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$R_X = \{1, 2, 3, 4, 5, 6\}$
$R_Y = \{1, 2, 3, 4, 5, 6\}$

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1. Find R_X, R_Y and the PMF of X and Y

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2. Find $P(X = 2, Y = 6)$
3. Find $P(X > 3 \mid Y = 2)$
4. Let $Z = X + Y$. Find the range and PMF of Z
5. Find $P(X = 4 \mid Z = 8)$

(a) $R_X = \{1, 2, \ldots, 6\}$

$R_Y = \{1, 2, \ldots, 6\}$

$$P_X(k) = \begin{cases} 1/6 & k \in \{1, 2, \cdots 6\} \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(k) = \begin{cases} 1/6 & k \in \{1, 2, \cdots 6\} \\ 0 & \text{otherwise} \end{cases}$$

(b) $P(X=2, Y=6) = P(X=2)\,P(Y=6)$

(X, Y are ind.)

$$= \frac{1}{6} \cdot \frac{1}{6} = 1/36$$

(c) $P(X>3 \mid Y=2)$.

Since X, Y are ind.

$$\Rightarrow P(X>3 \mid Y=2)$$

$$= P(X>3)$$

$$= P(X=4) + P(X=5) + P(X=6)$$

$$= 1/6 + 1/6 + 1/6 = 3/6 = 1/2$$

(d) $R_Z = \{2, 3, 4, \ldots, 12\}$

We need $P_Z(k)$, $k = 2, \ldots, 12$,

$$P_Z(2) = P(Z=2)$$

$$= P(X=1, Y=1)$$

$P_Z(2) = P(X=1, Y=1)$

$= P(X=1)\,P(Y=1)$

$= 1/6 \cdot 1/6 = 1/36$

$P_Z(3) = P(X=1, Y=2) + P(X=2, Y=1)$

$= P(X=1)\,P(Y=2) + P(X=2)\,P(Y=1)$

$= \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{18}$

$P_Z(4) = P(X=1, Y=3) + P(X=2, Y=2) + P(X=3, Y=1)$

$= 3 \cdot 1/36 = \frac{1}{12}$ and so on

$P_Z(5) = 4/36$

$P_Z(6) = 5/36$

$\vdots \qquad \vdots \qquad \vdots$

$P_Z(12) = \frac{1}{6}$

<u>Verify</u>: $\sum_k P_Z(k) = 1$

(2) $P(X=4 \mid Z=8)$

$$= \frac{P(X=4,\ Z=8)}{P(Z=8)}$$

X & Y ind

$$= \frac{\frac{1}{6} \cdot \frac{1}{6}}{5/36} = 1/5$$

» Problem 5

PMF of Sum of Poisson Random Variables

Let $X \sim \text{Poisson}(\alpha)$ and $Y \sim \text{Poisson}(\beta)$ be two independent Poisson random variables.

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Let $X \sim \text{Poisson}(\alpha)$ and $Y \sim \text{Poisson}(\beta)$ be two independent Poisson random variables. Let $Z = X + Y$ be a new random variable.

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PMF of Sum of Poisson Random Variables

Let $X \sim \text{Poisson}(\alpha)$ and $Y \sim \text{Poisson}(\beta)$ be two independent Poisson random variables. Let $Z = X + Y$ be a new random variable. Find the PMF of Z.

Sol^n: $R_X = \{0, 1, 2, \dots\}$, $R_Y = \{0, 1, 2, \dots\}$

$R_Z = \{0, 1, 2, \dots\}$

$$P_Z(k) = P(Z = k) = P(X + Y = k)$$

$$= \sum_{i=0}^{k} P(X + Y = k \mid X = i)\, P(X = i)$$ [Law of Total Prob]

» Answer to previous problem...

$$= \sum_{i=0}^{k} P(Y=k-i \mid X=i)\, P(X=i)$$

X, Y ind.

$$= \sum_{i=0}^{k} P(Y=k-i)\, P(X=i)$$

$$= \sum_{i=1}^{k} \frac{e^{-\beta}\beta^{k-i}}{(k-i)!} \cdot \frac{e^{-\alpha}\alpha^{i}}{i!}$$

$$= e^{-(\alpha+\beta)} \sum_{i=0}^{k} \frac{\alpha^i \beta^{k-i}}{(k-i)!\, i!} \Leftarrow$$

$$= \frac{e^{-(\alpha+\beta)}}{k!} \sum_{i=0}^{k} \underbrace{\frac{k!}{(k-i)!\, i!}\alpha^i \beta^{k-i}}$$

$$= \frac{e^{-(\alpha+\beta)}}{k!} \underbrace{\sum \binom{k}{i} \alpha^i \beta^{k-i}}$$

$$= \frac{e^{-(\alpha+\beta)}}{k!} (\alpha+\beta)^k$$

$$\Rightarrow Z \sim \text{Poisson}(\alpha+\beta)$$

» Problem 6

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PMF of a Function of a Random Variable

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$$P_X(k) = \begin{cases} 1/4 & \text{for } k = -2 \\ 1/8 & \text{for } k = -1 \\ 1/8 & \text{for } k = 0 \\ 1/4 & \text{for } k = 1 \\ 1/4 & \text{for } k = 2 \\ 0 & \text{otherwise} \end{cases}$$

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We define a new R.V. $Y = (X+1)^2$. What is PMF of Y?

Try! $\rightarrow$ Assign.

» Answer to previous problem...

» Definition of Cumulative Distribution Function

α $0.5 : \frac{1}{2}$

Definition of CDF

The cumulative distribution function (CDF) of a random variable X is defined as

$$F_X(x) = P(X \leq x), \quad \text{for all } x \in \mathbb{R}.$$

Measure theory

$F_X(1.5) = P(X \leq 1.5)$

$P(X \in \{1.5, 1.7, 1.9\})$

$R_X = [1, 2]$

$P(X = 1.5) = \frac{3}{\infty} = 0$

$P(X \in [1.5, 1.7])$

PMF: Prob. Mass Functn.

$R_X = \{1, 2, \dots, 6\}$

$P(X = 5) = 1/6$ $P(X \leq 3)$

Make sense for Discr.

» Definition of Cumulative Distribution Function

P_X

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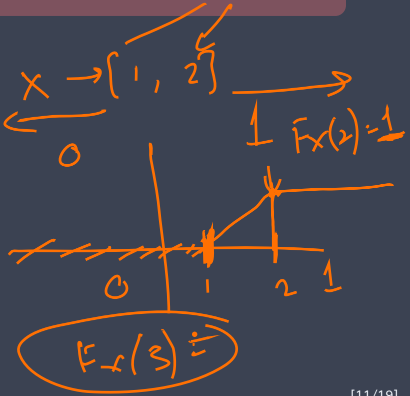

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Example of CDF

Consider an experiment where a coin is tossed twice.

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Consider an experiment where a coin is tossed twice. Let X denote the number of heads. Find the CDF of X.

$$P_X(0) = P(X = 0) = 1/4,$$
$$P_X(1) = P(X = 1) = 1/2,$$
$$P_X(2) = P(X = 2) = 1/4.$$

P(TT)

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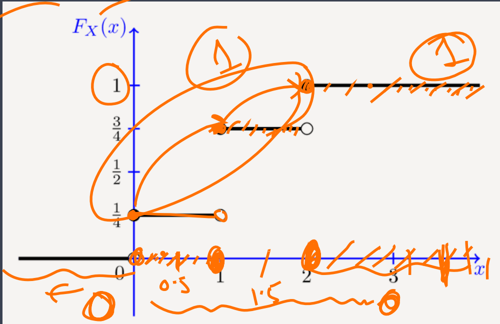

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$X = \{0, 1, 2\}$

» Remarks on CDF...

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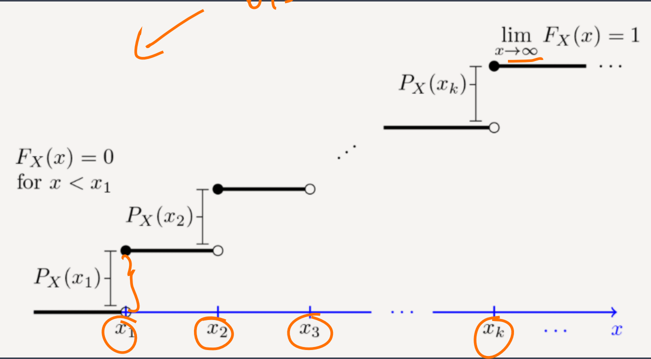
Discrete



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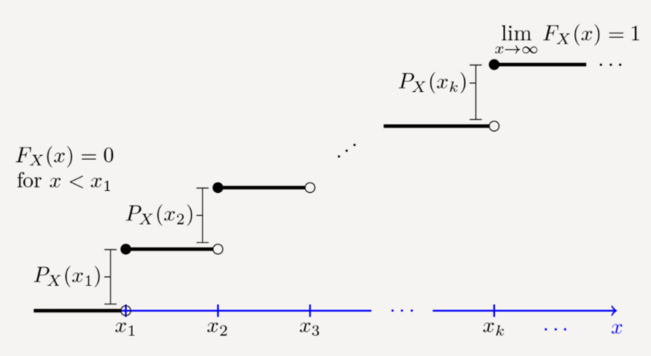
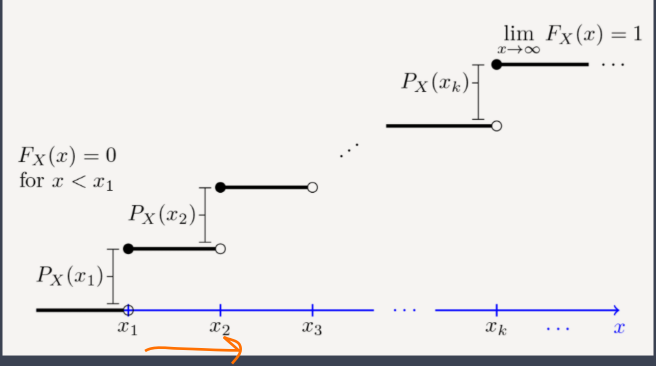


* Let X be a discrete R.V. with range $R_X = \{x_1, x_2, \ldots\}$

» Remarks on CDF...



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* Here we assume $x_1 < x_2 < \cdots$

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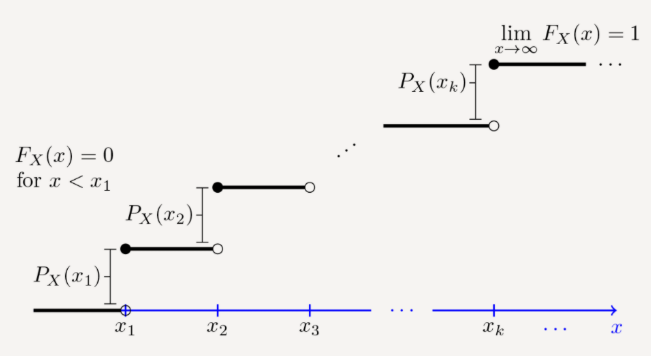

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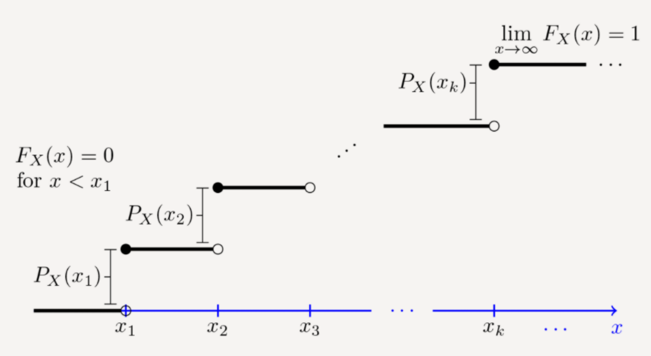

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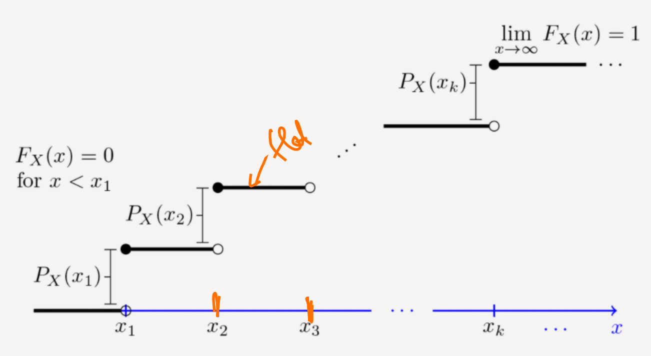

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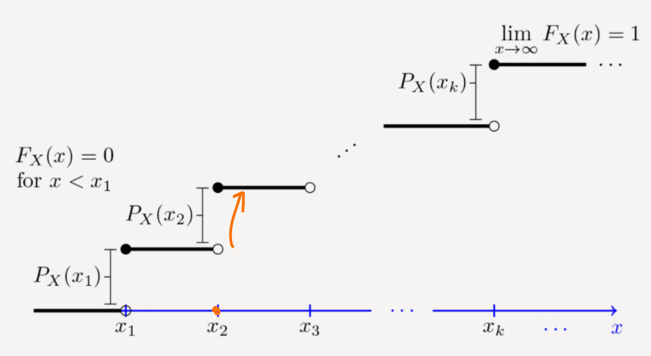
- CDF stays flat in x_k and x_{k+1}

$$F_X(x) = F_X(x_k), \quad x_k < x < x_{k+1}$$

discrete case

» Remarks on CDF...



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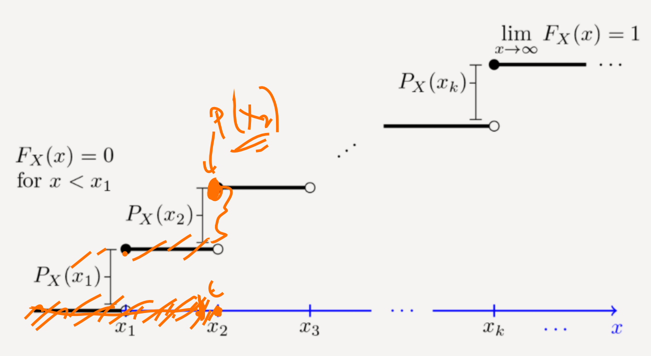

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- CDF jumps at each x_k. In particular, we have

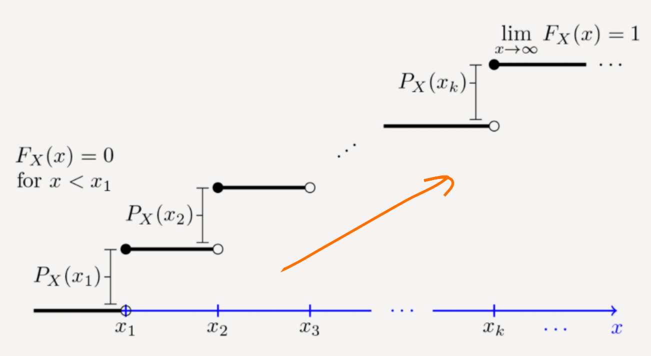
$k=2$

$$\underbrace{F_X(x_k) - F_X(x_k - \epsilon) = P_X(x_k)}, \quad \epsilon > 0 \text{ small}$$

$$F_X(x_2) - F_X(x_2 - \epsilon) = P_X(x_2)$$

$$P(X \le x_2) - P(X \le x_2 - \epsilon)$$

» Remarks on CDF...



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$$F_X(x_k) - F_X(x_k - \epsilon) = P_X(x_k), \quad \epsilon > 0 \text{ small}$$

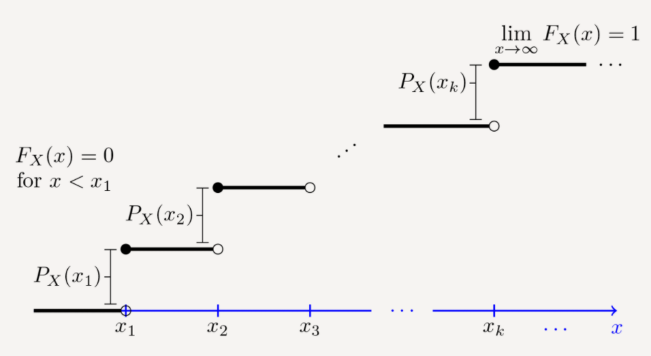
$F_X(x) \leq F_X(y)$

* Hence, CDF is a non-decreasing function: if $y > x$, then $\cancel{F_X(y) \leq F_X(x)}$

x y

$P_X(X \leq y)$

» Remarks on CDF...



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$x \to \infty$

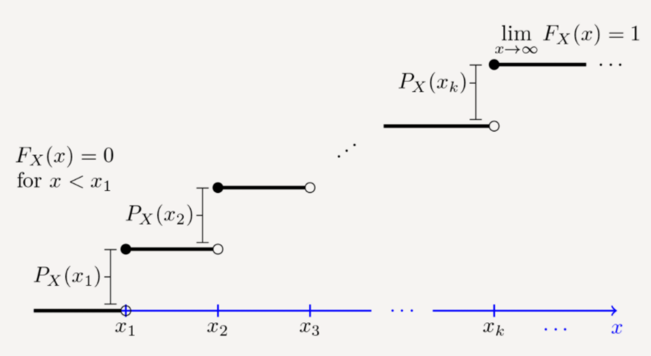
$P_X(X \leq x)$

* Hence, CDF is a non-decreasing function: if $y > x$, then $F_X(y) \leq F_X(x)$
* CDF approaches 1 as x becomes large, i.e., $\lim_{x \to \infty} F_X(x) = 1$

$x \to \infty$

As $x \to \infty$ $\{X \leq x\} \to$ Sample space

» Remarks on CDF...



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* CDF approaches 1 as x becomes large, i.e., $\lim_{x\to\infty} F_X(x) = 1$
* if $R_X = \{x_1, x_2, \dots\}$, $\quad F_X(x) = \sum_{x_k \leq x} P_X(x_k)$

» Properties of CDF...

» Properties of CDF...

A result

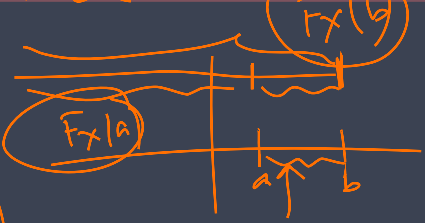
For all $a \leq b$, we have

$$P(a < X < b) = F_X(b) - F_X(a)$$

For $a \le b$.

$$P(X \le b) = P(X \le a) + P(a < X \le b)$$

$$\Rightarrow F_X(b) = F_X(a) + P(a < X \le b)$$



$<$ & $\le$

They make diff in discr. case.

$$P(X < x) = P(X \le x) - P(X = x)$$
$$= F_X(x) - P_X(x).$$

» Example of CDF...

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Let X be a discrete random variable with range $R_X = \{1, 2, 3, \dots\}$. Suppose the PMF of X is given by

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Let X be a discrete random variable with range $R_X = \{1, 2, 3, \dots\}$. Suppose the PMF of X is given by

$$P_X(k) = \frac{1}{2^k}, \quad \text{for } k = 1, 2, 3, \dots$$

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1. Find and plot the CDF of X, $F_X(x)$

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$F_X(5) = \sum_{k=1}^{5} \frac{1}{2^k}$

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

1. Find and plot the CDF of X, $F_X(x)$
2. Find $P(2 < X \leq 5)$
3. Find $P(X > 4)$

* Is this a valid PMF?

$\sum_{k=1}^{\infty} P_X(k) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$

» Answer to previous problem...

- Find and plot the CDF of X, $F_X(x)$

$\frac{1}{2} + \frac{1}{4}$

$P(2 < X \le 5)$ using CDF

$= F_X(5) - F_X(2)$

$= \frac{31}{32} - \frac{3}{4} = \frac{7}{32}$

$P(3) + P(4) + P(5).$

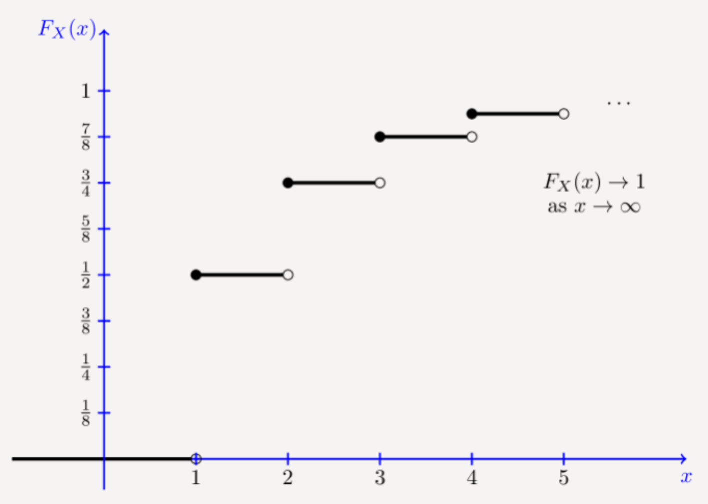
(c) $P(X > 4)$

$= 1 - P(X \le 4)$

$= 1 - F_X(4)$

$= 1 - \frac{15}{16} = \frac{1}{16}$

» Answer to previous problem...



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Let X be a continuous random variable.

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Let X be a continuous random variable. Let X denote a real number chosen uniformly at random.

Example

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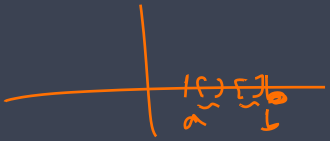

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* Note that $P(X = x) = 0$ for all x

» Example

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Let X be a continuous random variable. Let X denote a real number chosen uniformly at random. Here uniformly means that *all* intervals in $[a, b]$ that have same length must have same probability. Find the CDF of X.

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* Uniformity implies that

» Example

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Let X be a continuous random variable. Let X denote a real number chosen uniformly at random. Here uniformly means that all intervals in $[a, b]$ that have same length must have same probability. Find the CDF of X.

* Note that $P(X = x) = 0$ for all x
* Uniformity implies that

$$P(X \in [x_1, x_2]) \propto (x_2 - x_1), \quad \text{where } a \leq x_1 \leq x_2 \leq b$$

» Example

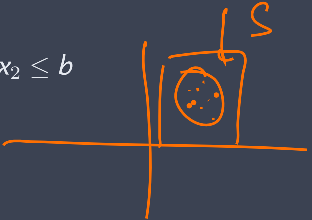
Example

Let X be a continuous random variable. Let X denote a real number chosen uniformly at random. Here uniformly means that all intervals in $[a, b]$ that have same length must have same probability. Find the CDF of X.

Sample sp

* Note that $P(X = x) = 0$ for all x
* Uniformity implies that

$$P(X \in [x_1, x_2]) \propto (x_2 - x_1), \quad \text{where } a \leq x_1 \leq x_2 \leq b$$



* Since $P(X \in [a, b]) = 1$, we have

$$P(X \in [x_1, x_2]) = \frac{x_2 - x_1}{b - a}, \quad \text{where } a \leq x_1 \leq x_2 \leq b$$

» Answer to previous problem...

$F_X(x) = P(X \le x)$.

$\Rightarrow F_X(x) = 0 \qquad x < a$

$F_X(x) = 1 \qquad x \geq b$

For $a \le x \le b$

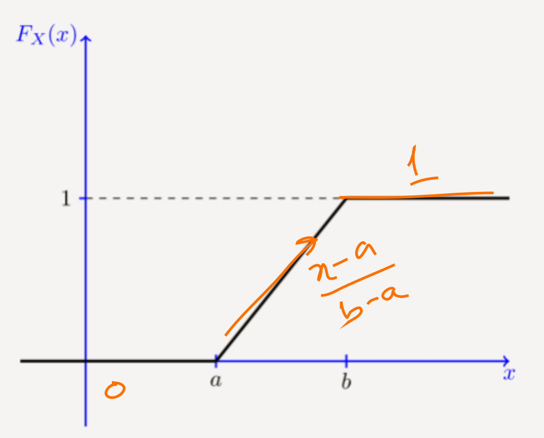
$F_X(x) = P(X \le x)$

$= P(X \in [a, x])$

$= \dfrac{x - a}{b - a}$

$$F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

» Answer to previous problem...



$\underline{F_X}$ ✓

» Continuous Random Variable

Definition: Continuous Random Variable

A random variable X with CDF $F_X(x)$ is said to be continuous if $F_X(x)$ is a continuous function for all $x \in \mathbb{R}$.