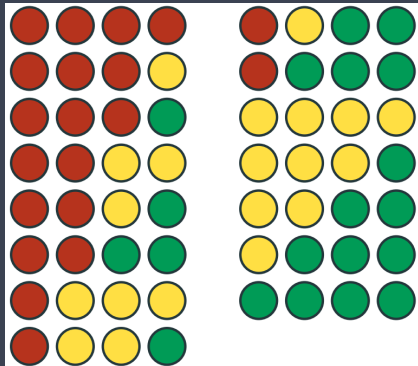
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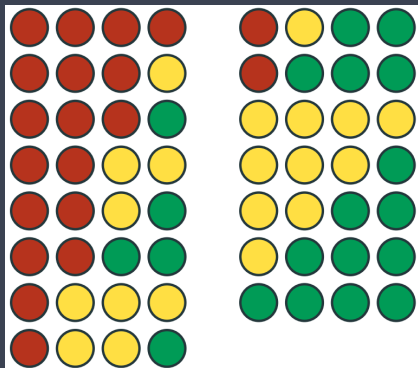
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* Let us list all possible combinations...



* There are 15 possible combinations. Do we see any structure?

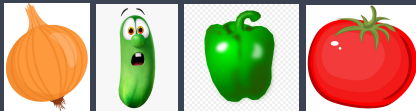
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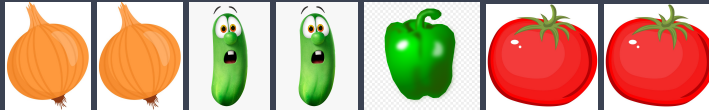


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- * Let us consider choosing 7 items out of unlimited supply of 4 salad items as follows...



- * One possible way to select 7 items out of 4 (with repetitions) is

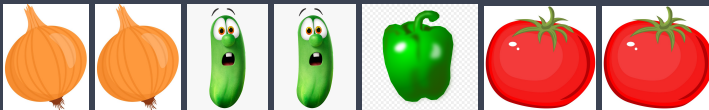


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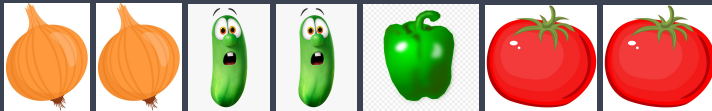
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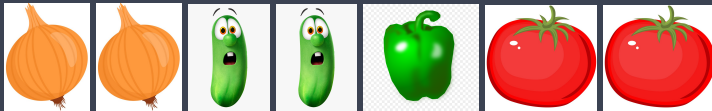
- * Do you already see a way to find all possible combinations?

» Combinations with Repetitions...

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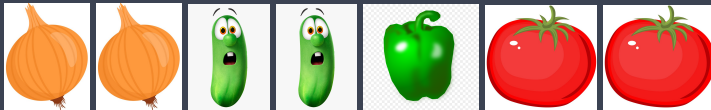


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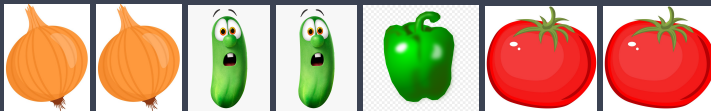
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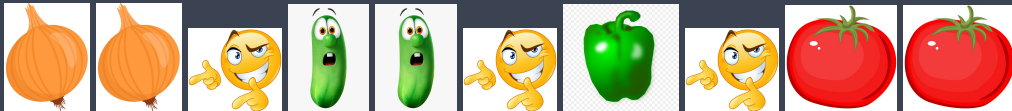


- * In the figure above, does the ordering of items matter? **No!**
- * If we indeed fix the ordering, then it is about putting delimiters, isn't it?

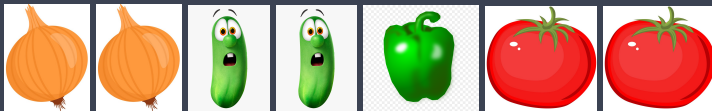
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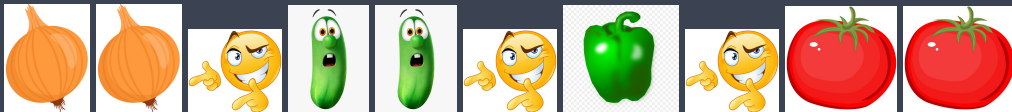
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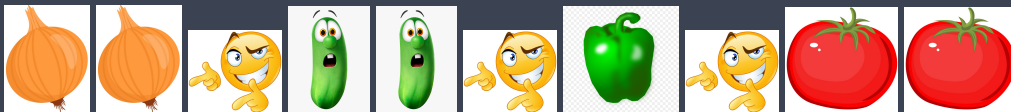


- * The number of ways we can put the delimiters determine the number of combinations

» **How many delimiters are needed?**

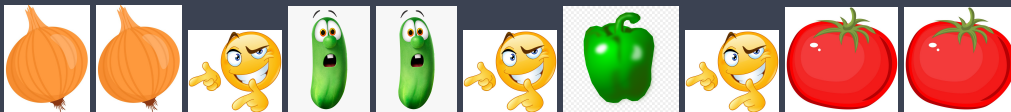
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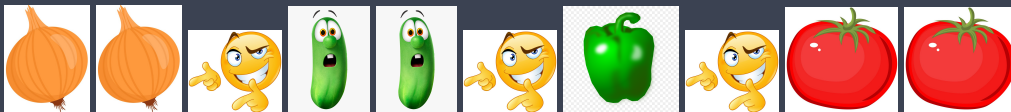


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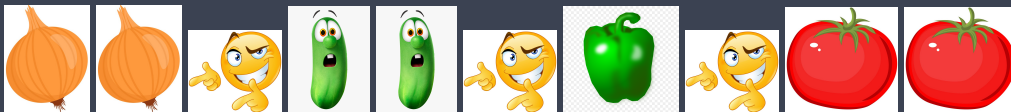
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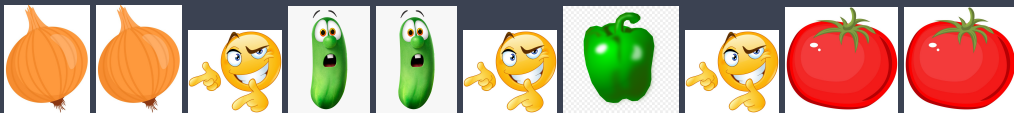
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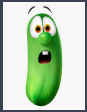
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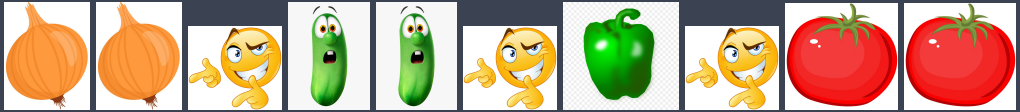
- * Form the example above, we need $3=(4-1)$ delimiters!
- * Total number of objects (7 items + 3 delimiters) is 10
- * The problem now reduces to arranging 3 delimiters among 10 items! Voila!

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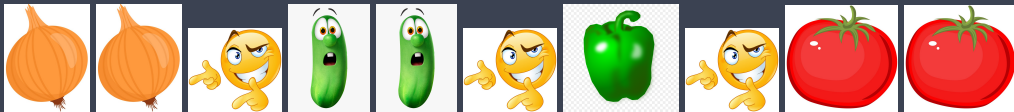


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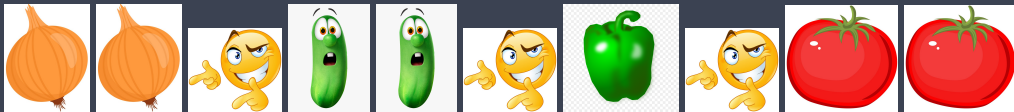


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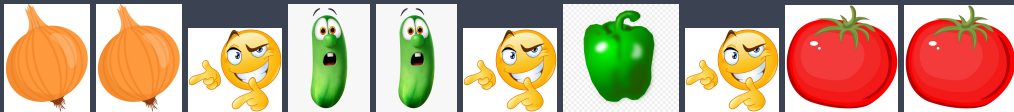
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Formula for combinations with repetitions

The formula for number of combinations of size k of n objects with repetitions is

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Quiz

Where does this problem fit in our combination and permutation table?

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- * This is a problem of selecting assignments for 4 people
- * Here assignments and people are different: so a ordered case
- * No repetitions: same assignment is not given to two persons
- * It is a case of ordered without repetitions, i.e., k -permutations

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* The answer is $\binom{9!}{(9-4)!} = 9 \times 8 \times 7 \times 6 = 3024$

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Problem with a twist!

There are 4 people and 9 different assignments. We need to distribute all assignments among people. No assignment should be assigned to two people. Every person can be given arbitrary number of assignments from 0 to 9. How many ways are there to do it?

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Which category this problem belongs to?



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Attempt to analyze the problem and fit it into a known case

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- * Same assignment can't be given to two persons, the number of subsets of assignments for second person depends on what we chose for first person! **How to attack this problem?**

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Options	4	4	4	4	4	4	4	4	4

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* There are $4^9 = 262144$ choices! This was a case of **Tuples**!

» Distributing Candies Among Kids

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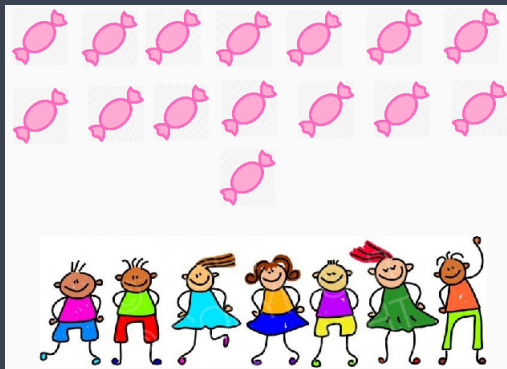
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Analyzing the problem, and fitting it into the known case

- * Number of combinations = size of combination
- * Number of options = Number of kids

» Distributing Candies Among Kids: Solution Finally!

Problem

There are 15 identical candies. How many ways are there to distribute them among 7 kids?

Analyzing the problem, and fitting it into the known case

- * Number of combinations = size of combination
- * Number of options = Number of kids
- * The answer is $\binom{15 + (7 - 1)}{(7 - 1)} = \binom{21}{3} = 54264$