Fourier Transform: From Time to Space

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February 4, 2023

I Introduction

This paper focuses on an intuitive understanding of time, frequency and how they are related using Fourier Transform. Math behind such relationship is explored, which gives amazing insights that extends the corners of relativistic dynamics and quantum world. How this relationship manifests to a fundamental trade off that exists in nature, and how it is beautifully encapsulated in the language of mathematics is described in the text. Using principles of relativity and quantum mechanics, this idea is extended to space and all particles, and is then used to prove Quantum Uncertainty Principle, which shows how far fetching Fourier Transform can be...

II The Fourier Transform

II.i Core Intuition

The classical aim of Fourier Transform is frequency decomposition of signals. Given a signal, we want to find out what pure frequencies it is made up of. Concretely, it asks the question "How well a particular frequency correlates with the given signal?".

Given some thought, this problem turns out to be similar to that of circular winding a mechanical spring. At first, this physical analogy may seem out of context, but it captures the idea of unmixing frequencies from a signal, in a way that strike to ones imagination. Consider for simplicity, a 2D linear spring made of metallic wire, which really is a projection of an actual 3D linear spring on a plane parallel to its axis (figure 1). It is characterized by number of turns (or loops) of the wire per unit length, without any applied load i.e ***free turn frequency*** $\boldsymbol{f_t}$, which in this case is 4 turns/cm . For argument sake, pretend that we don't know the turn frequency of this spring and our aim is to somehow find it out without using a measuring scale.

If we take this 2D linear spring and wrap it around the circumference of a circle in some plane, essentially turning its initially linear axis circular, we will end up with a wound up spring, with a circular axis in that plane. Resulting wound up spring is characterized by number of times the linear spring is coiled in circles (circular wraps) per unit length of the spring i.e the ***wrapping frequency*** $\boldsymbol{f_w}$. This is illustrated in figure 2. There are two fundamentally different frequencies in this setup. First is the turn frequency of spring, and second is the frequency with which we are wrapping the spring around a circle. The key idea is that, even though we cannot control turn frequency of the spring, we can choose to wrap it around a circle less or more tightly, however we want i.e we have full control over the wrapping frequency. If we start with some low initial wrapping frequency and increase it over time, the weight of wound up spring remains centered near origin most of the time (figure 2a, 2b, 2c, 2d, 2e). Wrapping frequencies very close to zero are neglected since they do not complete even a single rotation around the circle, and require very long spring to do so. The problem with very low wrapping frequencies only arises because the height of spring at any point above or below its axis is always taken as positive. Had we chosen to fix the spring axis at height 0 and represent part of the spring above and below its axis with positive and negative heights respectively, then this problem would not have occurred. However, negative heights do not make sense in practice.

Nonetheless, something peculiar happens when wrapping frequency matches turn frequency of the spring. Physically, this means that each turn of the linear spring ends up as a single rotation around the circle in its wrapped form. In that case, weight of the wound up spring is heavily *off-centered* from the origin, as shown in figure 2f.

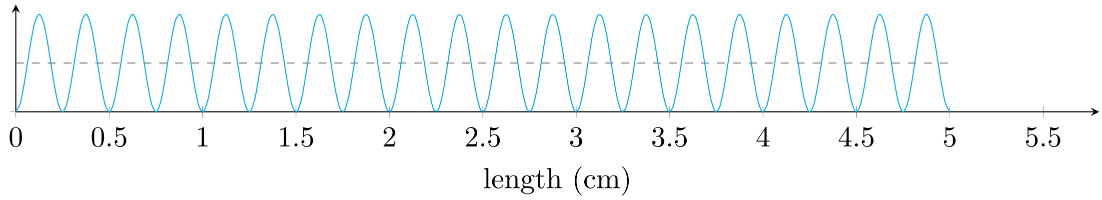
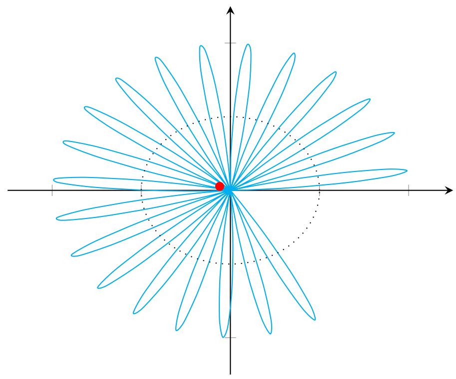
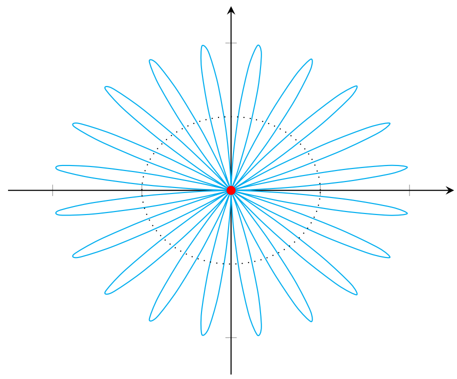


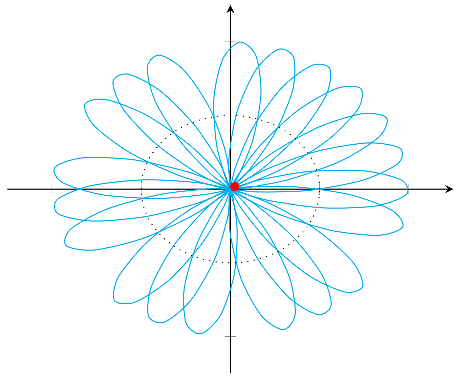
Figure 1: A 2D linear spring with free length 5 cm and free turn frequency $f_t = 4$ turns/cm. Dashed line represents spring axis.



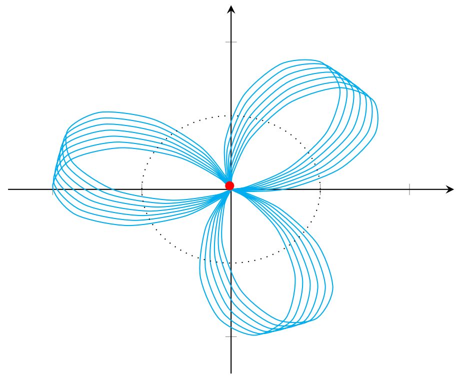
(a) $f_w = 0.17$ circular wraps/cm. incomplete wrapping around the circle



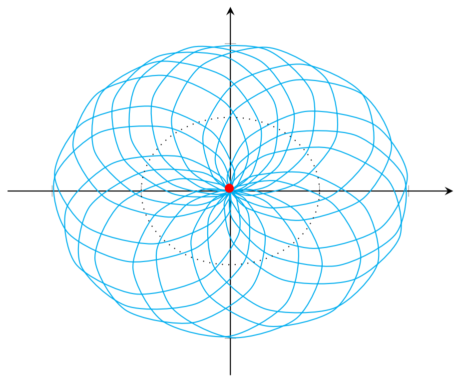
(b) $f_w = 0.20$ circular wraps/cm. One rotation around the circle



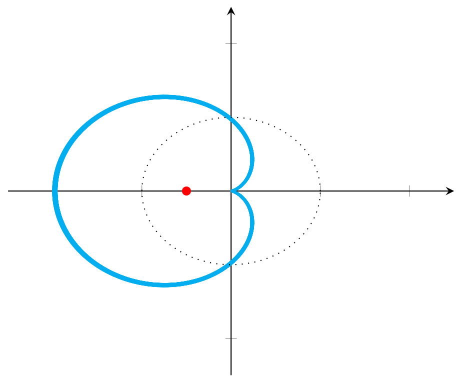
(c) $f_w = 0.64$ circular wrap/cm



(d) $f_w = 1.32$ circular wraps/cm



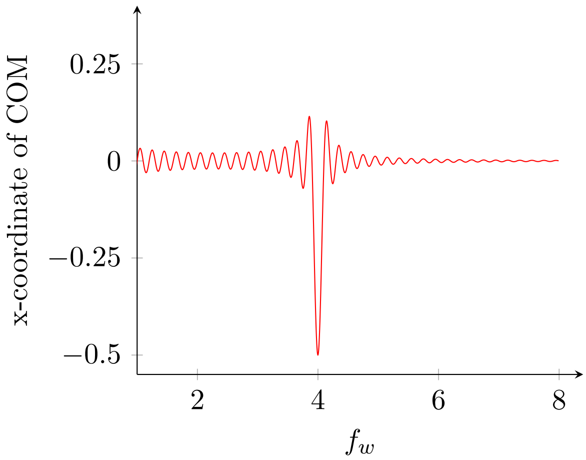
(e) $f_w = 2.52$ circular wraps/cm



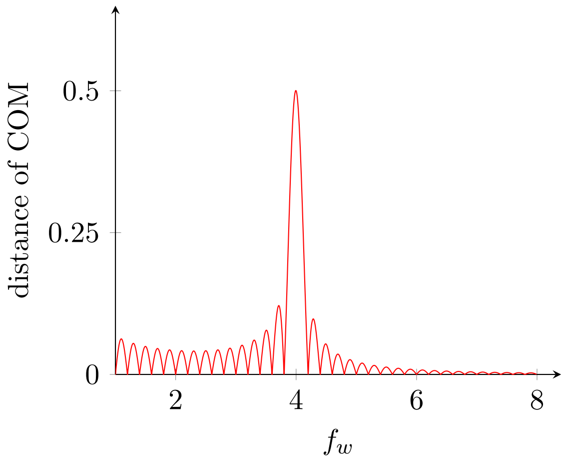
(f) $f_w = f_t = 4$ circular wrap/cm. COM is unusally far from origin

Figure 2: 2D linear spring of free length 5 cm and turn frequency $f_t = 4$ turns/cm wrapped around a circle centered at origin with various wrapping frequencies f_w. Positive f_w corresponds to anticlockwise rotation. Red dot represents center of mass (COM) and the dotted circle represents axis of the wound up spring

This observation can be expressed in terms of center of mass (COM) of the wound up spring, which is a 2D point on the reference plane. It is close to the origin for most of wrapping frequencies except when wrapping frequency is the same as free turn frequency, where it is significantly far from origin. Plot of the position of COM for various wrapping frequencies is shown in figure 3. The spike at $f_w = 4$ wraps/cm means that it is the dominant turn frequency of spring i.e it correlates pretty well with the spring. Hence, *position of center of mass of the wound up spring is a measure of how well a wrapping frequency correlates with the linear spring turn frequency.* Distance of COM from origin gives the strength while the angle COM makes off the positive horizon gives the phase of the wrapping frequency within the spring.



(a) x-coordinate of COM vs f_w

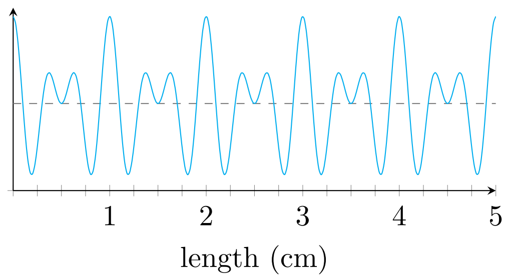


(b) Distance of COM from origin vs f_w

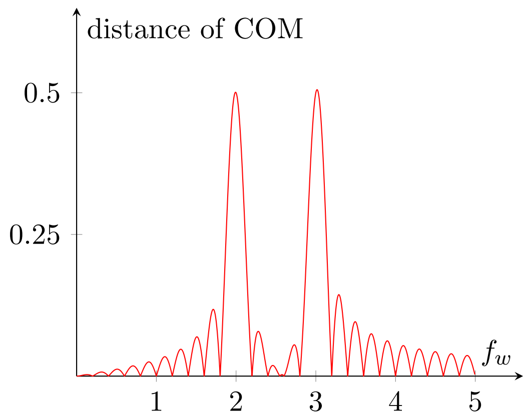
Figure 3: Center of mass (COM) of the wound up spring for various wrapping frequencies f_w

In this example, $f_w = 4$ wraps/cm gives COM at point $(-0.5, 0)$ (figure 2f), which corresponds to a relative strength of 0.5 and phase 180°. For f_w other than 4, COM is pretty close to $(0, 0)$ implying they do not correlate with the spring. With this information, one can say that the 2D linear spring shown in figure 1 resembles a pure cosine wave with frequency 4 Hz and phase offset of π rad i.e $\cos(2\pi f t + \theta)$ with $f = 4$ Hz and $\theta = \pi$ rad. The strength of 0.5 means that pure cosine with frequency -4 Hz also correlates with the spring to the same degree. Negative frequency only implies wrapping in clockwise direction, as positive angles corresponds to rotation off the positive x-axis in anticlockwise direction. Therefore, the spring is equivalent to a signal $0.5\cos(8\pi t + \pi) + 0.5\cos(-8\pi t + \pi) \equiv -\cos(8\pi)$

This wrapping of a 2D spring around a circle for the extraction of frequency information also works for deformed springs. This is illustrated in figure 4. Consequently, a deformed 2D spring correlates with multiple frequencies, as shown in figure 4b, where it correlates highly with 2 and 3 wraps/cm. It is analogous to a signal made up of combination of multiple pure frequencies.



(a) A 2D linear deformed spring



(b) Variation of distance of the COM from origin as wrapping frequency changes

Figure 4: Frequency analysis of a 2D linear deformed spring by wrapping it around a circle with frequency f_w and tracking the resulting center of mass.

II.ii General Mathematical Construct

Following our analogy of springs, a general algorithm of frequency decomposition of any signal (say $f(t)$) requires wrapping the given signal around a circle in some plane of our choice with some wrapping frequency f_w, and then tracking the *imaginary* center of mass of resulting wrapped up signal as f_w changes. To express this in the language of math is even more interesting.

First problem is the choice of a wrapping plane. Among coordinate planes, most common is the Cartesian plane, which represents each point uniquely as (x, y). However, when it comes to rotation and wrapping, Complex plane is a far better choice. The reason is Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, which says that any point on the complex plane i.e complex number $z = x+iy$ can be expressed as rotation off the positive real axis around the circumference of a circle with radius $r = \sqrt{x^2+y^2}$, implying $z = x + iy = re^{i\theta}$, where $\theta = \tan^{-1} y/x$ is the angle z makes off the positive real axis. $e^{2\pi it}$ thus represents a point rotating in the complex plane around the circumference of a unit circle at the rate of 1 rotation/s in anticlockwise direction starting from $1 + 0i$.

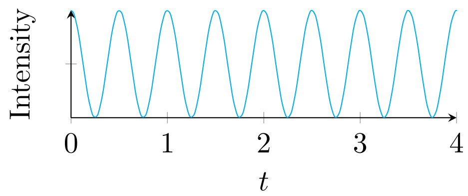
Next problem is to control the rate of rotation or wrapping. It can easily be done by multiplying wrapping frequency f_w in the exponent. $e^{2\pi i f_w t}$ now represents rotation around a unit circle with the rate of f_w rotations/s. For ex if $f_w = 1/5$ cycles/s, then time has to increase all the way to 5s for the exponent to rise from 0 to 2π, resulting in 1 cycle every 5s (fig 5b).

If we now multiply this rotating point $e^{2\pi i f_w t}$ with the signal $f(t)$, the rotating point at any instant in time will get scaled up or down by the intensity of the signal for that time. This is illustrated in figure 5. In effect, the point, rather that rotating at a fixed distance from the origin, will then shuffle near and far from the origin depending on how the signal intensity changes with time, The distance of the rotating point from origin at a particular time will then depend on the signal intensity at that time. $f(t)e^{2\pi i f_w t}$ thus represents a point in the complex plane that traces the wrapped up signal with certain wrapping frequency f_w, as the time passes by!

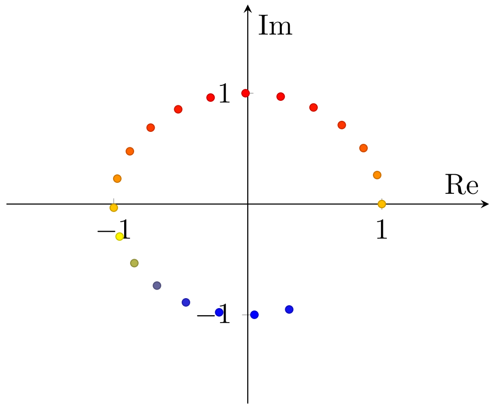
Only thing that is left is locating the *imaginary* center of mass of the wrapped up signal on the complex plane. For that, imagine the signal to be made of a metal wire, just like the analogy of 2D springs. Both the signal and its wrapped form will then have some mass. Generally speaking, the location of center of mass of any body with uniform mass distribution can be approximately by first sampling some points on the body, and then taking their average. Likewise, we can sample some points from the wrapped up signal in complex plane, and then take their average (as complex numbers) to get the location of COM (fig 5c and 5d). Sampling higher number of points will naturally give better result.

$$COM(f_w) = \frac{1}{N}\sum_{i=1}^{N} f(t_i)e^{2\pi i f_w t_i} \tag{2.1}$$

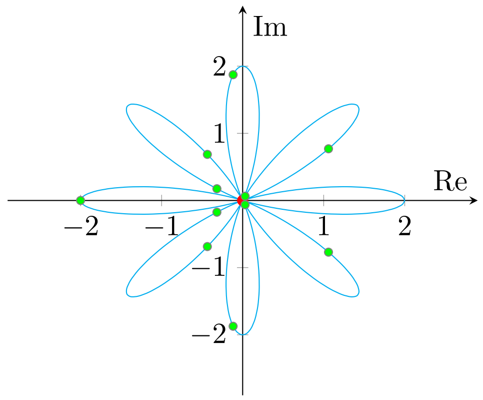
The expression under summation represents signal wrapped up in the complex plane with a controllable frequency f_w.



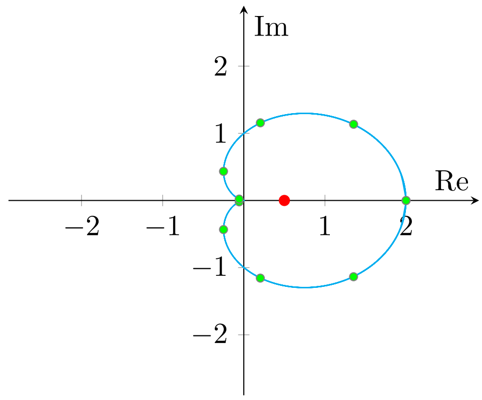
(a) Pure signal $f(t)$ with frequency 2 Hz



(b) Point rotating in complex plane around the circumference of unit circle anticlockwise expressed as $e^{2\pi i f_w t}$



(c) Wrapped up signal as the product of $f(t)$ and $e^{2\pi i f_w t}$. Here $f_w = 0.25$ cycles/s



(d) Wrapped up signal with frequency $f_w = 2$ rotations/s

Figure 5: Mathematics of wrapping a signal around a circle of unit radius in the complex plane with wrapping frequency f_w. The wrapped up signal can be traced by scaling a point which is rotating in complex plane (given by $e^{2\pi i f_w t}$) by the signal $f(t)$ i.e $f(t)e^{2\pi i f_w t}$ represents the wrapped signal. The location of center of mass of wrapped up signal (shown as red dot) is approximated by sampling some points from the graph (shown as green dots), and then taking their average.

In the limit, we can sample infinitely many points by replacing sum with an integral over certain time range and then dividing it by the same time range opposed to the number of points.

$$COM(f_w) = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t) e^{2\pi i f_w t} \, \mathrm{d}t \tag{2.2}$$

Another perspective of looking at equation (2.2) is that the integral in this equation is equal to the COM of the wrapped up signal, scaled by the time interval of integration

$$\int_{t_1}^{t_2} f(t) e^{2\pi i f_w t} \, \mathrm{d}t = COM(f_w) \times (t_2 - t_1) \tag{2.3}$$

Hence, this integral for a certain wrapping frequency f_w represents COM of wrapped signal $COM(f_w)$ scaled by the duration for which f_w persists within the input signal. If a certain frequency persist for a long duration in the signal, integral for that frequency will have a higher value compared to some other frequency that even though present, persist for a shorter time in the signal. In other words, relative value of the integral for any two frequencies is proportional to the time duration for which one frequency persist relative to the other in the signal. Thus, integral alone is a better measure of the strength of a certain frequency in the signal, compared to COM. However both gives the same phase information.

If clockwise rotation in complex plane should correspond to positive frequencies, then f_w needs to be replaced with $-f_w$, since up until now, positive f_w meant anticlockwise rotation. Incorporating this convention in the definition of $COM(f_w)$ gives

$$\hat{F}(f_w) = \int_{t_1}^{t_2} f(t)e^{-2\pi i f_w t}\,\mathrm{d}t = COM(f_w) \times (t_2 - t_1) \tag{2.4}$$

where $\hat{F}(f_w)$ is some function of wrapping frequency f_w whose output for a certain f_w is a complex number, equal to the location COM of wrapped up signal (for that f_w) scaled by the time interval of integration. Magnitude r (distance from origin) of the output complex number gives the *strength* while its argument θ (rotation off the positive real axis) gives the *phase* of a that f_w in the original signal. In effect, equation (2.4) encapsulate entire algorithm of decomposing any signal in its constituent frequencies!

Using $\omega = 2\pi f_w$ in place of f_w and integrating over entire time line (which does not seem practical at first but accounts for the values of integral for every possible finite time range) gives a theoretically general form known as the Fourier Transform (since it transforms information from time to frequency domain)

$$\hat{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t \tag{2.5}$$

This is like separation of gases using fractional distillation (a process based on differences in boiling points of constituents) after all of them have been mixed together. Such homogeneous mix of gases is analogous to a signal, while boiling point of individual gases are analogous to pure frequencies composing the signal. Amount of a certain gas in the mixture corresponds to the strength of a certain pure frequency within the signal.

II.iii Definition

Fourier transform of a function in time t, say a signal $f(t)$ (which can either be real or complex valued) is a new complex valued function in frequency ω, $\hat{F}(\omega)$ (i.e spectrum), whose output for a certain frequency is a complex number $\hat{F}(\omega) = re^{i\theta}$ that encodes the strength $(= r)$ and phase $(= \theta)$ of that frequency within the original signal. For a certain frequency ω, strength means its amplitude and for how long it persists within the signal, while phase represents shift in its position within the signal [5].

$$\boxed{\hat{F}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t} \tag{2.6}$$

In general, if y is in reciprocal space of x, then

$$\hat{F}(y) = \int_{-\infty}^{\infty} f(x)e^{-ixy}\,\mathrm{d}x$$

So if x is time in seconds, then y represents temporal frequency in s^{-1} (or Hz). Similarly if x is distance in meters, then y symbolizes spatial frequency in m^{-1}.

II.iv Origin of uncertainty

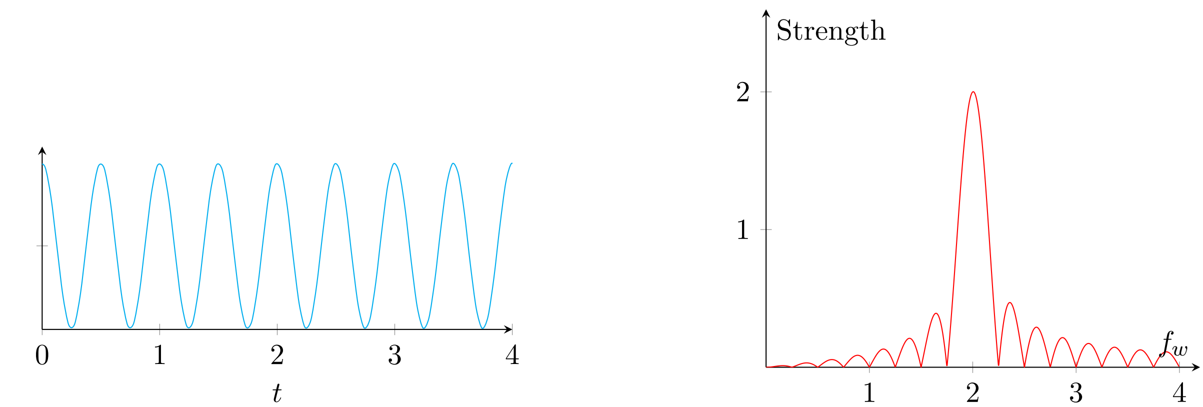
Following the physical analogy of 2D springs (sec II.i) and its mathematical formulation (sec II.ii) for frequency decomposition of signals, it can be easily noticed that the spectrum i.e plot of strength of various frequencies within the signal (given by the distance of center of mass of wrapped signal from origin) is NOT infinitely sharp, as it is expected to be (see figure 3b and 4b).

A signal made of a single pure frequency is expected to have a spectrum with only one sharp peak corresponding to that frequency. Similarly, a signal composed of many pure frequencies should have a spectrum with multiple sharp peaks corresponding to only those frequencies. However in reality, spectrum consists of broad peaks centered at constituent frequencies. This is illustrated in figure 6 for a pure signal spanning 4s, where the peak at 2 Hz suggest that it is the dominant frequency in the signal. However, the fact that peak at 2 Hz is broad also means that frequencies *near* 2 Hz also correlates pretty well with the signal, even though the signal is pure 2 Hz. This makes frequency decomposition really ambiguous. On the flip side, if the same signal persists for a longer period of time as depicted in figure 7, the peak at 2 Hz becomes relatively sharp, which reduces ambiguity in the frequency detection. This can be explained using an analogy similar to that of 2D springs.

Imagine the signal to be made of metal wire having some mass and finite length depending on the time duration for which signal persists. As we wrap the signal around a circle centered at origin in the complex plane with wrapping frequency $f_w = 2$ Hz, weight of the resulting wrapped up signal will be centered far from origin, which corresponds to high strength of $f_w = 2$ Hz within the signal.

For the signal that persist for short time (requires small length of metal wire), as the wrapping frequency shifts away from 2 Hz, the wrapped up signal stays off centered from the origin due to short length of the wire. It does not have much length to balance itself around the origin. f_w needs to significantly different from 2 Hz for it to start balancing off. This keeps center of mass of the wrapped up signal away from the origin for frequencies near 2 Hz resulting in a broad peak.

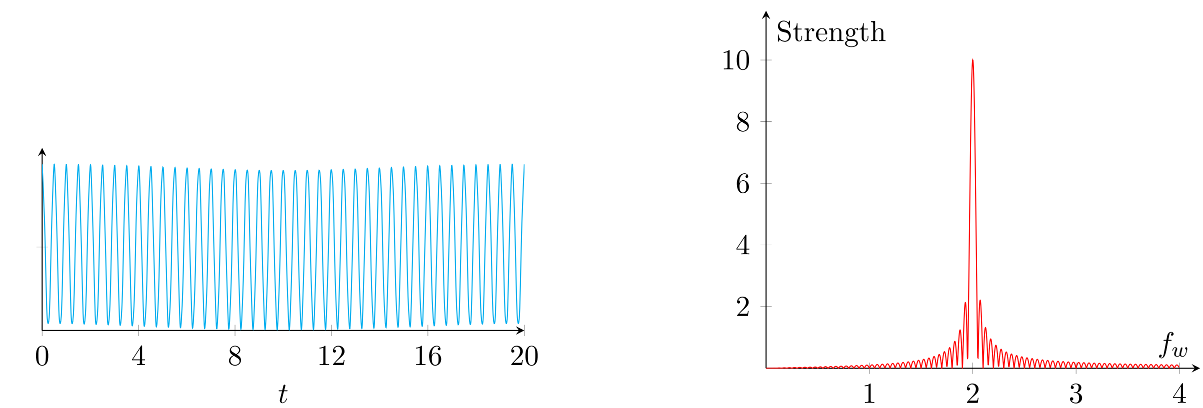
However, for the signal that persist for long time (requires long metal wire), as the wrapping frequency shifts even slightly from 2 Hz, the wrapped up signal easily balances itself around the origin due to long length of the wire. Center of mass is therefor pulled back to the origin quite effectively as wrapping frequency shifts away from 2 Hz, giving a relatively sharp peak at 2 Hz.



(a) Pure signal $f(t)$ with frequency 2 Hz persisting for 4 secs

(b) Fourier Transform spectrum of $f(t)$

Figure 6: Fourier Transform of a pure signal persisting for short time duration



(a) Pure signal $f(t)$ with frequency 2 Hz persisting for 20 secs

(b) Fourier Transform spectrum of $f(t)$

Figure 7: Fourier Transform of a pure signal persisting for long time duration

Overall, this means that a signal over short time interval has a spread out Fourier transform, while a signal that persists for long time gives a sharp Fourier Transform. Any attempt to decrease spread in one of the domain (time or frequency) inevitably increases the spread in another domain and vice versa. In other words, there exist a ***natural trade-off*** between the spread in time and frequency representations of a signal. It is *natural*, which signifies the fact that this trade-off is fundamental to the definition of a signal and its time and frequency representations, and is NOT caused by a flawed model or imperfect measurement techniques. It is explored in detail in section V

III Inverse Fourier Transform

Just like a signal can be decomposed into constituent frequencies using Fourier transform, the reverse process of constructing a signal from its frequency information is also possible. This can be done by first shifting and scaling every possible pure frequency signal by its phase and strength respectively (as given by frequency information), and then adding all of them together.

More formally, given the Fourier transform $\hat{F}(\omega)$ of a function $f(t)$, which encapsulates information about the strength and phase of every pure frequency in $f(t)$, the original function can be synthesized using

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(\omega)e^{i\omega t}\,\mathrm{d}\omega} \qquad \text{if } \hat{F}(\omega) \text{ is integrable} \tag{3.1}$$

This is known as **Fourier Inversion Theorem**, which allows a signal to be reconstructed from its frequency information. [5]

Proof: *(uses Appendix VIII.i and VIII.iii)* Multiplying both sides of equation (2.6) with $e^{i\omega t'}$ (note t' in place of t) and integrating with respect to ω gives

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{i\omega t'}\hat{F}(\omega)\,\mathrm{d}\omega &= \int_{-\infty}^{\infty} e^{i\omega t'}\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t\right)\mathrm{d}\omega \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)e^{-i\omega(t-t')}\,\mathrm{d}t\,\mathrm{d}\omega = \int_{-\infty}^{\infty} f(t)\left(\int_{-\infty}^{\infty} e^{-i\omega(t-t')}\,\mathrm{d}\omega\right)\mathrm{d}t \qquad \text{(using 8.1)} \\
&= 2\pi\int_{-\infty}^{\infty} f(t)\delta(t'-t)\,\mathrm{d}t = 2\pi f(t') \qquad \text{(using 8.6 and 8.5)}
\end{aligned}$$

replacing t' by t gives the final form as in equation (3.1)

IV Plancherel's Theorem

Fourier transform version of *Parseval's identity for Fourier Series*, given by Michel Plancherel (1885-1967) in 1910, called the Plancherel's Theorem is [5]

$$\int_{-\infty}^{\infty} |f(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{F}(\omega)|^2\,\mathrm{d}\omega \tag{4.1}$$

which means that area under the square modulus of a function $|f(t)|^2$ is equal to area under the square modulus of its spectrum $|\hat{F}(\omega)|^2$

Proof:

$$\int_{-\infty}^{\infty} |f(t)|^2\,\mathrm{d}t = \int_{-\infty}^{\infty} f(t)f^{\star}(t)\,\mathrm{d}t$$

$$
\begin{aligned}
&= \int\limits_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \hat{F}(\omega) e^{i\omega t} \, \mathrm{d}\omega \right) \left(\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \hat{F}^{\star}(\omega') e^{-i\omega' t} \, \mathrm{d}\omega' \right) \mathrm{d}t && \text{(using 3.1)} \\
&= \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \hat{F}(\omega) \hat{F}^{\star}(\omega') e^{i(\omega-\omega')t} \, \mathrm{d}\omega \, \mathrm{d}\omega' \, \mathrm{d}t \\
&= \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \hat{F}(\omega) \hat{F}^{\star}(\omega') \left(\int\limits_{-\infty}^{\infty} e^{i(\omega-\omega')t} \, \mathrm{d}t \right) \mathrm{d}\omega \, \mathrm{d}\omega' && \text{(using 8.1)} \\
&= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \hat{F}(\omega) \hat{F}^{\star}(\omega') \delta(\omega - \omega') \, \mathrm{d}\omega \, \mathrm{d}\omega' && \text{(using 8.6)} \\
&= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \hat{F}(\omega) \hat{F}^{\star}(\omega) \, \mathrm{d}\omega = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |\hat{F}(\omega)|^2 \, \mathrm{d}\omega && \text{(using 8.5)}
\end{aligned}
$$

Special Case: If $f(t)$ is a probability distribution, then $|f(t)|^2$ represents the probability of occurrence of t (like a wave function). In that case, $\hat{F}(\omega)$ will also be a probability distribution since $|\hat{F}(\omega)|^2$ represents strength of frequency ω within $f(t)$. If $f(t)$ is normalized, (4.1) gives

$$
\int\limits_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t = 1 \qquad \text{and} \qquad \int\limits_{-\infty}^{\infty} |\hat{F}(\omega)|^2 \, \mathrm{d}\omega = 2\pi \tag{4.2}
$$

V Fourier Uncertainty Principle

Following section II.iv that describes the origin of trade off between time and frequency, this section deals with mathematical aspect of this trade off. A signal $f(t)$ and its frequency representation $\hat{F}(\omega)$ are closely related. If a signal is localized (made out of observation over a short period of time), then its spectrum is spread out i.e it correlates well with wide range of frequencies. It gets really ambiguous as to what frequencies it is actually made up of. However, observation over long period of time corresponds to a spread out signal, which gives a concentrated frequency spectrum i.e it correlates only with narrow range of frequencies, which allows for easy detection of constituent frequencies. This gives rise to a ***natural trade off*** as to how concentrated a signal and its frequency representation can be. Qualitative analysis and quantitative description of this trade off is given in following subsections.

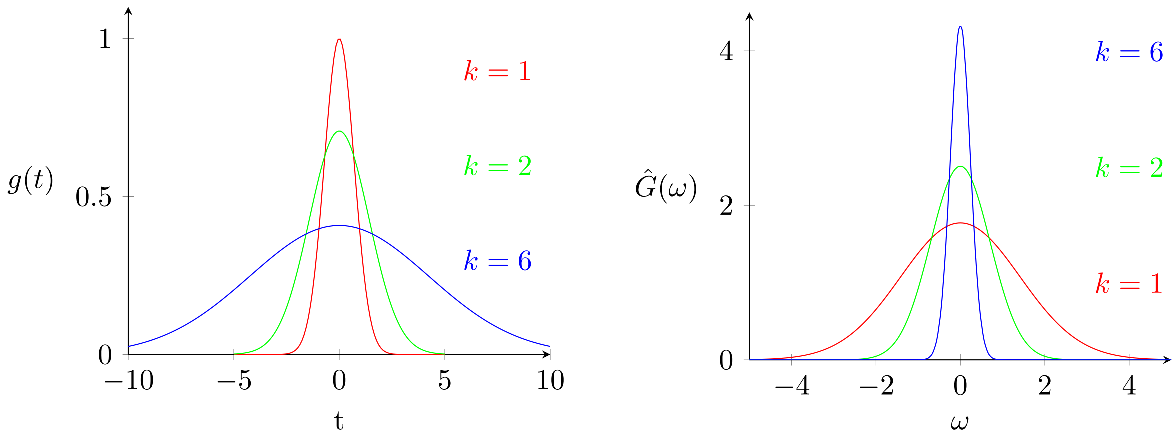
V.i Qualitative: The Fourier Trade Off

let $f(t)$ be a continuous and integrable function over $\mathbb{R}$. Since we do not have any control over the spread of $f(t)$, we define a new function $g(t) = \frac{1}{\sqrt{k}} f\left(\frac{t}{k}\right)$, which essentially is a wrapper over $f(t)$ with an additional parameter k. This parameter allows us to control the spread of $g(t)$. Hence k can be called as *spread constant* (see figure 8a) [3].

$$
\begin{aligned}
\hat{G}(\omega) &= \int\limits_{-\infty}^{\infty} g(t) e^{-i\omega t} \, \mathrm{d}t = \int\limits_{-\infty}^{\infty} \left[\frac{1}{\sqrt{k}} f\left(\frac{t}{k}\right) \right] e^{-i\omega t} \, \mathrm{d}t && \text{(using 2.6)} \\
&= \frac{1}{\sqrt{k}} \int\limits_{-\infty}^{\infty} f(u) e^{-i\omega(uk)} \, k \, \mathrm{d}u = \sqrt{k} \int\limits_{-\infty}^{\infty} f(u) e^{-i(k\omega)u} \, \mathrm{d}u = \sqrt{k} \hat{F}(k\omega) && \text{(where } u = \tfrac{t}{k}\text{)}
\end{aligned}
$$

$$
\boxed{\hat{G}(\omega) = \sqrt{k} \hat{F}(k\omega)} \tag{5.1}
$$

As k increases, $g(t)$ spreads out, but its spectrum $\hat{G}(\omega)$ gets localized (from eq 5.1). On the flip side, as k decreases, $g(t)$ localizes and $\hat{G}(\omega)$ spreads out. In either case, one of them is localized (certain) and other is inevitably spread (uncertain) (see figure 8). This is known as the ***Fourier Trade Off***



(a) $g(t)$ vs t

(b) $\hat{G}(\omega)$ vs ω

Figure 8: An illustration of *Fourier Trade Off* for a Gaussian $f(t) = e^{-t^2}$ and $\hat{F}(\omega) = \sqrt{\pi}e^{-\frac{\omega^2}{4}}$. Wrapper $g(t) = \frac{1}{\sqrt{k}} f\left(\frac{t}{k}\right)$, $\hat{G}(\omega) = \sqrt{k}\hat{F}(k\omega)$, where k is *spread constant*

V.ii Quantitative: The Uncertainty Principle

The idea of fundamental trade off (section V.i) gives rise to a interesting question: *Is there any quantitative aspect of this trade-off?*. More precisely: *Is there any lower bound to the total uncertainty in time and frequency?*.

This requires *uncertainty* to be defined, which is **probabilistic** and should not be confused with **possibility**. Latter is an absolute concept, where something being impossible eliminates its existence entirely, while probability is a relative concept, where something being highly probable still retains the possibility of occurrence of low probable event.

The key idea here is that ***f(t)* is a continuous probability distribution**, where $|f(t)|^2$ gives the probability (and not possibility) of occurrence of t. Likewise, its spectrum $\hat{F}(\omega)$ is also a probability distribution.

$$\text{Probability of occurrence of t: } P(t) = \frac{|f(t)|^2}{\int\limits_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t}$$

For a discrete random variable x, with probability $P(x)$, average $x = \bar{x} = \sum\limits_i x_i P(x_i)$ and variance (spread of distribution about the mean) $\sigma_x^2 = \sum\limits_i (x_i - \bar{x})^2 P(x_i)$. Similarly

$$\text{average } t = \bar{t} = \int\limits_{-\infty}^{\infty} tP(t)\,\mathrm{d}t = \frac{\int\limits_{-\infty}^{\infty} t|f(t)|^2 \,\mathrm{d}t}{\int\limits_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t} \tag{5.2a}$$

$$\text{average } \omega = \bar{\omega} = \int\limits_{-\infty}^{\infty} \omega P(\omega)\,\mathrm{d}\omega = \frac{\int\limits_{-\infty}^{\infty} \omega|\hat{F}(\omega)|^2 \,\mathrm{d}\omega}{\int\limits_{-\infty}^{\infty} |\hat{F}(\omega)|^2 \,\mathrm{d}\omega} \tag{5.2b}$$

$$\text{variance in t} = \sigma_t^2 = \int\limits_{-\infty}^{\infty} (t-\bar{t})^2 P(t)\,\mathrm{d}t = \frac{\int\limits_{-\infty}^{\infty} (t-\bar{t})^2|f(t)|^2 \,\mathrm{d}t}{\int\limits_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t} \tag{5.2c}$$

$$\text{variance in } \omega = \sigma_\omega^2 = \int_{-\infty}^{\infty} (\omega - \bar{\omega})^2 P(\omega)\,\mathrm{d}\omega = \frac{\int_{-\infty}^{\infty} (\omega - \bar{\omega})^2 |\hat{F}(\omega)|^2\,\mathrm{d}\omega}{\int_{-\infty}^{\infty} |\hat{F}(\omega)|^2\,\mathrm{d}\omega} \tag{5.2d}$$

In the context of distributions, ***Uncertainty*** *means the extent of spread of a distribution about its mean value*, which is represented by standard deviation σ_t about the mean. Intuitively, high uncertainty means that if the measurement is repeated, their is a high chance of getting a value other than mean, giving a spread out probability distribution (and vice-versa).

Our goal is to find a lower bound to the total uncertainty (or spread) in t (signal) and ω (spectrum), if that even exist. Mathematically, $\sigma_t\,\sigma_\omega \geq ?$. For simplicity, let $f(t)$ be a continuous, normalized probability distribution centered at $t = 0$

$$avg(t) = \bar{t} = 0 \qquad \text{(centered at } t = 0) \tag{5.3a}$$

$$\int_{-\infty}^{\infty} |f(t)|^2\,\mathrm{d}t = 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty} |\hat{F}(\omega)|^2\,\mathrm{d}\omega = 2\pi \qquad \text{(using 4.2)} \tag{5.3b}$$

$$\sigma_\omega^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\omega - \bar{\omega})^2 |\hat{F}(\omega)|^2\,\mathrm{d}\omega \qquad \text{(using 5.2d and 5.3b)} \tag{5.3c}$$

Defining a function (which at first looks random, but have the ability to relate σ_t with σ_ω)[3]

$$h_k(\omega) = \frac{(\omega - \bar{\omega})}{k\sigma_\omega^2} \hat{F}(\omega) + \frac{\mathrm{d}}{\mathrm{d}\omega} \hat{F}(\omega) \qquad \text{where } k \in \mathbb{R} \text{ and } k \neq 0 \tag{5.4}$$

and the integral

$$I(k) = \int_{-\infty}^{\infty} |h_k(\omega)|^2\,\mathrm{d}\omega \tag{5.5}$$

The key idea here is that $I(k)$ is an integral of square modulus, hence $I(k) \geq 0$

$$I(k) = \int_{-\infty}^{\infty} |h_k(\omega)|^2\,\mathrm{d}\omega = \int_{-\infty}^{\infty} h_k(\omega)\, h_k^\star(\omega)\,\mathrm{d}\omega \geq 0 \tag{5.6}$$

$$\begin{aligned}
I(k) &= \int_{-\infty}^{\infty} \left[\frac{(\omega - \bar{\omega})}{k\sigma_\omega^2} \hat{F}(\omega) + \frac{\mathrm{d}}{\mathrm{d}\omega}\hat{F}(\omega)\right] \left[\frac{(\omega - \bar{\omega})}{k\sigma_\omega^2} \hat{F}^\star(\omega) + \frac{\mathrm{d}}{\mathrm{d}\omega}\hat{F}^\star(\omega)\right] \mathrm{d}\omega \qquad \text{(using 5.4)} \\
&= \int_{-\infty}^{\infty} \left[\frac{(\omega - \bar{\omega})^2}{k^2\sigma_\omega^4} |\hat{F}(\omega)|^2 + \frac{(\omega - \bar{\omega})}{k\sigma_\omega^2} \left(\hat{F}(\omega)\frac{\mathrm{d}\hat{F}^\star}{\mathrm{d}\omega} + \hat{F}^\star(\omega)\frac{\mathrm{d}\hat{F}}{\mathrm{d}\omega}\right) + \frac{\mathrm{d}\hat{F}}{\mathrm{d}\omega}\frac{\mathrm{d}\hat{F}^\star}{\mathrm{d}\omega}\right] \mathrm{d}\omega \\
&= \int_{-\infty}^{\infty} \left[\frac{(\omega - \bar{\omega})^2}{k^2\sigma_\omega^4} |\hat{F}(\omega)|^2 + \frac{(\omega - \bar{\omega})}{k\sigma_\omega^2}\frac{\mathrm{d}}{\mathrm{d}\omega} \left(\hat{F}(\omega)\hat{F}^\star(\omega)\right) + \frac{\mathrm{d}\hat{F}\,(\omega)}{\mathrm{d}\omega}\frac{\mathrm{d}\hat{F}^{\star\,(\omega)}}{\mathrm{d}\omega}\right] \mathrm{d}\omega
\end{aligned}$$

$$\text{Hence,} \qquad I(k) = I_1(k) + I_2(k) + I_3 \tag{5.7}$$

Computing these terms separately, first term is

$$I_1(k) = \frac{1}{k^2\sigma_\omega^4} \int_{-\infty}^{\infty} (\omega - \bar{\omega})^2 |\hat{F}(\omega)|^2\,\mathrm{d}\omega = \frac{2\pi\sigma_\omega^2}{k^2\sigma_\omega^4} = \frac{2\pi}{k^2\sigma_\omega^2} \qquad \text{(using 5.3c)} \tag{5.8}$$

Second term contains derivative and can be integrated by parts

$$I_2(k) = \frac{1}{k\sigma_\omega^2} \int_{-\infty}^{\infty} (\omega - \bar{\omega}) \frac{\mathrm{d}}{\mathrm{d}\omega} \left(\hat{F}(\omega)\hat{F}^\star(\omega)\right) \mathrm{d}\omega = \frac{1}{k\sigma_\omega^2} \int_{-\infty}^{\infty} (\omega - \bar{\omega}) \frac{\mathrm{d}}{\mathrm{d}\omega} |\hat{F}(\omega)|^2\,\mathrm{d}\omega$$

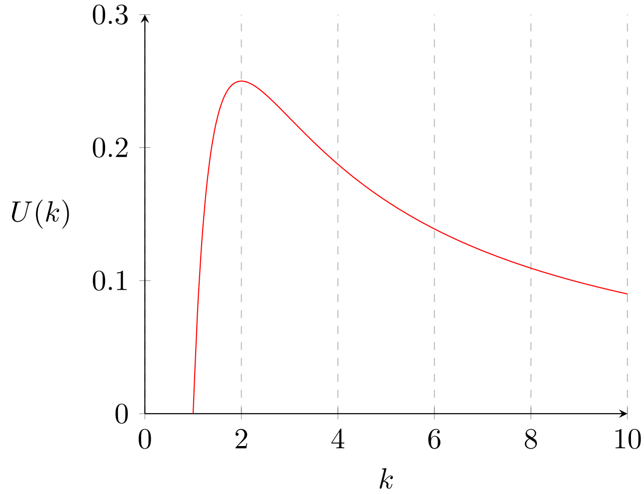


Figure 9: Graph of $U(k) = \frac{1}{k} - \frac{1}{k^2}$ for $k \in \mathbb{R}$, $k \neq 0$

$$= \frac{1}{k\sigma_\omega^2}\left(\left[(\omega - \bar{\omega})|\hat{F}(\omega)|^2\right]_{-\infty}^{+\infty} - \int_{-\infty}^{\infty} |\hat{F}(\omega)|^2 \,\mathrm{d}\omega\right)$$

$$I_2(k) = \frac{1}{k\sigma_\omega^2}(0 - 2\pi) = -\frac{2\pi}{k\sigma_\omega^2} \qquad \text{since } \hat{F}(\omega) \text{ vanishes at } \pm\infty, \text{ and using (5.3b)} \tag{5.9}$$

Third term is tricky, and is solved using Appendix VIII.ii

$$I_3 = \int_{-\infty}^{\infty} \frac{\mathrm{d}\hat{F}}{\mathrm{d}\omega}(\omega) \frac{\mathrm{d}\hat{F}^\star(\omega)}{\mathrm{d}\omega} \,\mathrm{d}\omega$$

$$= \int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,\mathrm{d}t\right) \cdot \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\int_{-\infty}^{\infty} f^\star(t')e^{i\omega t'}\,\mathrm{d}t'\right)\mathrm{d}\omega \qquad \text{(using 2.6)}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)f^\star(t')\,(-i^2tt')e^{i\omega(t'-t)}\,\mathrm{d}t\,\mathrm{d}t'\,\mathrm{d}\omega \qquad \text{(using 8.2 and 8.1)}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} tt' f(t)f^\star(t')\left(\int_{-\infty}^{\infty} e^{i\omega(t'-t)}\,\mathrm{d}\omega\right)\mathrm{d}t\,\mathrm{d}t' \qquad \text{(using 8.1)}$$

$$= 2\pi\int_{-\infty}^{\infty} tf(t)\left(\int_{-\infty}^{\infty} t' f^\star(t')\delta(t'-t)\,\mathrm{d}t'\right)\mathrm{d}t \qquad \text{(using 8.6)}$$

$$I_3 = 2\pi\int_{-\infty}^{\infty} t^2|f(t)|^2\,\mathrm{d}t = 2\pi\sigma_t^2 \qquad \text{(using 8.5 and 5.3a)} \tag{5.10}$$

using (5.8), (5.9) and (5.10) in (5.7)

$$I(k) = \frac{2\pi}{k^2\sigma_\omega^2} - \frac{2\pi}{k\sigma_\omega^2} + 2\pi\sigma_t^2 = 2\pi\left(\frac{1}{\sigma_\omega^2}\left(\frac{1}{k^2} - \frac{1}{k}\right) + \sigma_t^2\right) \geq 0 \qquad \text{(using inequality 5.6)}$$

$$\boxed{\sigma_t^2\sigma_\omega^2 \geq \frac{1}{k} - \frac{1}{k^2} = U(k)} \quad \text{where } k \in \mathbb{R} \text{ and } k \neq 0 \tag{5.11}$$

This will still be true for maximum value of $U(k)$, so $\sigma_t^2\sigma_\omega^2 \geq max(U(k))$

$$\text{setting} \quad U'(k) = \frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{1}{k} - \frac{1}{k^2}\right) = -\frac{1}{k^2} + \frac{2}{k^3} = 0 \quad \text{gives critical point } k = 2$$

$$U''(k) = \frac{2}{k^3} - \frac{6}{k^4} \implies U''(2) = -\frac{1}{8} \leq 0 \quad \text{(maxima)}$$

Hence, $U(k)$ attains maximum value at $k = 2$, $max(U(k)) = U(2) = \frac{1}{4}$ (fig 9). Finally

$$\textbf{Fourier Uncertainty Principle:} \quad \boxed{\sigma_t^2\sigma_\omega^2 \geq \frac{1}{4} \quad \text{or} \quad \sigma_t\sigma_\omega \geq \frac{1}{2}} \tag{5.12}$$

VI Towards Quantum World

Fourier trade off [eq 5.12] gives a *natural* lower bound to the total spread (or uncertainty) in signal and its spectrum. *Natural* signifies it is not a consequence of imperfect measurements, but rather it's a spread fundamental to what a signal even is (spread over time).

There is nothing special about time here. With carefully crafted relativistic intuition and some quantum principles, this idea can be extended way beyond time to waves spread over space! [1]. To understand how a particle can be regarded as wave, we have to consider energy, and how it is carried over space, the idea to which is encapsulated in principles listed below

1. **Inertia of energy**: Following Einstein, energy is equivalent to mass, and mass represents energy. They are always proportional to each other by

$$Energy = mass \times c^2$$

   where c is the speed of light, which according to de Broglie is more precisely *Limit speed of energy*.

2. **Quantum Relation**: Following the description of a Quantum Mechanics, the basic idea behind quanta is that an isolated packet of energy is meaningless without a frequency associated with it

$$Energy = h \times frequency$$

   where h is plank's constant

Following these principles, one can associate each portion of energy with a proper mass m_0 (mass as measured by an observer at rest relative to the body), as well as with a periodic phenomenon with frequency ν_0

$$E_0 = m_0c^2 = h\nu_0$$

where m_0 and ν_0 are measured in the rest frame of the energy packet.

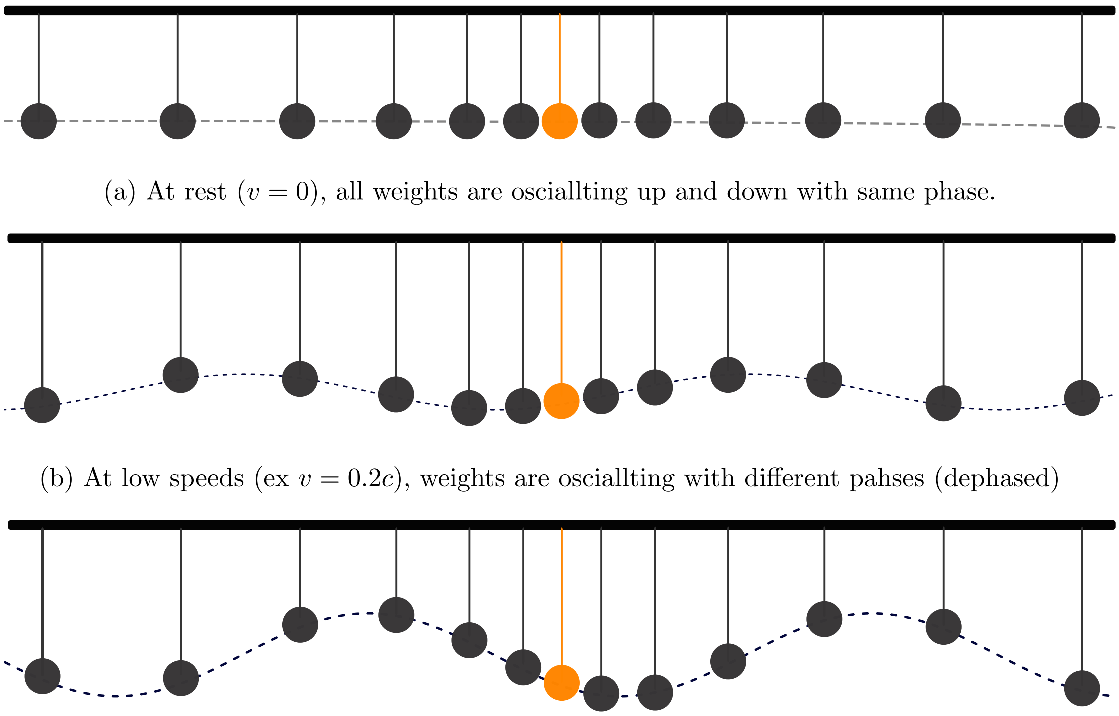
So if mass is the same as energy, and that energy is carried by some periodic phenomenon, then a particle can be considered as a little wave packet dispersed over space, energy of which is carried in some form of oscillations. Photons show this behavior, as proven by Einstein in Photoelectric effect. De Broglie hypothesis extend this idea to all particles, which can be made clear by a simple analogy.

VI.i Mechanical analogy of Wave Nature

Following is a modified version of the analogy de Broglie had originally proposed [1]. Consider a rigid horizontal wire from which identical weights are suspended using springs, such that density of such weights (number of weights per unit length) decreases rapidly as one moves out from the mid point of wire i.e weights are highly concentrated at the center of the wire. Imagine that all weights are oscillating up and down with same frequency, phase and amplitude i.e they are in perfect sync. A virtual thread passing through the center of mass of the weights would then be a straight line, oscillating up and down with same frequency as any of the weight. This ensemble of suspended weights is *analogous to a energy packet*, where energy is carried in the oscillations of the virtual thread (Figure 10a).

$$\text{Proper frequency of oscillation of weights/thread:} \quad \nu_0 = \frac{m_0c^2}{h} \tag{6.1}$$

Up till now, the description is for an observer at rest relative to the system. However, if an observer is moving relative to this system with a uniform velocity $v = \beta c$, where β is a dimensionless constant representing speed as fraction of speed of light c, then each weight to him would be like a clock, showing Einstein ***time dilation***. Also, distribution of weights along



(a) At rest ($v = 0$), all weights are osciallting up and down with same phase.

(b) At low speeds (ex $v = 0.2c$), weights are osciallting with different pahses (dephased)

(c) At high speeds (ex $v = 0.6c$), dephasing in the oscillation of weights is highly pronounced

Figure 10: Mechanical analogue of a particle dispersed over space as observed in different frames of reference. Dashed line joining the centers of suspended weights is the *virtual thread*

the length of wire would no longer be isotropic about the center due to ***Lorentz or length contraction***. As a consequence, weights will fall out of phase (Figure 10b), and their oscillations appear to slow down by a factor of $\sqrt{1-\beta^2}$ from their proper frequency. An observer in motion relative to the system will then measure the frequency of oscillations as

$$\nu_1 = \nu_0\sqrt{1-\beta^2} = \frac{m_0c^2}{h}\sqrt{1-\beta^2} \tag{6.2}$$

From this moving viewpoint, the virtual thread connecting center of mass of weights will a sinusoid, parallel to the motion of the system. Faster the motion of system relative to observer, higher will be the effects of time dilation and length contraction causing dephasing of weights to be more pronounced, resulting in higher frequency of the sinusoidal virtual thread (Figure 10c). Hence, the frequency of sinusoidal thread can be imagined to be associated with the kinetic energy of the system.

From relativistic dynamics, if a body with proper mass m_0 is in uniform motion with velocity $v = \beta c$ relative to an observer, then its mass (and consequently energy) as measured by the observer would be

$$m_{relativistic} = \frac{m_0}{\sqrt{1-\beta^2}} \implies E_{relativistic} = m_{relativistic} \times c^2 = \frac{m_0c^2}{\sqrt{1-\beta^2}} \tag{6.3}$$

where $\gamma = 1/\sqrt{1-\beta^2}$ is the ***Lorentz Factor***. Since kinetic energy is the same as energy gained by a body when brought from rest to velocity $v = \beta c$

$$E_{kinetic} = E_{relativistic} - E_0 = m_0c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) \tag{6.4}$$

which for small β reduces to classical form $E_{kinetic} = m_0v^2/2$.
If we associate kinetic energy with the frequency of sinusoidal virtual thread, then

$$\nu = \frac{E_{kinetic}}{h} = \frac{m_0c^2}{h}\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) \tag{6.5}$$

Notice that the sinusoidal frequency (6.5) is fundamentally different from oscillation frequency ν_1 (6.2), in the way Lorentz factor combines with proper frequency $\nu_0 = \frac{m_0 c^2}{h}$.

VI.ii The Energy Momentum Relation

Lorentz Factor $\gamma = 1/\sqrt{1-\beta^2}$ is the key to relativistic dynamics. It can also be expressed in terms of momentum p, a classical property describing how a body moves through space. [4]

$$\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{6.6}$$

$$p = m_{relativistic} \times v = \frac{m_0 v}{\sqrt{1-\beta^2}} = \frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{(using 6.3)}$$

$$\text{squaring both sides:} \quad p^2\left(1-\frac{v^2}{c^2}\right) = m_0^2 v^2$$

$$\text{solving for } v^2\text{:} \quad p^2 = v^2\left(m_0^2 + \frac{p^2}{c^2}\right) \implies v^2 = \frac{p^2}{m_0^2 + \frac{p^2}{c^2}} \tag{6.7}$$

using v^2 from (6.7) in Lorentz factor (6.6) gives

$$\gamma^2(v) = \frac{1}{1-\dfrac{v^2}{c^2}} = \frac{1}{1-\dfrac{p^2}{m_0^2c^2+p^2}} = 1 + \frac{p^2}{m_0^2c^2}$$

$$\boxed{\gamma(p) = \sqrt{1+\frac{p^2}{m_0^2c^2}}} \quad \text{where} \quad \gamma(p) \equiv \gamma(m_{rel}v) \equiv \gamma \tag{6.8}$$

using eq (6.8) in relativistic mass (6.3)

$$m_{relativistic} = m_0\gamma = \sqrt{m_0^2 + \frac{p^2}{c^2}}$$

$$E_{relativistic} = m_{relativistic} \times c^2$$

$$\boxed{E_{relativistic} = \sqrt{m_0^2c^4 + p^2c^2}} \tag{6.9}$$

eq (6.9) is the Energy Momentum Relation for a free particle in flat spacetime. First term is rest energy (invariant), while second term is Kinetic energy (due to relative motion through space). If p is very small, this reduces to the classical form

$$E_{relativistic} \approx m_0c^2 + \frac{p^2}{2m_0}$$

Special cases are

1. **At Rest**: If particle is at rest relative to the observer, than $p = 0$ and (6.9) simplifies to

$$E_0 = m_0c^2 \qquad \text{(mass-energy equivalence)}$$

2. **Massless Particle**: Particles having no proper or rest mass ($m_0 = 0$) like photons composing light are *massless*. They always travel at the speed of light $v = c$, so that no observer can ever catch up to them and see *nothing*. For them, rest mass has no meaning since they are never at rest. The relativistic mass would be [4]

$$m_{relativistic} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{0}{0} \qquad \text{(using 6.3)}$$

which is indeterminant and can still be non-zero. Using the general energy-momentum relation (6.9) with $m_0 = 0$

$$\boxed{E_{relativistic} = pc} \qquad \text{(all Kinetic)} \tag{6.10}$$

From the principle of inertia of energy, $E_{relativistic} = m_{relativistic} \times c^2$

$$m_{relativistic} \times c^2 = pc \implies \boxed{m_{relativistic} = \frac{p}{c}} \tag{6.11}$$

which means that *relativistic mass of a massless particle is purely kinetic and proportional to its momentum*! This agrees with the fact that massless particles experience gravitational fields.

3. **High Kinetic Energy limit**: For particles with low proper mass m_0 and moving at very high speeds (accelerated subatomic particles like electrons), kinetic mass gain can be enormous relative to their proper mass. In that case, rest mass (or energy) can be neglected compared to kinetic mass (or energy).

$$\boxed{E_{relativistic} \approx pc} \qquad \text{(predominantly Kinetic)} \tag{6.12}$$

VI.iii From Time to Space

Following the mechanical analogy in section VI.i, if we associate frequency of sinusoidal virtual thread with the kinetic energy from energy-momentum relation (eq 6.9, 6.10 and 6.12)

$$\nu = \frac{E}{h} = \frac{pc}{h}$$

Analogous to temporal frequency $\nu = 1/T$ cycles per unit time, where T is the time period (time required for a wave to complete one cycle), ***spatial frequency*** $\bar{\nu} = 1/\lambda$ cycles per unit distance, where λ is the distance period (distance traveled by wave in one cycle). Using $c = \nu\lambda$

$$\nu = \frac{c}{\lambda} = c\bar{\nu} = \frac{pc}{h} \implies \boxed{p = h\bar{\nu}} \tag{6.13}$$

Equation (6.13) has a highly significant message. *The momentum of a particle is proportional to the spatial frequency of the wave describing its motion through space*, the one we have been calling sinusoidal virtual thread!. This is the infamous Louis de Broglie hypothesis, saying that *momentum is the same as spatial frequency*. Following relativistic mass of photons (eq 6.11)

$$m_{relativistic} = \frac{p}{c} \implies \boxed{m_{relativistic} = \frac{h}{c}\bar{\nu}} \tag{6.14}$$

equation (6.14) implies that relativistic mass of a massless particle ($m_0 = 0$) is proportional to the spatial frequency of the wave describing its motion through space.

VI.iv From Fourier to Quantum Uncertainty Principle

If a is a continuous random variable with probability distribution $A(a)$, such that $a = kb$ where k is proportionality constant, $a, b \in \mathbb{R}$, then their variance are related as

$$\sigma_a^2 = \int_{-\infty}^{\infty} (a-\bar{a})^2 P(a)\,\mathrm{d}a = \frac{\int_{-\infty}^{\infty} (a-\bar{a})^2 |A(a)|^2\,\mathrm{d}a}{\int_{-\infty}^{\infty} |A(a)|^2\,\mathrm{d}a}$$

$$= \frac{\int_{-\infty}^{\infty} k^2(b-\bar{b})^2 |A(b)|^2\,\mathrm{d}b}{\int_{-\infty}^{\infty} |A(b)|^2\,\mathrm{d}b} = k^2\sigma_b^2$$

$$\sigma_a^2 = k^2\sigma_b^2 \implies \boxed{\sigma_a = k\sigma_b} \tag{6.15}$$

since standard deviation is a measure of spread from the mean, negative root is neglected.

From Fourier uncertainty principle (equation 5.12)

$$\sigma_t\sigma_\omega \geq \frac{1}{2} \implies \boxed{\sigma_t\sigma_\nu \geq \frac{1}{4\pi}} \qquad \text{(using } \omega = 2\pi\nu \text{ and eq 6.15)} \tag{6.16}$$

equation (6.16) is the Time-Temporal frequency uncertainty principle. For a particle in uniform motion with velocity $v = x/t$ and associated spatial wave with frequency $\bar{\nu} = \nu/v$, where $\bar{\nu} = 1/\lambda$ is the spatial frequency, eq (6.15) gives

$$x = vt \implies \sigma_x = v\sigma_t$$
$$\bar{\nu} = \frac{\nu}{v} \implies \sigma_{\bar{\nu}} = \frac{\sigma_\nu}{v}$$
$$\sigma_x\sigma_{\bar{\nu}} = \sigma_t\sigma_\nu \implies \boxed{\sigma_x\sigma_{\bar{\nu}} \geq \frac{1}{4\pi}} \qquad \text{(using 6.16)} \tag{6.17}$$

equation (6.17) is the Space-Spatial frequency uncertainty principle for waves spread over space. Using proportionality of momentum and spatial frequency (eq 6.13)

$$p = h\bar{\nu} \implies \sigma_{\bar{\nu}} = \frac{\sigma_p}{h}$$
$$\sigma_x\sigma_{\bar{\nu}} = \frac{\sigma_x\sigma_p}{h} \geq \frac{1}{4\pi} \qquad \text{(using 6.17)}$$
$$\boxed{\sigma_x\sigma_p \geq \frac{h}{4\pi}} \tag{6.18}$$

equation (6.18) is the infamous **Heisenberg's uncertainty principle**!. It's astonishing how a simple idea of frequency decomposition of a signal can manifest to such huge implications.

VII Conclusion

The lesson from this is that Heisenberg's uncertainty principle is not an artifact of randomness or imperfect measurements in the quantum realm. Rather, it's a fundamental trade-off between how concentrated a wave and its frequency representation can be, applied in the context of wave nature of the particle (i.e saying particle is a wave and spread out over space). Space and spatial frequency (which is the same as momentum $p = h\bar{\nu}$) share the same trade-off as time and temporal frequency. The trade-off in itself is however quite general and shows up in many real world non-quantum cases as well!

VIII Appendix

VIII.i Fubini's Theorem

Multiple integrals can be replaced by iterated integrals (or vice-versa), and the order of integration can be switched, provided multiple integral of the absolute integrand converges. Mathematically

If $\displaystyle\iint\limits_{X\times Y} |f(x,y)|\,\mathrm{d}x\,\mathrm{d}y < \infty$

$$\text{Then} \quad \iint\limits_{X\times Y} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_X \left(\int_Y f(x,y)\,\mathrm{d}y\right)\mathrm{d}x = \int_Y \left(\int_X f(x,y)\,\mathrm{d}x\right)\mathrm{d}y \tag{8.1}$$

VIII.ii Leibniz integral rule

Leibniz integral rule of differentiation under the integral sign is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^b f(x,t)\,\mathrm{d}t\right) = \int_a^b \frac{\partial}{\partial x} f(x,t)\,\mathrm{d}t \tag{8.2}$$

Method that uses this rule to compute integrals is known as *Feynman's method*

VIII.iii Dirac Delta Function

The Dirac delta function $\delta(x)$ is a generalized distribution over real numbers defined by two key properties [2]

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases} \qquad \text{(only turns on at 0)} \tag{8.3}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1 \qquad \text{(area under distribution is 1)} \tag{8.4}$$

This leads to the following

1. Integration Property

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathrm{d}xv = \int_{-\infty}^{\infty} f(a)\delta(x-a)\,\mathrm{d}x = f(a)\int_{-\infty}^{\infty}\delta(x-a)\,\mathrm{d}x = f(a) \tag{8.5}$$

where second integral comes from the fact that $\delta(x-a)$, and consequently the integrand is 0 everywhere except when $x=a$, so that only $f(x=a)$ contributes to the integral, which is constant and can be moved outside

2. Another independent definition of delta function, following Appendix VIII.v

$$\int_{-\infty}^{\infty} e^{i\omega t}\,\mathrm{d}\omega = 2\pi\delta(t) \tag{8.6}$$

VIII.iv Definite Integral of sin(x)/x

$$\int_0^{\infty} \frac{\sin(x)}{x}\,\mathrm{d}x = \frac{\pi}{2} \tag{8.7}$$

Proof: [5] This is an improper integral since the integrand $\sin(x)/x$ has x in the denominator, but still converges. In order to actually integrate this, we need a way to somehow eliminate the x in denominator while keeping the integrand convergent. One way to do that is to multiply the integrand with some function of x and a parameter b, which when differentiated with respect to b under the integral sign eliminates x (Feynman's Technique, Appendix VIII.ii).

One such function is e^{bx}, $b \in \mathbb{R}$. However, this diverges in the limit $x \to \infty$, and can only converge if the exponent is negative. So the best choice is e^{-bx}, $b > 0$ which converges in the limit $x \to \infty$. Defining the parameterized integral

$$I(b) = \int_0^{\infty} \frac{\sin(x)e^{-bx}}{x}\,\mathrm{d}x \tag{8.8}$$

our goal is to find $I(b=0)$. Differentiating with respect to b

$$I'(b) = \frac{\mathrm{d}}{\mathrm{d}b} I(b) = \frac{\mathrm{d}}{\mathrm{d}b}\left(\int_0^{\infty} \frac{\sin(x)e^{-bx}}{x}\,\mathrm{d}x\right)$$

$$= \int_0^\infty \frac{\partial}{\partial b}\left(\frac{\sin(x)e^{-bx}}{x}\right) \mathrm{d}x = -\int_0^\infty \sin(x)e^{-bx}\,\mathrm{d}x \qquad \text{(using 8.2)}$$

integrating by parts, taking $\sin(x)$ as first and e^{-bx} as second function gives

$$I'(b) = \left[\frac{e^{-bx}}{1+b^2}\left(b\sin(x) + \cos(x)\right)\right]_{x=0}^{x=\infty} = -\frac{1}{1+b^2}$$

integrating with respect to b

$$\int I'(b)\,\mathrm{d}b = I(b) = -\tan^{-1}(b) + c$$

$$\implies I(b=0) = c, \quad I(b=\infty) = -\frac{\pi}{2} + c = -\frac{\pi}{2} + I(b=0) \tag{8.9}$$

where c is the constant of integration. Since by definition of $I(b)$ (eq 8.8)

$$I(b) = \int_0^\infty \frac{\sin(x)e^{-bx}}{x}\,\mathrm{d}x \implies I(b=\infty) = 0$$

$$\text{So} \quad I(b=\infty) = 0 = -\frac{\pi}{2} + I(b=0) \qquad \text{(using 8.9)}$$

$$\boxed{I(b=0) = \frac{\pi}{2}}$$

which is the required integral. In general, if $k \in \mathbb{R}, k > 0$, then

$$\int_{x=0}^{x=\infty} \frac{\sin(kx)}{x}\,\mathrm{d}x = \int_{y=0}^{y=\infty} \frac{\sin(y)}{y}\,\mathrm{d}y = \frac{\pi}{2} \tag{8.10}$$

where second integral results from substitution $y = kx$

VIII.v Definite Integral of Complex Exponential

$$\int_{-\infty}^{\infty} e^{i\omega t}\,\mathrm{d}\omega = 2\pi\delta(t) \tag{8.11}$$

Proof: [5] let $I(t) = \int_{-\infty}^{\infty} e^{i\omega t}\,\mathrm{d}\omega$ be the required integral, and $f_k(t), k \in \mathbb{R}, k > 0$ be its generalized form defined as

$$f_k(t) = \int_{-k}^{k} e^{i\omega t}\,\mathrm{d}\omega \implies I(t) = \lim_{k\to\infty} f_k(t) \tag{8.12}$$

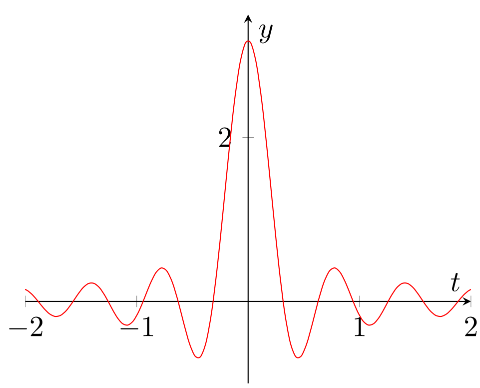
However, $f_k(t)$ can be easily computed as

$$f_k(t) = \int_{-k}^{k} e^{i\omega t}\,\mathrm{d}\omega = \left[\frac{e^{i\omega t}}{it}\right]_{\omega=-k}^{\omega=k} = \frac{e^{ikt} - e^{-ikt}}{it} = \frac{2\sin(kt)}{t} \tag{8.13}$$

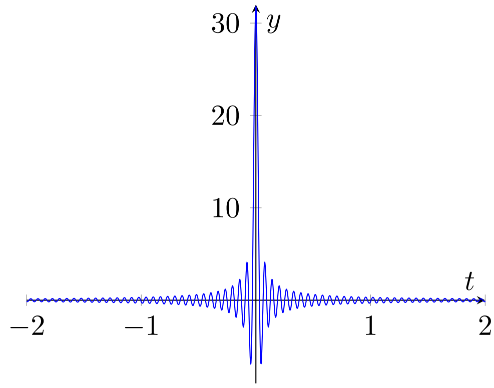
where $e^{ikt} = \cos(kt) + i\sin(kt)$ is the Euler's formula. Equation (8.13) shows that $f_k(t)$ is an *even function* (as $k > 0$) containing the improper integrand $\sin(kt)/t$ (section VIII.iv)

$$\int_{-\infty}^{\infty} f_k(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} \frac{2\sin(kt)}{t}\,\mathrm{d}t = 4\int_0^\infty \frac{\sin(kt)}{t}\,\mathrm{d}t = 2\pi \qquad \text{(using 8.10)}$$

$$\text{divide by } 2\pi\text{:} \quad \int_{-\infty}^{\infty} \frac{f_k(t)}{2\pi}\,\mathrm{d}t = \int_{-\infty}^{\infty} \frac{\sin(kt)}{\pi t}\,\mathrm{d}t = 1 \tag{8.14}$$



(a) $k = 10$



(b) $k = 100$

Figure 11: Plot of $\sin(kt)/\pi t$ against t for different $k > 0, k \in \mathbb{R}$. As k increases, function approaches k/π for $t = 0$ and tends to 0 for $t \neq 0$

equation (8.14) looks similar to the area property of delta function (eq 8.4). The graph of the integrand $\sin(kt)/\pi t$ (figure 11) suggests that at high k values, it is essentially 0 everywhere except a huge peak at $t = 0$. In the limit of $k \to \infty$

$$\lim_{k\to\infty} \frac{\sin(kt)}{\pi t} = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0 \end{cases} \tag{8.15}$$

from the two properties of $\sin(kt)/\pi t$ in equation (8.15) and (8.14), it is clear that

$$\lim_{k\to\infty} \frac{\sin(kt)}{\pi t} = \delta(t)$$

Using the definitions of $f_k(t)$ from equation (8.12) and (8.13)

$$\frac{\sin(kt)}{\pi t} = \frac{f_k(t)}{2\pi} \implies \lim_{k\to\infty} \frac{\sin(kt)}{\pi t} = \lim_{k\to\infty} \frac{f_k(t)}{2\pi} = \delta(t)$$
$$\lim_{k\to\infty} f_k(t) = 2\pi\delta(t) \implies \boxed{I(t) = 2\pi\delta(t)}$$

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