Probability and Statistics: Lecture-12

Monsoon-2020

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$$P(E \cap F) = P(E)P(F).$$

Otherwise, E and F are called dependent events.
If E and F are independent, then

$$P(E \mid F) = P(E).$$

Solution:

$$P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{P(E)\cancel{P(F)}}{\cancel{P(F)}}$$

$$= P(E)$$

» Examples of independent events...

$E \cap G = \{(1,4)\}$

$P(E \cap G) = \frac{1}{36} \times$

$P(G) = \frac{4}{36}$

Example

Roll two 6-sided dice, yielding values D_1 and D_2. Let us consider the following events:

- $E : D_1 = 1$
- $F : D_2 = 6$
- $G : D_1 + D_2 = 5$
- ~~Taht~~ That is, $G = \{(1,4),(2,3),(3,2),(4,1)\}$

1. Are E and F independent?

$E = \{(1,1), (1,2), \cdots, (1,6)\}$

$F = \{(1,6), (2,6), \cdots (6,6)\}$

$E \cap F = \{(1,6)\}$

$P(E \cap F) = \frac{1}{36}$, $P(E) = \frac{6}{36} = \frac{1}{6}$

$P(F) = \frac{1}{6}$

$\therefore P(E \cap F) = P(E) \cdot P(F)$

» Examples of independent events...

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1. Are E and F independent? Yes
2. Are E and G independent? No

» General Definition of Independence...

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Definition of independence for 3 events

Three events $E, F,$ and G are independent if

* $P(E \cap F \cap G) = P(E)P(F)P(G)$

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Definition of independence for 3 events

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General Definition for many events

The n events $E_1, E_2, \ldots, E_n$ are independent if $1 \le r \le n$

for $r = 1, \ldots, n$:

for every subset $E_1, E_2, \ldots, E_r$:

$$P(E_1 \cap E_2 \cap \cdots \cap E_r) = P(E_1)P(E_2)\cdots P(E_r)$$

» Example of general independence...

» Example of general independence...

$\boxed{F \cap G} = \{(1,6)\}$ $P(F \cap G) = \frac{1}{36} = P(F)\,P(G)$

Question

Each roll of 6-sided die is an independent trial. Two rolls with output D_1 and D_2. Consider the following events:

- $E : D_1 = 1$
- $F : D_2 = 6$
- $G : D_1 + D_2 = 7$
- $G = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$

$\boxed{E \cap G} = \{(1,6)\}$

$P(G) = \frac{6}{36} = \frac{1}{6}$

$P(E) = \frac{6}{36} = \frac{1}{6}$

Answer the following:

1. Are E and F independent? Yes
2. Are E and G independent? Yes
3. Are F and G independent? Yes
4. Are E, F, G independent? No

$P(E \cap G) = \frac{1}{36} = P(G)\,P(E)$

$\Rightarrow$ E & G are ind.

$E \cap F \cap G = \{(1,6)\}$

$P(E \cap F \cap G) = \frac{1}{36} \neq \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6}$

» Solution to problem on previous slide...

We have $E : D_1 = 1 \quad F : D_2 = 6 \quad G : D_1 + D_2 = 7$

» Independent Trials...

Definition of Independent Trials

A set of n trials are called independent trials if

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Definition of Independent Trials

A set of n trials are called independent trials if

1. Each of the n trials have same set of possible outcomes

» Independent Trials...

Event : Toss a coin → {H, T}

T_1 : Toss a coin ← {H, T}

T_2 : Toss a coin ← {H, T}

T_3 :

T_4 :

Definition of Independent Trials

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Examples of Independent Trials...

* Flip a coin n times

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Examples of Independent Trials...

* Flip a coin n times
* Roll a die n times

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Examples of Independent Trials...

* Flip a coin n times
* Roll a die n times
* Send a multiple choice survey to n people

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Examples of Independent Trials...

* Flip a coin n times
* Roll a die n times
* Send a multiple choice survey to n people
* Send n web requests to k different servers

» Examples involving independent trials...

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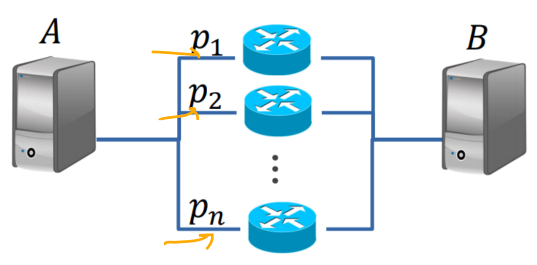
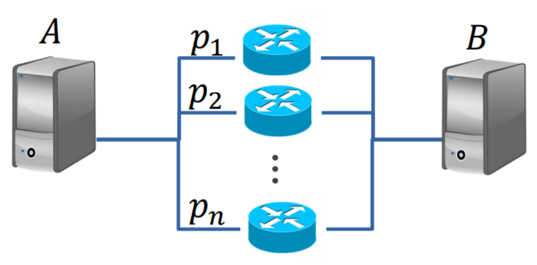


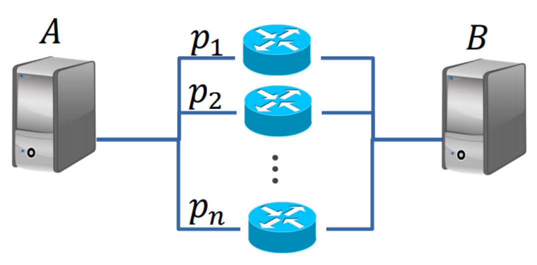
» Examples involving independent trials...



Problem

Consider the parallel network above:

» Examples involving independent trials...



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Consider the parallel network above:

* n independent routers, each with probability p_i of functioning, where $1 \leq i \leq n$

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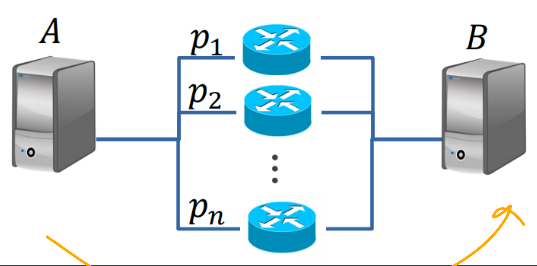

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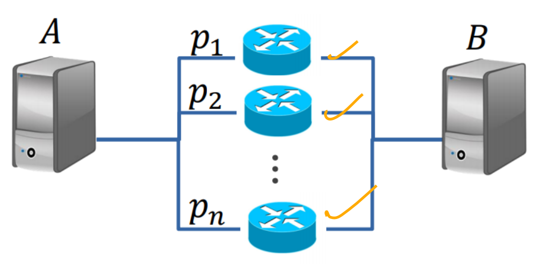

Problem

Consider the parallel network above:

* n independent routers, each with probability p_i of functioning, where $1 \leq i \leq n$
* $E =$ functional path from A to B exists.

What is $P(E)$?

$P(E) = P(\text{at least 1 router works})$

$= 1 - P(\text{no router works})$

$= 1 - P(R_1 \text{ Fail} \wedge R_2 \text{ Fail} \wedge \dots \wedge R_n \text{ Fail})$

$= 1 - P(R_1 \text{ Fail}) P(R_2 \text{ Fail}) \dots P(R_n \text{ Fail})$

$= 1 - (1-p_1)(1-p_2) \dots (1-p_n)$

» Examples involving independent trials...

Problem: coin toss

Suppose we flip a coin n times. Each coin flip is an independent trial with probability p of coming up heads. Write an expression for the following:

» Examples involving independent trials...

H_i : Head on the ith trial

Problem: coin toss

Suppose we flip a coin n times. Each coin flip is an independent trial with probability p of coming up heads. Write an expression for the following:

* P(n heads on n coin flips) $= P(H_1 \cap H_2 \cdots \cap H_n)$

$= P(H_1)\,P(H_2) \cdots P(H_n)$

$= p \cdot p \cdots p$

$= p^n$

$F_i := H_i^c$

Problem: coin toss

Suppose we flip a coin n times. Each coin flip is an independent trial with probability p of coming up heads. Write an expression for the following:

* P(n heads on n coin flips)
* P(n tails on n coin flips) $= P(H_1)\,P(H_2)\cdots P(H_n)$

$= (1-P(H_1))(1-P(H_2))\cdots(1-P(H_n))$

$= \underbrace{(1-p)\cdots(1-p)}_{n}$

$= (1-p)^n$.

» Examples involving independent trials...

Problem: coin toss

Suppose we flip a coin n times. Each coin flip is an independent trial with probability p of coming up heads. Write an expression for the following:

- P(n heads on n coin flips)
- P(n tails on n coin flips)
- P(first k heads, then $n-k$ tails)

$$\underbrace{P(H_1)\,P(H_2)\cdots P(H_k)}_{\text{1st } k \text{ heads}}\;\underbrace{P(T_1)\,P(T_2)\cdots P(T_{n-k})}$$

$$p^k\,(1-p)^{n-k}$$

Problem: coin toss

Suppose we flip a coin n times. Each coin flip is an independent trial with probability p of coming up heads. Write an expression for the following:

* P(n heads on n coin flips)
* P(n tails on n coin flips)
* P(first k heads, then $n-k$ tails)
* P(exactly k heads on n coin flips)

T H T H . . . H ___

n coin flips

$\binom{n}{k} p^k (1-p)^{n-k}$

k - H

n - k T

» Solution to parallel network problem...

» Biased Coin, Independence, Infinite Sample Space, Total Probability, and Bayes Theorem...

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Problem

A biased coin (with probability of obtaining a Head equal to $p > 0$ is tossed repeatedly and independently until the first head is observed. Compute the probability that the first head appears at an even numbered toss.

$k \to 2m$, $2m+1$

T T T . . . . T H T T . . . H

even numbers

» Biased Coin, Independence, Infinite Sample Space, Total Probability, and Bayes Theorem...

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Solution to the problem...

What are sample space and events in this problem?

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3. Consider event E : first head on even numbered toss.

We want to compute $P(E)$.

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We want to compute $P(E)$.
How do we solve problems like this?

» Solution to problem on previous slide...Part-1

$E_k = \underbrace{T \cdot T \cdots T}_{2k-1}\, \underset{2k}{H}$

Consider the partition of E into $E_1, E_2, \ldots$

where E_k = event that 1^{st} head occurs on the $(2k)^{th}$ toss

$$E = \bigcup_{k=1}^{\infty} E_k$$

E_k's are Mutually exclusive

$$\Rightarrow \quad P(E) = P\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} P(E_k)$$

$$P(E_k) = (1-p)^{2k-1}\, p \qquad P(E) = \sum_{k=1}^{\infty} (1-p)^{2k-1}\, p$$

$$= \frac{1-p}{2-p}$$

» Solution to problem on previous slide...Part-2

» Coin Toss Example...

- - - - - → ∞

Problem

A coin for which P(Heads) = p is tossed until two successive Tails are obtained. Find the probability that the experiment is completed on the nth toss.

TT

Ω = all possible infinite seq^n of tosses

E_1 : first toss is H

E_2 : first two tosses are TH

E_3 : " " " " TT

F_n : experiment completes on the n^{th} toss.

» Coin Toss Example...

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3. event E_2 : first two tosses are TH
4. event E_3 : first two tosses are TT

» Coin Toss Example...

E_2: T H — — — — — — —

E_3: T T — — — — · E_1: H — — — — — — — —

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A coin for which P(Heads) = p is tossed until two successive Tails are obtained. Find the probability that the experiment is completed on the nth toss.

Solution

What are sample space and events?

1. Sample space, S: all possible infinite sequences of tosses
2. event E_1 : first toss is H
3. event E_2 : first two tosses are TH
4. event E_3 : first two tosses are TT
5. event F_n : experiment completed on the nth toss.

» Solution to problem in previous slide...part-1

$\{E_1, E_2, E_3\}$ is a partition of Ω

We want $P(F_n)$, $n = 2, 3, \ldots$

For $n = 2$: $P(F_2) = P(E_3) = (1-p)(1-p) = (1-p)^2$

and for $n > 2$:

$P(F_n \mid E_1) = P(F_{n-1})$

$P(F_n \mid E_2) = P(F_{n-2})$

$P(F_n \mid E_3) = 0$

E_1: H | – – – – – – – ($(n-1)^{th}$), n^{th} ↑ $P(F_n)$

E_2: T H | – – – – – – ($(n-1)^{th}$), n^{th}

$\underbrace{\ \ }_{1}\ \underbrace{\ \ }_{2}$

TT $\underbrace{\qquad\qquad\qquad}_{n-2}$ ↑ n^{th}

» Solution to problem in previous slide...part-2

$p_n = P(F_n)$, $p_2 = P(F_2) = (1-p)^2$

p_n satisfies