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Uncertainty Principles in Fourier Analysis

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1. Introduction

In this paper we consider some inequalities of the type of Heisenberg's uncertainty relation. Some of them will be formulated in terms of classical Fourier theory, others will be expressed in terms of a notion that will be called the *musical score* of a time function.

We shall consider a complex-valued function f of the real variable t, defined for $-\infty < t < \infty$. The variable t will be referred to as the *time*, and f will be called a *signal*. For the moment we shall assume that f belongs to L_2 [L_2 stands for $L_2(-\infty, \infty)$], whence Planchérel's theorem can be applied. We define the Fourier transform g of f by

$$g(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i t\omega} f(t)\, dt, \tag{1.1}$$

where the integral has to be interpreted carefully as the limit in the mean of $\int_{-T}^{T}$ as $T \to \infty$. We shall write

$$g = \mathscr{F} f.$$

Plancherel's theorem states that f can be obtained from g in a similar way:

$$f(t) = \int_{-\infty}^{\infty} e^{2\pi i t\omega} g(\omega)\, d\omega, \tag{1.2}$$

and Parseval's theorem says that

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |g(\omega)|^2\, d\omega. \tag{1.3}$$

With the notation chosen here, the Heisenberg relation can be expressed as

$$\left[\int_{-\infty}^{\infty} (t-a)^2 |f(t)|^2\, dt\right]^{1/2} \cdot \left[\int_{-\infty}^{\infty} (\omega - b)^2 |g(\omega)|^2\, d\omega\right]^{1/2} \geq \|f\|/4\pi, \tag{1.4}$$

where, as usual, $\|f\|$ denotes $[\int_{-\infty}^{\infty} |f(t)|^2\, dt]^{1/2}$.

Here a and b are arbitrary real numbers. The factors on the left-hand side of (1.4) can be called the *time spread* around the time a, and the *frequency spread* around the frequency b, respectively.

For convenience we shall only consider $a = b = 0$, from which the general case can be derived. This is achieved by introducing f^* and g^* instead of f and g, where

$$f^*(t) = f(t+a)\, e^{-2\pi i b t}, \quad g^*(\omega) = g(\omega + b)\, e^{2\pi i a \omega}, \tag{1.5}$$

where $g^* = \mathscr{F} f^*$. Now (1.4) reduces to

$$\left[\int_{-\infty}^{\infty} t^2 |f^*(t)|^2\, dt\right]^{1/2} \cdot \left[\int_{-\infty}^{\infty} \omega^2 |g^*(\omega)|^2\, d\omega\right]^{1/2} \geq \|f^*\|/4\pi.$$

So henceforth we shall consider the inequality

$$\left[\int_{-\infty}^{\infty} t^2 |f(t)|^2\, dt\right]^{1/2} \cdot \left[\int_{-\infty}^{\infty} \omega^2 |g(\omega)|^2\, d\omega\right]^{1/2} \geq \|f\|/4\pi, \tag{1.6}$$

where $g = \mathscr{F} f$. This was derived by Weyl ([*6*], Appendix 1) from the inequality

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt \leq 4 \int_{-\infty}^{\infty} t^2 |f(t)|^2\, dt \cdot \int_{-\infty}^{\infty} |df(t)/dt|^2\, dt. \tag{1.7}$$

Equality in (1.6) and in (1.7) is attained if and only if f has the form $f(t) = A \exp(-\alpha t^2)$, where A and α are constants, $\alpha > 0$.

We deal with (1.6) in a different way in Sec. 2.

The uncertainty principle expresses, roughly speaking, that if a signal is confined to a small time interval, then its Fourier transform cannot be confined to a small frequency interval. A more detailed version of the uncertainty principle was given by Fuchs [*2*] and by Landau and Pollak [*3*]. These authors proved quantitative statements relating the percentage of the energy that lies in a given time interval (compared to the total energy $\|f\|^2$) to the percentage of the energy that lies in a given frequency interval. We shall not deal with this kind of work in this paper. Instead, we have an entirely different way (Sec. 2) of considering time and frequency simultaneously.

2. The Musical Score of a Signal

Usually we describe sound by a single function f, defined for $-\infty < t < \infty$, i.e. by a signal. For some applications it is natural to discuss mathematical and physical properties in terms of f itself, and for others it is more appropriate to speak in terms of its Fourier transform g. Between those extremes, however, there is a class of questions where it is desirable to consider time and frequency simultaneously.

For example, if f represents a piece of music, then the composer does not produce f itself; he does not even define it. He may try to prescribe the exact frequency and the exact time interval of a note (although the uncertainty principle says that he can never be completely successful in this effort), but he does not try to prescribe the phase. The composer does not deal with f; it is only the gramophone company which produces and sells an f. On the other hand, the composer certainly does not want to describe the Fourier transform. This Fourier transform is very useful for solving mathematical and physical problems, but it gives an absolutely unreadable picture of the given piece of music.

What the composer really does, or thinks he does, or should think he does, is something entirely different from describing either f or $\mathscr{F}f$. Instead, he constructs a function of two variables. The variables are the time and the frequency, the function describes the intensity of the sound. He describes the function by a complicated set of dots on score paper. His way of describing time is slightly different from what a mathematician would do, but certainly vertical lines denote constant time, and horizontal lines denote constant frequency.

We shall give a mathematical description of such an intensity function. In some respect our choice will be arbitrary and somewhat unrealistic (for example the fact that we use the future for the description of the present), but it has the advantage of being easy to handle and having several useful invariance and symmetry properties. It is essentially the same expression as the one for the phase-space distribution introduced in quantum mechanics by Wigner [7] and elaborated by Moyal [*4*]. Both in music and in quantum mechanics we have the situation of a function of a single variable, which appears to be a function of two variables as long as the observation is not too precise. The parallel between quantum mechanics and music can be carried a little further by comparing the composer to the classical physicist. The way the composer writes an isolated note as a dot, and thinks of it as being completely determined in time and frequency, is similar to the classical physicist's conception of a particle with well-determined position and momentum.

The possibility of describing energy density in the time-frequency plane by an expression which is essentially Wigner's, was pointed out by Ville [*5*].

In this paper's presentation we shall assume that our time functions are in L_2. The scope of the notion of the score is considerably wider, but presently we are dealing mainly with inequalities which are meaningless if f has infinite total energy. Therefore, not much is lost by assuming $f \in L_2$ in our present discussion.

If $f_1 \in L_2$, $f_2 \in L_2$, and if x, y are real numbers, then we define

$$H(x, y; f_1, f_2) = 2\int_{-\infty}^{\infty} f_1(x+t)\overline{f_2(x-t)}e^{-4\pi iyt}\, dt. \tag{2.1}$$

We shall call $H(x, y; f, f)$ the *energy density* of f at time x and frequency y. Considered as a function of x and y we call it the *musical score* of f, or *score* for short.

Of course we owe the reader some explanation for this definition. First we remark that (2.1) is related to

$$H(0, 0; f_1, f_2) = 2\int_{-\infty}^{\infty} f_1(t)\overline{f_2(-t)}\, dt \tag{2.2}$$

in the following way. If f_j^* is defined by

$$f_j^*(t) = f_j(t+x)e^{-2\pi iyt} \qquad (j = 1, 2)$$

[cf. (1.5)], then we have

$$H(x, y; f_1, f_2) = H(0, 0; f_1^*, f_2^*), \tag{2.3}$$

and this is why we have the right to refer to $H(x, y; f_1, f_2)$ as something related to the moment x and the frequency y. The words "energy density" can be partly explained by formulas (2.4)–(2.6) which we mention without proof: If $-\infty \le p < q \le \infty$, and if both f and $g\,(=\mathscr{F}f)$ are in $L_1 \cap L_2$, then

$$\int_p^q dx \int_{-\infty}^{\infty} H(x, y; f, f)\, dy = \int_p^q |f(t)|^2\, dt, \tag{2.4}$$

$$\int_p^q dy \int_{-\infty}^{\infty} H(x, y; f, f)\, dx = \int_p^q |g(\omega)|^2\, d\omega. \tag{2.5}$$

If, moreover, f is what is called band-limited, i.e. if f has the form

$$f(t) = \int_c^d g(\omega)e^{2\pi i\omega t}\, d\omega,$$

with finite c and d $(c < d)$, then we have, for all p and q

$$\int_p^q dx \int_c^d H(x, y; f, f)\, dy = \int_p^q |f(t)|^2\, dt. \tag{2.6}$$

On the other hand, there is also a serious objection against the name "energy density," namely the fact that $H(x, y; f, f)$ is not ≥ 0 for all f [for example, if $tf(t) > 0$ for all t, we have, by (2.2), $H(0, 0; f, f) < 0$]. But this objection does not hold against certain "moving averages" $H_{\alpha\beta}$, which we are introducing presently.

Let $\alpha > 0, \beta > 0$. We shall form the Gaussian average (often called Gauss transform, or Weierstrass transform, with slightly different notation)

$$H_{\alpha\beta}(x, y; f_1, f_2)$$
$$= (\alpha\beta)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{\pi}{\alpha}(x-\xi)^2 - \frac{\pi}{\beta}(y-\eta)^2\right] H(\xi, \eta; f_1, f_2)\, d\xi\, d\eta. \tag{2.7}$$

This integral converges rapidly, since $|H(\xi, \eta; f_1, f_2)| \leq 2\|f_1\| \cdot \|f_2\|$, which follows from (2.1) by application of the Cauchy–Bunyakovski inequality.

From (2.7) we obtain by application of Fubini's theorem

$$H_{\alpha\beta}(x, y; f_1, f_2) = \alpha^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{\pi}{4\alpha}(s+t-2x)^2\right.$$
$$\left. - \eta\beta(s-t)^2 - 2\pi i y(s-t)\right] f_1(s)\overline{f_2(t)}\, ds\, dt. \tag{2.8}$$

Owing to the strong convergence factors in (2.8), we can define $H_{\alpha\beta}$ in cases where the score itself does not exist, for example if f is a periodic function, or a delta function.

We can consider $H_{\alpha\beta}$ as a blurred picture of the score. It follows immediately from a well-known semigroup property of the Gauss transform, that if we make a blurred picture of $H_{\alpha\beta}$, with "blurring parameters" γ and δ, then what we obtain is $H_{\alpha+\gamma, \beta+\delta}$.

There is a simple relation between score and Fourier transform, which we mention without proof. The score of the Fourier transform of f is obtained by turning the score of f over 90°, and the same thing holds for the blurred score (provided that we interchange the blurring parameters):

$$H_{\alpha\beta}(x, y; f_1, f_2) = H_{\beta\alpha}(y, -x; \mathscr{F}f_1, \mathscr{F}f_2). \tag{2.9}$$

This is a formula of Parseval type, and indeed, if $\alpha \to \infty$, $\beta \to \infty$, then (2.9) turns into the Parseval formula.

We finally emphasize that $H_{\alpha\beta}(x, y; f, f)$ is positive-definite if and only if $\alpha\beta > \frac{1}{4}$ (see Sec. 4). This means that if a physical experiment produces the value of $H_{\alpha\beta}(x, y; f, f)$, and if the nature of the experiment is such that it measures an amount of energy so that the result can never be negative, then the product of the blurring parameters α and β has to exceed $\frac{1}{4}$. This amounts to saying that if a measurement is very accurate in the sense that it needs the signal f in a small time interval only, then it cannot be very accurate about the frequencies in that signal.

3. The Heisenberg Inequality

We shall use the Hermite polynomials H_n with the following notation (see Erdélyi [*1*, p. 193])

$$H_n(t) = (-1)^n \exp(t^2) \left(\frac{d}{dt}\right)^n \exp(-t^2) \qquad (n = 0, 1, 2, \ldots). \quad (3.1)$$

For the corresponding orthogonal system on $L_2 = L_2(-\infty, \infty)$ we take

$$\psi_n(t) = (2^{n-1/2}n!)^{-1/2}\, H_n[(2\pi)^{1/2}t] \exp(-\pi t^2) \qquad (n = 0, 1, 2, \ldots), \quad (3.2)$$

whence $\psi_n(t)$ equals $n!\, C_n$ times the coefficient of w^n in the power series development of $\exp[\pi t^2 - 2\pi(t-w)^2]$, with

$$C_n = [2^{n-1/2}n!\,(2\pi)^n]^{-1/2}.$$

We have $(\psi_n, \psi_m) = \delta_{nm}$ $(n, m = 0, 1, 2, \ldots)$, where δ_{nm} is the Kronecker symbol, and the inner product is defined by $(f, g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt$.

It is well known that the ψ_m are eigenfunctions of the Fourier transform: $\mathscr{F}\psi_m = i^{-m}\psi_m$. Therefore, if $f \in L_2$, and $g = \mathscr{F}f$, then the Fourier coefficients of f and g are related by $(f, \psi_m) = (\mathscr{F}f, \mathscr{F}\psi_m) = i^m(g, \psi_m)$. This plays its role in the following theorem.

Theorem 3.1. *If* $f \in L_2(-\infty, \infty)$, $g = \mathscr{F}f$, $\gamma_m = (f, \psi_m)$, *then we have*

$$\int_{-\infty}^{\infty} t^2[|f(t)|^2 + |g(t)|^2]\, dt = (2\pi)^{-1} \sum_{m=0}^{\infty} |\gamma_m|^2(2m+1). \quad (3.3)$$

Proof. It follows from the recurrence relation for the Hermite polynomials that for $m = 0, 1, 2, \ldots$

$$(4\pi)^{1/2}t\psi_m(t) = (m+1)^{1/2}\psi_{m+1}(t) + m^{1/2}\psi_{m-1}(t)$$

[if we define $\psi_{-1}(t) = 0$]. Hence, putting $\gamma_{-1} = 0$,

$$(4\pi)^{1/2}[tf(t), \psi_m(t)] = (m+1)^{1/2}\gamma_{m+1} + m^{1/2}\gamma_{m-1},$$
$$(4\pi)^{1/2}[tg(t), \psi_m(t)] = i^{-m-1}(m+1)^{1/2}\gamma_{m+1} + i^{-m+1}m^{1/2}\gamma_{m-1}.$$

Applying Parseval's formula both to $tf(t)$ and $tg(t)$, we obtain that the left-hand side of (3.3) equals

$$(4\pi)^{-1}\left[2\sum_{m=0}^{\infty}(m+1)|\gamma_{m+1}|^2 + 2\sum_{m=0}^{\infty} m|\gamma_{m-1}|^2\right],$$

and this is equal to the right-hand side of (3.3).

Theorem 3.2. *If $f \in L_2(-\infty, \infty)$, $g = \mathscr{F}f$, then*

$$\int_{-\infty}^{\infty} t^2|f(t)|^2\,dt + \int_{-\infty}^{\infty} t^2|g(t)|^2\,dt \geq (2\pi)^{-1}\int_{-\infty}^{\infty} |f(t)|^2\,dt,$$

with equality only if $f(t)$ is almost everywhere equal to a constant multiple of $\exp(-\pi t^2)$.

Proof. This follows directly from (3.3), since

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = |\gamma_0|^2 + |\gamma_1|^2 + \cdots,$$

and since $\psi_0(t)$ is a multiple of $\exp(-\pi t^2)$.

From Theorem 3.2 we can derive Heisenberg's inequality as follows. We take some constant $p > 0$ and we consider the functions f_1, g_1 defined by

$$f_1(t) = p^{-1/2}f(t/p), \qquad g_1(t) = p^{1/2}g(tp) \tag{3.4}$$

(whence again $g_1 = \mathscr{F}f_1$), and we apply (3.3) to f_1, g_1. This leads to

$$p^2\int_{-\infty}^{\infty} t^2|f(t)|^2\,dt + p^{-2}\int_{-\infty}^{\infty} t^2|g(t)|^2\,dt \geq (2\pi)^{-1}\int_{-\infty}^{\infty} |f(t)|^2\,dt. \tag{3.5}$$

Taking, on the left, the minimum with respect to p, Heisenberg's inequality (1.6) follows at once.

We shall explore this idea a little further, by proving that if Heisenberg's inequality is almost an equality, then f is almost equal to one of the functions for which it is an exact equality.

Theorem 3.3. *Let $f \in L_2(-\infty, \infty)$, $g = \mathscr{F}f$, $\int_{-\infty}^{\infty}|f(t)|^2\,dt = 1$. Assume that δ is a nonnegative number with the property that for every $c > 0$ and for every complex number λ with $|\lambda| = 1$ we have*

$$\left[\int_{-\infty}^{\infty} |f(t) - \lambda c^{-1/2}\psi_0(ct)|^2\,dt\right]^{1/2} \geq \delta \tag{3.6}$$

[*where* $\psi_0(t) = 2^{1/4}\exp(-\pi t^2)$]. *Then we have*

$$\left[\int_{-\infty}^{\infty} t^2|f(t)|^2\,dt\right]^{1/2} \cdot \left[\int_{-\infty}^{\infty} \omega^2|g(\omega)|^2\,d\omega\right]^{1/2} \geq (4\pi)^{-1}[3 - 2(1 - \tfrac{1}{2}\delta^2)^2]. \tag{3.7}$$

Proof. We put

$$\int_{-\infty}^{\infty} t^2|f(t)|^2\,dt = A, \qquad \int_{-\infty}^{\infty} \omega^2|g(\omega)|^2\,d\omega = B.$$

From (3.3) it follows that

$$\begin{aligned} A+B &\geq (2\pi)^{-1}[|\gamma_0|^2+3(|\gamma_1|^2+|\gamma_2|^2+\cdots)] \\ &= (2\pi)^{-1}(3-2|\gamma_0|^2). \end{aligned} \tag{3.8}$$

On the other hand we have, by (3.6), if $|\lambda|=1$,

$$\delta^2 \leq \|f-\lambda\psi_0\|^2 = |\gamma_0-\lambda|^2+|\gamma_1|^2+|\gamma_2|^2+\cdots.$$

Taking λ such that $\lambda^{-1}\gamma_0 \geq 0$, $|\lambda|=1$, we obtain

$$\delta^2 \leq (|\gamma_0|-1)^2+(1-|\gamma_0|^2) = 2-2|\gamma_0|.$$

Hence, by (3.8),

$$A+B \geq (2\pi)^{-1}[3-2(1-\tfrac{1}{2}\delta^2)^2].$$

If f, g satisfy the conditions of the theorem, then it is not difficult to show that also f_1, g_1 [defined by (3.4)] satisfy those conditions. Therefore

$$p^2A+p^{-2}B \geq (2\pi)^{-1}[3-2(1-\tfrac{1}{2}\delta^2)^2].$$

Choosing $p=A^{-1/4}B^{1/4}$, we infer (3.7).

As a corollary we mention:

Theorem 3.4. *If $f \in L_2(-\infty, \infty)$, $g=\mathscr{F}f$, $\int_{-\infty}^{\infty}|f(t)|^2\,dt=1$, and if f is odd* [i.e. $f(t)=f(-t)$ *for all* t], *then we have*

$$\left[\int_{-\infty}^{\infty} t^2|f(t)|^2\,dt\right]^{1/2} \cdot \left[\int_{-\infty}^{\infty} \omega^2|g(\omega)|^2\,d\omega\right]^{1/2} \geq 3/(4\pi).$$

Proof. Since f is odd, and $\psi_0(ct)$ is even, the inner product of $f(t)$ and $\psi_0(ct)$ vanishes for all c. Hence, the left-hand side of (3.6) equals $2^{1/2}$ (i.e. the length of the difference of two orthogonal unit vectors). Now taking $\delta=2^{1/2}$, formula (3.7) gives the required result.

A further result is:

Theorem 3.5. *If the conditions of Theorem 3.3 hold, and if f is even, then*

$$\left[\int_{-\infty}^{\infty} t^2|f(t)|^2\,dt\right]^{1/2} \cdot \left[\int_{-\infty}^{\infty} \omega^2|g(\omega)|^2\,d\omega\right]^{1/2} \geq (4\pi)^{-1}[5-4(1-\tfrac{1}{2}\delta^2)^2].$$

Proof. The following modification can now be made in the beginning of the proof of Theorem 3.3. Since f is even, and ψ_1 is odd, we have $\gamma_1=0$.

Hence (3.8) can be refined:

$$\begin{aligned} A + B &= (2\pi)^{-1}[|\gamma_0|^2 + 5|\gamma_2|^2 + 7|\gamma_3|^2 + \cdots] \\ &\geq (2\pi)^{-1}[|\gamma_0|^2 + 5(|\gamma_2|^2 + |\gamma_3|^2 + \cdots)] \\ &\geq (2\pi)^{-1}(5 - 4|\gamma_0|^2). \end{aligned}$$

The rest of the proof can be copied from Theorem 3.3.

4. Inequalities Concerning the Score

Before stating any inequality, we first remark that it is often sufficient to restrict the discussion of the blurred score $H_{\alpha\beta}$ [see (2.7) and (2.8)] to the case $\alpha = \beta$. For, the transformation (3.4) has a simple effect:

$$H_{\alpha\beta}(x, y; f, f) = H_{\gamma\delta}(u, v; f_1, f_1), \tag{4.1}$$

with $u = xp$, $v = y/p$, $\gamma = \alpha p^2$, $\beta = \delta p^{-2}$. Thus by taking $p = (\beta/\alpha)^{1/4}$ the general case is reduced to a case with equal parameters. The advantage of the case with equal parameters lies in the fact that it is especially adapted to the Hermite functions (with the normalization given in Sec. 2). For example, it can be shown that the score of ψ_m has complete rotational symmetry with respect to the origin of the score plane. And the following simple result will show its usefulness in the sequel.

Theorem 4.1. *If* $\alpha \geq 0$, $m, n = 0, 1, 2, \ldots$, *then*

$$H_{\alpha\alpha}(0, 0; \psi_m, \psi_n) = \delta_{mn}(\alpha + \tfrac{1}{2})^{-1}[(2\alpha - 1)/(2\alpha + 1)]^m.$$

Proof. From the definition of ψ_m (see the beginning of Sec. 3) it follows that $H_{\alpha\alpha}(0, 0; \psi_m, \psi_n)$ equals $C_m C_n m!\, n!$ times the coefficient of $w^m z^n$ in the power series development of $H_{\alpha\alpha}(0, 0; h_w, h_z)$, where

$$h_w(t) = \exp[\pi t^2 - 2\pi(t - w)^2], \qquad h_z(t) = \exp[\pi t^2 - 2\pi(t - z)^2].$$

If we express this $H_{\alpha\alpha}(0, 0; h_w, h_z)$ by means of (2.8), we get an integral of the type

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-\pi Q(s, t) + 4\pi(sw + tz)]\, ds\, dt, \tag{4.2}$$

where Q is a binary quadratic form. The value of such an integral is

$$(\det Q)^{-1/2} \exp[\pi Q^{-1}(2s, 2w)],$$

where det Q stands for the determinant, and Q^{-1} for the inverse form. Explicit calculation leads to

$$H_{\alpha\alpha}(0, 0; h_w, h_z) = (\alpha + \tfrac{1}{2})^{-1/2} \exp[4\pi wz(2\alpha - 1)/(2\alpha + 1)].$$

If we expand this in powers of w and z the theorem follows in a few lines.

We can now settle the question whether $H_{\alpha\beta}(x, y; f, f)$ is positive-definite. (The words positive-definite refer to the dependence upon f. If x, y, α, β are given, we call $H_{\alpha\beta}(x, y; f, f)$ positive-definite if $H_{\alpha\beta}(x, y; f, f) > 0$ for all $f \in L_2(-\infty, \infty)$ unless f vanishes almost everywhere.)

Theorem 4.2. *Let x, y, α, β be real numbers, $\alpha > 0$, $\beta > 0$. Then $H_{\alpha\beta}(x, y; f, f)$ is positive-definite if and only if $\alpha\beta > \frac{1}{4}$. If $\alpha\beta = \frac{1}{4}$, it is semidefinite.*

Proof. Putting $f^*(t) = f(t+x)\exp(-2\pi iyt)$ we have [cf. (2.3)], for all ξ, η,

$$H(\xi + x, \eta + y; f, f) = H(\xi, \eta; f^*, f^*).$$

Hence by (2.7),

$$H_{\alpha\beta}(x, y; f, f) = H_{\alpha\beta}(0, 0; f^*, f^*),$$

and therefore it suffices to consider the case $x = y = 0$ from now on. Next, we remark that by (4.1) it suffices to consider the special case $\alpha = \beta$. So we only have to investigate whether $H_{\alpha\alpha}(0, 0; f, f)$ is definite.

Again putting $(f, \psi_m) = \gamma_m$, we have, by Theorem 4.1,

$$H_{\alpha\alpha}(0, 0; f, f) = \sum_{m=0}^{\infty} (\alpha + \tfrac{1}{2})^{-1}[(2\alpha - 1)/(2\alpha + 1)]^m |\gamma_m|^2.$$

If $\alpha > \frac{1}{2}$, the right-hand side is positive provided that at least one γ_m is nonzero. If $\alpha = \frac{1}{2}$, the right-hand side is ≥ 0 for all f, but $= 0$ as soon as $(f, \psi_0) = 0$. If $0 < \alpha < \frac{1}{2}$, we can have $H_{\alpha\alpha}(0, 0; f, f) < 0$, for example if $f = \psi_m$ and m is odd. This proves the theorem.

The fact that $H_{\alpha\beta}(x, y; f, f) \geq 0$ if $\alpha\beta \geq \frac{1}{4}$ can also be deduced directly from (2.8). Putting $2\beta - (2\alpha)^{-1} = \sigma$, expanding $\exp(\sigma st)$ in its power series, and interchanging summation and integration, we obtain

$$H_{\alpha\beta}(x,y; f, f) = \alpha^{-1/2} \sum_{k=0}^{\infty} (\sigma^k/k!) \left| \int_{-\infty}^{\infty} s^k \exp\left[-\frac{\pi}{4\alpha}(s^2 - 4xs + 2x^2) - \pi\beta s^2 - 2\pi iys \right] ds \right|^2,$$

and this is ≥ 0 if $\sigma \geq 0$.

We next deal with some moments of H. For these we have inequalities expressing that the score cannot be concentrated upon a small neighborhood of a single point, and that is again something of the type of the uncertainty relation. The simplest case is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x, y; f, f)(x^2+y^2)\,dx\,dy \geq (2\pi)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x, y; f, f)\,dx\,dy. \tag{4.3}$$

This is the special case $k=1$ of Theorem 4.4, but it is already equivalent to Theorem 3.2. For, we have

$$\int_{-\infty}^{\infty} H(x, y; f, f)\,dy = |f(x)|^2$$

for almost all x, whence

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x, y; f, f)\,dx\,dy = \int_{-\infty}^{\infty} |f(x)|^2\,dx, \tag{4.4}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x, y; f, f)x^2\,dx\,dy = \int_{-\infty}^{\infty} x^2|f(x)|^2\,dx. \tag{4.5}$$

Using (2.9), we can derive from (4.5) that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x, y; f, f)y^2\,dx\,dy = \int_{-\infty}^{\infty} y^2|g(y)|^2\,dy.$$

Thus the left-hand side of (4.3) is equal to the left-hand side in Theorem 3.2. The right-hand sides are equal according to (4.4), and so (4.3) is equivalent to the inequality in Theorem 3.2.

Our proof of the inequality for the kth moment of $H(x, y; f, f)$ will start from a result that is similar to Theorem 4.1:

Theorem 4.3. *Let, for $k, m = 0, 1, 2, \ldots$, the coefficient of $u^k v^m$ in the power series development of $(1-u-v-uv)^{-1}$ be denoted by p_{km}. Then we have, for $k, m, n = 0, 1, 2, \ldots$,*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x, y; \psi_m, \psi_m)(x^2+y^2)^k\,dx\,dy = \delta_{mn}k!(2\pi)^{-k}p_{mk}. \tag{4.6}$$

Proof. Using the method of proof of Theorem 4.3, and the functions h_w, h_z introduced there, we remark that the left-hand side of (4.6) equals $C_m C_n m!\,n!$ times the coefficient of $w^m z^n$ in the development of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x, y; h_w, h_z)(x^2+y^2)^k\,dx\,dy. \tag{4.7}$$

We can evaluate $H(x, y; h_w, h_z)$ by means of (2.1). Integrating with respect to t, we obtain

$$H(x, y; h_w, h_z) = \\ 2^{-1/2} \exp[-4\pi x(w+z) + 4\pi i y(w-z) - 2\pi(x^2+y^2) - 4\pi wz].$$

It follows that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[2\pi u(x^2+y^2)]H(x, y; h_w, h_z)\,dx\,dy$ converges absolutely if $|u| < 1$, and that (4.6) equals $(2\pi)^{-k}k!$ times the coefficient of u^k in the power series expansion of that double integral. This double integral is of the type (4.2). Evaluation gives

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[2\pi u(x^2+y^2)]H(x, y; h_w, h_z)\,dx\,dy \\ = 2^{-1/2}(1-u)^{-1}\exp[4\pi wz(1+u)/(1-u)].$$

Hence the left-hand side of (4.6) equals $2^{-1/2}C_m C_n m!\,n!(2\pi)^{-k}k!$ times the coefficient of $w^m z^n u^k$ in the development of $(1-u)^{-1}\exp[4\pi wz(1+u)/(1-u)]$. This coefficient vanishes if $m \neq n$; if $m = n$, it equals $(m!)^{-1}(4\pi)^m$ times the coefficient of u^k in the expansion of $(1+u)^m(1-u)^{-m-1}$. The latter coefficient is equal to the p_{km} defined in our theorem, for

$$\sum_{m=0}^{\infty} v^m(1+u)^m(1-u)^{-m-1} = (1-u-v-uv)^{-1} = \sum_{m=0}^{\infty} v^m \sum_{k=0}^{\infty} p_{km}u^k.$$

Finally, since

$$2^{-1/2}C_m C_n m!\,n!(2\pi)^{-k}k!\,\delta_{mn}(m!)^{-1}(4\pi)^m p_{km} = \delta_{mn}k!(2\pi)^{-k}p_{km},$$

the theorem follows at once.

Theorem 4.4. *Let $f \in L_2$, $k = 0, 1, 2, \ldots$. Then we have*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x, y; f, f)(x^2+y^2)^k\,dx\,dy \geq k!(2\pi)^{-k}\int_{-\infty}^{\infty} |f(t)|^2\,dt, \tag{4.8}$$

where the left-hand side is to be interpreted as the limit of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x, y; f, f)\exp[-\pi\varepsilon(x^2+y^2)](x^2+y^2)^k\,dx\,dy, \tag{4.9}$$

as ε tends to zero from the right.

If f is an odd function, and $k \geq 1$, then the constant $k!(2\pi)^{-k}$ on the right-hand side of (4.8) can be replaced by $3k!(2\pi)^{-k}$.

Proof. We first show that (4.9) is positive-definite (if k and ε are fixed, $0<\varepsilon<2$). It follows from Theorem 4.1 and (2.7) that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x, y; \psi_m, \psi_n) \exp[-\pi\varepsilon(x^2+y^2)]\, dx\, dy$$
$$=\delta_{mn}(1-\tfrac{1}{2}\varepsilon)^m(1+\tfrac{1}{2}\varepsilon)^{-m-1} \qquad (\varepsilon>0).$$

We can show that the kth derivative of $(1-\frac{1}{2}\varepsilon)^m(1+\frac{1}{2}\varepsilon)^{-m-1}$ has the sign of $(-1)^k$. It is the kth derivative (with respect to ε) of the coefficient of u^m in the development of $[1-u+\frac{1}{2}\varepsilon(1+u)]^{-1}$, that is, $(-1)^k$ times the coefficient of u^m in the development of

$$k!\,2^{-k}(1+u)^k[1+\tfrac{1}{2}\varepsilon-(1-\tfrac{1}{2}\varepsilon)u]^{-k-1},$$

and that coefficient is positive if $0<\varepsilon<2$. Therefore, if $c_{mk}(\varepsilon)$ is defined by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x, y; \psi_m, \psi_n) \exp[-\pi\varepsilon(x^2+y^2)](x^2+y^2)^k\, dx\, dy$$
$$=\delta_{mn}k!(2\pi)^{-k}c_{mk}(\varepsilon),$$

then $c_{mk}(\varepsilon)>0$ if $0<\varepsilon<2$, $m=0, 2, 1, \ldots$. It easily follows that (4.9) is positive-definite.

Applying the fact that (4.9) is positive-definite to $f-\sum_{m=0}^{M}\gamma_k\psi_k$ instead of f [with $\gamma_k=(f, \psi_k)$], we obtain that (4.9) is at least

$$k!\,(2\pi)^{-k}\sum_{m=0}^{M} c_{mk}(\varepsilon)|\gamma_m|^2.$$

As $c_{mk}(\varepsilon)\to p_{mk}$ [see (4.6)] if $\varepsilon\to 0$, it follows that the left-hand side of (4.8) (with the interpretation given in the theorem) is at least

$$k!(2\pi)^{-k}\sum_{m=0}^{M} p_{mk}|\gamma_m|^2.$$

Making $M\to\infty$ we obtain that it is at least

$$k!(2\pi)^{-k}\sum_{m=0}^{\infty} p_{mk}|\gamma_m|^2.$$

(Actually it is not difficult to show that the left-hand side of (4.8) is exactly equal to that sum.)

By the definition of p_{mk} (Theorem 4.3), we have $p_{00}=p_{10}=p_{20}=\cdots=1$ and $p_{m+1,k+1}=p_{m,k}+p_{m,k+1}+p_{m+1,k}$. It follows that p_{mk} increases with

k as well as with m. In particular $p_{mk} \geq 1$ for all $m, k \geq 0$, and $p_{mk} \geq 3$ if $m \geq 1$, $k \geq 1$. Hence

$$\sum_{m=0}^{\infty} p_{mk} |\gamma_m|^2 \geq \sum_{m=0}^{\infty} |\gamma_m|^2 = \int_{-\infty}^{\infty} |f(t)|^2 \, dt.$$

If f is odd we have $\gamma_0 = 0$, whence, for $k \geq 1$,

$$\sum_{m=0}^{\infty} p_{mk} |\gamma_m|^2 \geq 3 \sum_{m=1}^{\infty} |\gamma_m|^2 = 3 \int_{-\infty}^{\infty} |f(t)|^2 \, dt.$$

Properly speaking, inequalities of the type (4.8) do not prove the impossibility of a very strong concentration of the score upon a small neighborhood of the origin. For example, if 99% of the energy lies very close to the origin, and the remaining 1% lies in a part of the xy plane where $(x^2 + y^2)^k$ is very large, then that 1% can make the left-hand side of (4.8) large.

This objection does not hold against the inequality

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x, y; f, f) \left[1 - \exp\left(-\frac{\pi x^2}{\alpha} - \frac{\pi y^2}{\beta} \right) \right] dx \, dy$$
$$\geq [1 + 2(\alpha\beta)^{1/2}]^{-1} \int_{-\infty}^{\infty} |f(t)|^2 \, dt$$

(which holds for all $\alpha > 0$, $\beta > 0$). Using (2.7) and (4.4), we can derive this inequality from the following theorem.

Theorem 4.5. *If $\alpha > 0$, $\beta > 0$, we have for all real values of x and y*

$$H_{\alpha\beta}(x, y; f, f) \leq [\tfrac{1}{2} + (\alpha\beta)^{1/2}]^{-1} \int_{-\infty}^{\infty} |f(t)|^2 \, dt. \tag{4.10}$$

Proof. As in the proof of Theorem 4.2, it suffices to specialize:

$$x = y = 0, \qquad \alpha = \beta > 0.$$

We have, if $\alpha > 0$, $m = 0, 1, 2, \ldots$,

$$|(\alpha + \tfrac{1}{2})^{-1}[(2\alpha - 1)/(2\alpha + 1)]^m| \leq (\alpha + \tfrac{1}{2})^{-1},$$

whence, by Theorem 4.1,

$$H_{\alpha\alpha}(0, 0; f, f) \leq \sum_{m=0}^{\infty} (\alpha + \tfrac{1}{2})^{-1} |\gamma_m|^2.$$

The theorem now follows from $\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \sum_{m=0}^{\infty} |\gamma_m|^2$.

We briefly mention a second method for proving (4.10). If we put

$$K(s, t) = \alpha^{-1/2} \exp\left[-\frac{\pi(s+t)^2}{4\alpha} - \pi\beta(s-t)^2 \right],$$

then (4.10) can be considered, according to (2.8), as a statement concerning the largest positive eigenvalue of the symmetric integral equation

$$\int_{-\infty}^{\infty} K(s, t) f(t)\, dt = \lambda f(s).$$

It is not hard to show that

$$f_0(t) = \exp[-\pi t^2 (\beta/\alpha)^{1/2}]$$

is an eigenfunction, with eigenvalue $\lambda_0 = [\frac{1}{2} + (\alpha\beta)^{1/2}]^{-1}$. Taking into consideration that both $f_0(t) > 0$ and $K(s, t) \geq 0$ for all s and t, we know (in direct analogy to a theorem by Perron on symmetric matrices) that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(s, t) f(s) \overline{f(t)}\, ds\, dt \leq \lambda_0 \int_{-\infty}^{\infty} |f(t)|^2\, dt \tag{4.11}$$

for every $f \in L_2$. This is equivalent to (4.10).

A proof of (4.11) can be given in a few lines. We have

$$|f(s)\overline{f(t)}| \leq \frac{1}{2} \left\{ \frac{|f(s)|^2}{[f_0(s)]^2} + \frac{|f(t)|^2}{[f_0(t)]^2} \right\} f_0(s) f_0(t). \tag{4.12}$$

Replacing in (4.11) the product $f(s)\overline{f(t)}$ by the right-hand side of (4.12), we obtain the right-hand side of (4.11).

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