

Lab 8: Data Augmentation

Olivier Binette and Rebecca C. Steorts

Friday October 16, 2020

Agenda

- ▶ Reminders
- ▶ Problem statement
- ▶ Go through the lab's tasks
- ▶ Office hours

Reminders

- ▶ Feel free to send emails for questions or anything
- ▶ We're there for you during office ours
 - ▶ We can also find other times
- ▶ Do let us know if you run into issues, if you have concerns or comments about the class or anything.
 - ▶ I'm (Olivier) not a grader and I can bring things up anonymously with Beka or with other TAs.
- ▶ The goal of the class is for you to learn and to be prepared for what comes next.

Problem statement

Problem statement

Data points Y_1, Y_2, \dots, Y_n coming from a **mixture model**:

$$Y_i \sim \sum_{j=1}^n w_j N(\mu_j, \varepsilon^2).$$

What does this mean?

For every data point:

- ▶ let Z_i be a random variable such that $\mathbb{P}(Z_i = j) = w_j$ for $j = 1, 2, 3$,
- ▶ let $Y_i \mid Z_i \sim N(\mu_{Z_i}, \varepsilon^2)$.

Problem statement

Data points Y_1, Y_2, \dots, Y_n coming from a **mixture model**:

$$Y_i \sim \sum_{j=1}^n w_j N(\mu_j, \varepsilon^2).$$

What does this mean?

For every data point:

- ▶ let Z_i be a random variable such that $\mathbb{P}(Z_i = j) = w_j$ for $j = 1, 2, 3$,
- ▶ let $Y_i \mid Z_i \sim N(\mu_{Z_i}, \varepsilon^2)$.

Problem statement

Data points Y_1, Y_2, \dots, Y_n coming from a **mixture model**:

$$Y_i \sim \sum_{j=1}^n w_j N(\mu_j, \varepsilon^2).$$

What does this mean?

For every data point:

- ▶ let Z_i be a random variable such that $\mathbb{P}(Z_i = j) = w_j$ for $j = 1, 2, 3$,
- ▶ let $Y_i \mid Z_i \sim N(\mu_{Z_i}, \varepsilon^2)$.

Problem statement

Data points Y_1, Y_2, \dots, Y_n coming from a **mixture model**:

$$Y_i \sim \sum_{j=1}^n w_j N(\mu_j, \varepsilon^2).$$

What does this mean?

For every data point:

- ▶ let Z_i be a random variable such that $\mathbb{P}(Z_i = j) = w_j$ for $j = 1, 2, 3$,
- ▶ let $Y_i \mid Z_i \sim N(\mu_{Z_i}, \varepsilon^2)$.

Problem statement

Data points Y_1, Y_2, \dots, Y_n coming from a **mixture model**:

$$Y_i \sim \sum_{j=1}^n w_j N(\mu_j, \varepsilon^2).$$

What does this mean?

For every data point:

- ▶ let Z_i be a random variable such that $\mathbb{P}(Z_i = j) = w_j$ for $j = 1, 2, 3$,
- ▶ let $Y_i \mid Z_i \sim N(\mu_{Z_i}, \varepsilon^2)$.

Problem statement

Data points Y_1, Y_2, \dots, Y_n coming from a **mixture model**:

$$Y_i \sim \sum_{j=1}^n w_j N(\mu_j, \varepsilon^2).$$

What does this mean?

For every data point:

- ▶ let Z_i be a random variable such that $\mathbb{P}(Z_i = j) = w_j$ for $j = 1, 2, 3$,
- ▶ let $Y_i \mid Z_i \sim N(\mu_{Z_i}, \varepsilon^2)$.

Problem statement

In other words:

$$p(Y_i \mid w_{1:3}, \mu_{1:3}, \varepsilon^2) = \sum_{j=1}^3 w_j N(Y_i; \mu_j, \varepsilon^2)$$

and

$$p(Y_i \mid Z_i, w_{1:3}, \mu_{1:3}, \varepsilon^2) = N(Y_i; \mu_{Z_i}, \varepsilon^2)$$

Problem statement

Let's see what this mixture model could look like in an example.

Let $\mu_1 = -5$, $\mu_2 = 0$ and $\mu_3 = 5$, and let $\varepsilon = 1$. Let $w_j = 1/3$ for $j = 1, 2, 3$.

Now let's generate data from the mixture model:

```
n = 100
mu = c(-5, 0, 5)

Z = sample(1:3, size=n, replace=TRUE)
Y = rnorm(n, mean=mu[Z], sd=1)

hist(Y, breaks=20)
```

Problem statement

Let's see what this mixture model could look like in an example.

Let $\mu_1 = -5$, $\mu_2 = 0$ and $\mu_3 = 5$, and let $\varepsilon = 1$. Let $w_j = 1/3$ for $j = 1, 2, 3$.

Now let's generate data from the mixture model:

```
n = 100
mu = c(-5, 0, 5)

Z = sample(1:3, size=n, replace=TRUE)
Y = rnorm(n, mean=mu[Z], sd=1)

hist(Y, breaks=20)
```

Problem statement

Let's see what this mixture model could look like in an example.

Let $\mu_1 = -5$, $\mu_2 = 0$ and $\mu_3 = 5$, and let $\varepsilon = 1$. Let $w_j = 1/3$ for $j = 1, 2, 3$.

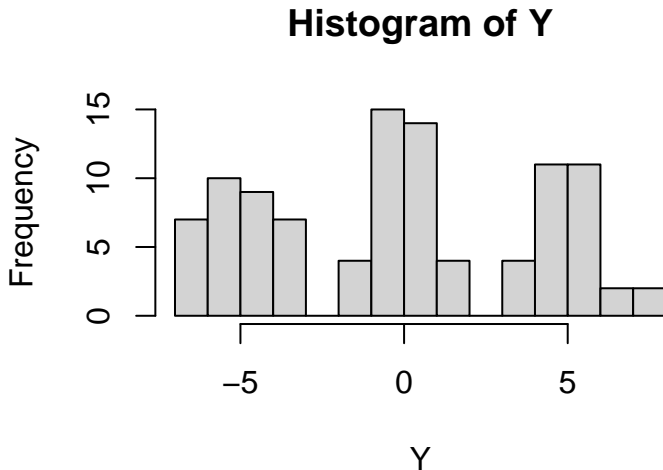
Now let's generate data from the mixture model:

```
n = 100
mu = c(-5, 0, 5)

Z = sample(1:3, size=n, replace=TRUE)
Y = rnorm(n, mean=mu[Z], sd=1)

hist(Y, breaks=20)
```

Problem statement



Problem statement

We have the model

$$Y_i \sim \sum_{j=1}^n w_j N(\mu_j, \varepsilon^2),$$

now we need priors on the unknown parameters μ_j , w_j and ε .

Priors

For the means:

$$\begin{aligned}\mu_j \mid \mu_0, \sigma_0^2 &\sim N(\mu_0, \sigma_0^2) \\ \mu_0 &\sim N(0, 3) \\ \sigma_0^2 &\sim IG(2, 2)\end{aligned}$$

and recall that $\sigma_0^2 \sim IG(2, 2)$ means that $\phi_0 = 1/\sigma_0^2 \sim \text{Gamma}(2, 2)$.

Problem statement

We have the model

$$Y_i \sim \sum_{j=1}^n w_j N(\mu_j, \varepsilon^2),$$

now we need priors on the unknown parameters μ_j , w_j and ε .

Priors

For the means:

$$\mu_j \mid \mu_0, \sigma_0^2 \sim N(\mu_0, \sigma_0^2)$$

$$\mu_0 \sim N(0, 3)$$

$$\sigma_0^2 \sim IG(2, 2)$$

and recall that $\sigma_0^2 \sim IG(2, 2)$ means that $\phi_0 = 1/\sigma_0^2 \sim \text{Gamma}(2, 2)$.

Problem statement

We have the model

$$Y_i \sim \sum_{j=1}^n w_j N(\mu_j, \varepsilon^2),$$

now we need priors on the unknown parameters μ_j , w_j and ε .

Priors

For the means:

$$\mu_j \mid \mu_0, \sigma_0 \sim N(\mu_0, \sigma_0^2)$$

$$\mu_0 \sim N(0, 3)$$

$$\sigma_0^2 \sim IG(2, 2)$$

and recall that $\sigma_0^2 \sim IG(2, 2)$ means that $\phi_0 = 1/\sigma_0^2 \sim \text{Gamma}(2, 2)$.

Problem statement

We have the model

$$Y_i \sim \sum_{j=1}^n w_j N(\mu_j, \varepsilon^2),$$

now we need priors on the unknown parameters μ_j , w_j and ε .

Priors

For the means:

$$\begin{aligned}\mu_j \mid \mu_0, \sigma_0 &\sim N(\mu_0, \sigma_0^2) \\ \mu_0 &\sim N(0, 3) \\ \sigma_0^2 &\sim IG(2, 2)\end{aligned}$$

and recall that $\sigma_0^2 \sim IG(2, 2)$ means that $\phi_0 = 1/\sigma_0^2 \sim \text{Gamma}(2, 2)$.

Problem statement

We have the model

$$Y_i \sim \sum_{j=1}^n w_j N(\mu_j, \varepsilon^2),$$

now we need priors on the unknown parameters μ_j , w_j and ε .

Priors

For the means:

$$\begin{aligned}\mu_j \mid \mu_0, \sigma_0 &\sim N(\mu_0, \sigma_0^2) \\ \mu_0 &\sim N(0, 3) \\ \sigma_0^2 &\sim IG(2, 2)\end{aligned}$$

and recall that $\sigma_0^2 \sim IG(2, 2)$ means that $\phi_0 = 1/\sigma_0^2 \sim \text{Gamma}(2, 2)$.

Problem statement

We have the model

$$Y_i \sim \sum_{j=1}^n w_j N(\mu_j, \varepsilon^2),$$

now we need priors on the unknown parameters μ_j , w_j and ε .

Priors

For the means:

$$\begin{aligned}\mu_j \mid \mu_0, \sigma_0 &\sim N(\mu_0, \sigma_0^2) \\ \mu_0 &\sim N(0, 3) \\ \sigma_0^2 &\sim IG(2, 2)\end{aligned}$$

and recall that $\sigma_0^2 \sim IG(2, 2)$ means that $\phi_0 = 1/\sigma_0^2 \sim \text{Gamma}(2, 2)$.

Problem statement

Priors

For the mixture weights:

$$(w_1, w_2, w_3) \sim \text{Dirichlet}(\mathbf{1})$$

which means that $p(w_1, w_2, w_3) \propto 1$.

Recall that, in general,

$$(w_1, w_2, w_3) \sim \text{Dirichlet}(\alpha_1, \alpha_2, \alpha_3) \\ \Rightarrow p(w_1, w_2, w_3) \propto w_1^{\alpha_1-1} w_2^{\alpha_2-1} w_3^{\alpha_3-1}.$$

Problem statement

Priors

For the mixture weights:

$$(w_1, w_2, w_3) \sim \text{Dirichlet}(\mathbf{1})$$

which means that $p(w_1, w_2, w_3) \propto 1$.

Recall that, in general,

$$\begin{aligned}(w_1, w_2, w_3) &\sim \text{Dirichlet}(\alpha_1, \alpha_2, \alpha_3) \\ \Rightarrow p(w_1, w_2, w_3) &\propto w_1^{\alpha_1-1} w_2^{\alpha_2-1} w_3^{\alpha_3-1}.\end{aligned}$$

Problem statement

Priors

For the mixture weights:

$$(w_1, w_2, w_3) \sim \text{Dirichlet}(\mathbf{1})$$

which means that $p(w_1, w_2, w_3) \propto 1$.

Recall that, in general,

$$\begin{aligned}(w_1, w_2, w_3) &\sim \text{Dirichlet}(\alpha_1, \alpha_2, \alpha_3) \\ \Rightarrow p(w_1, w_2, w_3) &\propto w_1^{\alpha_1-1} w_2^{\alpha_2-1} w_3^{\alpha_3-1}.\end{aligned}$$

Problem statement

Priors

For the variance:

$$\varepsilon^2 \sim IG(2, 2)$$

This means that

$$\tau = 1/\varepsilon^2 \sim \text{Gamma}(2, 2)$$

Problem statement

Priors

For the variance:

$$\varepsilon^2 \sim IG(2, 2)$$

This means that

$$\tau = 1/\varepsilon^2 \sim \text{Gamma}(2, 2)$$

Problem statement

In summary,

$$p(Y_i | \mu_1, \mu_2, \mu_3, w_1, w_2, w_3, \tau) = \sum_{j=1}^3 w_j N(\mu_j, \tau^{-1})$$

$$\mu_j | \mu_0, \sigma_0^2 \sim N(\mu_0, \phi_0^{-1})$$

$$\mu_0 \sim N(0, 3)$$

$$\phi_0 \sim \text{Gamma}(2, 2)$$

$$(w_1, w_2, w_3) \sim \text{Dirichlet}(\mathbf{1})$$

$$\tau \sim \text{Gamma}(2, 2),$$

for $i = 1, \dots, n$.

Task 1

Derive the joint posterior $p(w_1, w_2, w_3, \mu_1, \mu_2, \mu_3, \varepsilon^2, \mu_0, \sigma_0^2 | Y_{1:N})$ up to a normalizing constant.

Let's do the derivations using $\tau = 1/\varepsilon^2$ and $\phi_0 = 1/\sigma_0^2$.

The posterior distribution is always proportional to the full joint distribution:

$$\begin{aligned} & p(Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(w_{1:3}) p(\tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ &= \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau). \end{aligned}$$

Task 1

Derive the joint posterior $p(w_1, w_2, w_3, \mu_1, \mu_2, \mu_3, \varepsilon^2, \mu_0, \sigma_0^2 | Y_{1:N})$ up to a normalizing constant.

Let's do the derivations using $\tau = 1/\varepsilon^2$ and $\phi_0 = 1/\sigma_0^2$.

The posterior distribution is always proportional to the full joint distribution:

$$\begin{aligned} & p(Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(w_{1:3}) p(\tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ &= \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau). \end{aligned}$$

Task 1

Derive the joint posterior $p(w_1, w_2, w_3, \mu_1, \mu_2, \mu_3, \varepsilon^2, \mu_0, \sigma_0^2 | Y_{1:N})$ up to a normalizing constant.

Let's do the derivations using $\tau = 1/\varepsilon^2$ and $\phi_0 = 1/\sigma_0^2$.

The posterior distribution is always proportional to the full joint distribution:

$$\begin{aligned} & p(Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(w_{1:3}) p(\tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ &= \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau). \end{aligned}$$

Task 1

Derive the joint posterior $p(w_1, w_2, w_3, \mu_1, \mu_2, \mu_3, \varepsilon^2, \mu_0, \sigma_0^2 | Y_{1:N})$ up to a normalizing constant.

Let's do the derivations using $\tau = 1/\varepsilon^2$ and $\phi_0 = 1/\sigma_0^2$.

The posterior distribution is always proportional to the full joint distribution:

$$\begin{aligned} & p(Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(w_{1:3}) p(\tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ &= \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau). \end{aligned}$$

Task 1

Derive the joint posterior $p(w_1, w_2, w_3, \mu_1, \mu_2, \mu_3, \varepsilon^2, \mu_0, \sigma_0^2 | Y_{1:N})$ up to a normalizing constant.

Let's do the derivations using $\tau = 1/\varepsilon^2$ and $\phi_0 = 1/\sigma_0^2$.

The posterior distribution is always proportional to the full joint distribution:

$$\begin{aligned} & p(Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(w_{1:3}) p(\tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ &= \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau). \end{aligned}$$

Task 1

Derive the joint posterior $p(w_1, w_2, w_3, \mu_1, \mu_2, \mu_3, \varepsilon^2, \mu_0, \sigma_0^2 | Y_{1:N})$ up to a normalizing constant.

Let's do the derivations using $\tau = 1/\varepsilon^2$ and $\phi_0 = 1/\sigma_0^2$.

The posterior distribution is always proportional to the full joint distribution:

$$\begin{aligned} & p(Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(w_{1:3}) p(\tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ &= \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau). \end{aligned}$$

Task 1

Derive the joint posterior $p(w_1, w_2, w_3, \mu_1, \mu_2, \mu_3, \varepsilon^2, \mu_0, \sigma_0^2 | Y_{1:N})$ up to a normalizing constant.

Let's do the derivations using $\tau = 1/\varepsilon^2$ and $\phi_0 = 1/\sigma_0^2$.

The posterior distribution is always proportional to the full joint distribution:

$$\begin{aligned} & p(Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(w_{1:3}) p(\tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ &= \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau). \end{aligned}$$

Task 1

Derive the joint posterior $p(w_1, w_2, w_3, \mu_1, \mu_2, \mu_3, \varepsilon^2, \mu_0, \sigma_0^2 | Y_{1:N})$ up to a normalizing constant.

Let's do the derivations using $\tau = 1/\varepsilon^2$ and $\phi_0 = 1/\sigma_0^2$.

The posterior distribution is always proportional to the full joint distribution:

$$\begin{aligned} & p(Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(w_{1:3}) p(\tau) \\ &= p(Y_{1:n} \mid \mu_{1:3}, w_{1:3}, \tau) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ &= \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau). \end{aligned}$$

Task 1

The full joint:

$$\left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau).$$

And

$$p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) = \sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau),$$

$$p(\mu_j \mid \mu_0, \phi_0) = N(\mu_j; \mu_0, \phi_0^{-1}),$$

$$p(\mu_0) = N(\mu_0; 0, 3),$$

$$p(\phi_0) = \text{Gamma}(\phi_0; 2, 2),$$

$$p(\tau) = \text{Gamma}(\tau; 2, 2).$$

Task 1

The full joint:

$$\left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau).$$

And

$$p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) = \sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau),$$

$$p(\mu_j \mid \mu_0, \phi_0) = N(\mu_j; \mu_0, \phi_0^{-1}),$$

$$p(\mu_0) = N(\mu_0; 0, 3),$$

$$p(\phi_0) = \text{Gamma}(\phi_0; 2, 2),$$

$$p(\tau) = \text{Gamma}(\tau; 2, 2).$$

Task 1

The full joint:

$$\left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau).$$

And

$$p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) = \sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau),$$

$$p(\mu_j \mid \mu_0, \phi_0) = N(\mu_j; \mu_0, \phi_0^{-1}),$$

$$p(\mu_0) = N(\mu_0; 0, 3),$$

$$p(\phi_0) = \text{Gamma}(\phi_0; 2, 2),$$

$$p(\tau) = \text{Gamma}(\tau; 2, 2).$$

Task 1

The full joint:

$$\left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau).$$

And

$$p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) = \sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau),$$

$$p(\mu_j \mid \mu_0, \phi_0) = N(\mu_j; \mu_0, \phi_0^{-1}),$$

$$p(\mu_0) = N(\mu_0; 0, 3),$$

$$p(\phi_0) = \text{Gamma}(\phi_0; 2, 2),$$

$$p(\tau) = \text{Gamma}(\tau; 2, 2).$$

Task 1

The full joint:

$$\left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau).$$

And

$$p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) = \sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau),$$

$$p(\mu_j \mid \mu_0, \phi_0) = N(\mu_j; \mu_0, \phi_0^{-1}),$$

$$p(\mu_0) = N(\mu_0; 0, 3),$$

$$p(\phi_0) = \text{Gamma}(\phi_0; 2, 2),$$

$$p(\tau) = \text{Gamma}(\tau; 2, 2).$$

Task 1

The full joint:

$$\left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau).$$

And

$$p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) = \sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau),$$

$$p(\mu_j \mid \mu_0, \phi_0) = N(\mu_j; \mu_0, \phi_0^{-1}),$$

$$p(\mu_0) = N(\mu_0; 0, 3),$$

$$p(\phi_0) = \text{Gamma}(\phi_0; 2, 2),$$

$$p(\tau) = \text{Gamma}(\tau; 2, 2).$$

Task 2

Derive the full conditionals for all the parameters up to a normalizing constant.

$$\begin{aligned} & p(w_{1:3} \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \\ & \propto \prod_{i=1}^n \left(\sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau) \right) \end{aligned}$$

Task 2

Derive the full conditionals for all the parameters up to a normalizing constant.

$$\begin{aligned} & p(w_{1:3} \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \\ & \propto \prod_{i=1}^n \left(\sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau) \right) \end{aligned}$$

Task 2

Derive the full conditionals for all the parameters up to a normalizing constant.

$$\begin{aligned} & p(w_{1:3} \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \\ & \propto \prod_{i=1}^n \left(\sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau) \right) \end{aligned}$$

Task 2

Derive the full conditionals for all the parameters up to a normalizing constant.

$$\begin{aligned} & p(w_{1:3} \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \\ & \propto \prod_{i=1}^n \left(\sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau) \right) \end{aligned}$$

Task 2

$$\begin{aligned} & p(\mu_j \mid Y_{1:n}, \mu_{(-j)}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) p(\mu_j \mid \mu_0, \phi_0) \\ & \propto \left(\prod_{i=1}^n \left(\sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau) \right) \right) p(\mu_j \mid \mu_0, \phi_0) \end{aligned}$$

Task 2

$$\begin{aligned} & p(\mu_j \mid Y_{1:n}, \mu_{(-j)}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) p(\mu_j \mid \mu_0, \phi_0) \\ & \propto \left(\prod_{i=1}^n \left(\sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau) \right) \right) p(\mu_j \mid \mu_0, \phi_0) \end{aligned}$$

Task 2

$$\begin{aligned} & p(\mu_j \mid Y_{1:n}, \mu_{(-j)}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) p(\mu_j \mid \mu_0, \phi_0) \\ & \propto \left(\prod_{i=1}^n \left(\sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau) \right) \right) p(\mu_j \mid \mu_0, \phi_0) \end{aligned}$$

Task 2

$$\begin{aligned} & p(\mu_j \mid Y_{1:n}, \mu_{(-j)}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) p(\mu_j \mid \mu_0, \phi_0) \\ & \propto \left(\prod_{i=1}^n \left(\sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau) \right) \right) p(\mu_j \mid \mu_0, \phi_0) \end{aligned}$$

Task 2

$$\begin{aligned} & p(\tau \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) p(\tau) \\ & \propto \left(\prod_{i=1}^n \left(\sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau) \right) \right) \tau^{2-1} \exp\{-2\tau\}. \end{aligned}$$

Task 2

$$\begin{aligned} & p(\tau \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) p(\tau) \\ & \propto \left(\prod_{i=1}^n \left(\sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau) \right) \right) \tau^{2-1} \exp\{-2\tau\}. \end{aligned}$$

Task 2

$$\begin{aligned} & p(\tau \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) p(\tau) \\ & \propto \left(\prod_{i=1}^n \left(\sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau) \right) \right) \tau^{2-1} \exp\{-2\tau\}. \end{aligned}$$

Task 2

$$\begin{aligned} & p(\tau \mid Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) p(\tau) \\ & \propto \left(\prod_{i=1}^n \left(\sum_{j=1}^3 w_j N(Y_i; \mu_j, \tau) \right) \right) \tau^{2-1} \exp\{-2\tau\}. \end{aligned}$$

Task 2

$$\begin{aligned} & p(\mu_0 \mid Y_{1:n}, \mu_{1:3}, \phi_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) \\ & \propto \exp \left\{ -\frac{\phi_0}{2} \sum_{j=1}^3 (\mu_j - \mu_0)^2 \right\} \exp \{ \mu_0^2 / 6 \} \\ & \propto \exp \left\{ -\frac{1}{2} \left[(3\phi_0 + \frac{1}{3}) \mu_0^2 - 2\mu_0 \phi_0 \sum_{j=1}^3 \mu_j \right] \right\} \\ & \Rightarrow \mu_0 \mid - \sim N \left((3\phi_0 + \frac{1}{3})^{-1} \phi_0 \sum_{j=1}^3 \mu_j, (3\phi_0 + \frac{1}{3})^{-1} \right). \end{aligned}$$

Task 2

$$\begin{aligned} & p(\mu_0 \mid Y_{1:n}, \mu_{1:3}, \phi_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) \\ & \propto \exp \left\{ -\frac{\phi_0}{2} \sum_{j=1}^3 (\mu_j - \mu_0)^2 \right\} \exp \{ \mu_0^2 / 6 \} \\ & \propto \exp \left\{ -\frac{1}{2} \left[(3\phi_0 + \frac{1}{3}) \mu_0^2 - 2\mu_0 \phi_0 \sum_{j=1}^3 \mu_j \right] \right\} \\ & \Rightarrow \mu_0 \mid - \sim N \left((3\phi_0 + \frac{1}{3})^{-1} \phi_0 \sum_{j=1}^3 \mu_j, (3\phi_0 + \frac{1}{3})^{-1} \right). \end{aligned}$$

Task 2

$$\begin{aligned} & p(\mu_0 \mid Y_{1:n}, \mu_{1:3}, \phi_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) \\ & \propto \exp \left\{ -\frac{\phi_0}{2} \sum_{j=1}^3 (\mu_j - \mu_0)^2 \right\} \exp \{ \mu_0^2 / 6 \} \\ & \propto \exp \left\{ -\frac{1}{2} \left[(3\phi_0 + \frac{1}{3}) \mu_0^2 - 2\mu_0 \phi_0 \sum_{j=1}^3 \mu_j \right] \right\} \\ & \Rightarrow \mu_0 \mid - \sim N \left((3\phi_0 + \frac{1}{3})^{-1} \phi_0 \sum_{j=1}^3 \mu_j, (3\phi_0 + \frac{1}{3})^{-1} \right). \end{aligned}$$

Task 2

$$\begin{aligned} & p(\mu_0 \mid Y_{1:n}, \mu_{1:3}, \phi_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) \\ & \propto \exp \left\{ -\frac{\phi_0}{2} \sum_{j=1}^3 (\mu_j - \mu_0)^2 \right\} \exp \{ \mu_0^2 / 6 \} \\ & \propto \exp \left\{ -\frac{1}{2} \left[(3\phi_0 + \frac{1}{3}) \mu_0^2 - 2\mu_0 \phi_0 \sum_{j=1}^3 \mu_j \right] \right\} \\ & \Rightarrow \mu_0 \mid - \sim N \left((3\phi_0 + \frac{1}{3})^{-1} \phi_0 \sum_{j=1}^3 \mu_j, (3\phi_0 + \frac{1}{3})^{-1} \right). \end{aligned}$$

Task 2

$$\begin{aligned} & p(\mu_0 \mid Y_{1:n}, \mu_{1:3}, \phi_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) \\ & \propto \exp \left\{ -\frac{\phi_0}{2} \sum_{j=1}^3 (\mu_j - \mu_0)^2 \right\} \exp \{ \mu_0^2 / 6 \} \\ & \propto \exp \left\{ -\frac{1}{2} \left[(3\phi_0 + \frac{1}{3}) \mu_0^2 - 2\mu_0 \phi_0 \sum_{j=1}^3 \mu_j \right] \right\} \\ & \Rightarrow \mu_0 \mid - \sim N \left((3\phi_0 + \frac{1}{3})^{-1} \phi_0 \sum_{j=1}^3 \mu_j, (3\phi_0 + \frac{1}{3})^{-1} \right). \end{aligned}$$

Task 2

$$\begin{aligned} & p(\mu_0 \mid Y_{1:n}, \mu_{1:3}, \phi_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) \\ & \propto \exp \left\{ -\frac{\phi_0}{2} \sum_{j=1}^3 (\mu_j - \mu_0)^2 \right\} \exp \{ \mu_0^2 / 6 \} \\ & \propto \exp \left\{ -\frac{1}{2} \left[(3\phi_0 + \frac{1}{3}) \mu_0^2 - 2\mu_0 \phi_0 \sum_{j=1}^3 \mu_j \right] \right\} \\ & \Rightarrow \mu_0 \mid - \sim N \left((3\phi_0 + \frac{1}{3})^{-1} \phi_0 \sum_{j=1}^3 \mu_j, (3\phi_0 + \frac{1}{3})^{-1} \right). \end{aligned}$$

Task 2

$$\begin{aligned} & p(\phi_0 \mid Y_{1:n}, \mu_{1:3}, \mu_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\phi_0) \\ & \propto \left(\prod_{j=1}^3 \sqrt{\frac{\phi_0}{2\pi}} \exp \left\{ -\frac{\phi_0}{2} (\mu_0 - \mu_j)^2 \right\} \right) \phi_0^{2-1} \exp \{-2\phi_0\} \\ & \propto \phi_0^{7/2-1} \exp \left\{ -\phi_0 \left(2 + \frac{1}{2} \sum_{j=1}^3 (\mu_0 - \mu_j)^2 \right) \right\} \\ & \Rightarrow \phi_0 \mid - \sim \text{Gamma} \left(7/2, 2 + \frac{1}{2} \sum_{j=1}^3 (\mu_0 - \mu_j)^2 \right) \end{aligned}$$

Task 2

$$\begin{aligned} & p(\phi_0 \mid Y_{1:n}, \mu_{1:3}, \mu_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\phi_0) \\ & \propto \left(\prod_{j=1}^3 \sqrt{\frac{\phi_0}{2\pi}} \exp \left\{ -\frac{\phi_0}{2} (\mu_0 - \mu_j)^2 \right\} \right) \phi_0^{2-1} \exp \{-2\phi_0\} \\ & \propto \phi_0^{7/2-1} \exp \left\{ -\phi_0 \left(2 + \frac{1}{2} \sum_{j=1}^3 (\mu_0 - \mu_j)^2 \right) \right\} \\ & \Rightarrow \phi_0 \mid - \sim \text{Gamma} \left(7/2, 2 + \frac{1}{2} \sum_{j=1}^3 (\mu_0 - \mu_j)^2 \right) \end{aligned}$$

Task 2

$$\begin{aligned} & p(\phi_0 \mid Y_{1:n}, \mu_{1:3}, \mu_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\phi_0) \\ & \propto \left(\prod_{j=1}^3 \sqrt{\frac{\phi_0}{2\pi}} \exp \left\{ -\frac{\phi_0}{2} (\mu_0 - \mu_j)^2 \right\} \right) \phi_0^{2-1} \exp \{-2\phi_0\} \\ & \propto \phi_0^{7/2-1} \exp \left\{ -\phi_0 \left(2 + \frac{1}{2} \sum_{j=1}^3 (\mu_0 - \mu_j)^2 \right) \right\} \\ & \Rightarrow \phi_0 \mid - \sim \text{Gamma} \left(7/2, 2 + \frac{1}{2} \sum_{j=1}^3 (\mu_0 - \mu_j)^2 \right) \end{aligned}$$

Task 2

$$\begin{aligned} & p(\phi_0 \mid Y_{1:n}, \mu_{1:3}, \mu_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\phi_0) \\ & \propto \left(\prod_{j=1}^3 \sqrt{\frac{\phi_0}{2\pi}} \exp \left\{ -\frac{\phi_0}{2} (\mu_0 - \mu_j)^2 \right\} \right) \phi_0^{2-1} \exp \{ -2\phi_0 \} \\ & \propto \phi_0^{7/2-1} \exp \left\{ -\phi_0 \left(2 + \frac{1}{2} \sum_{j=1}^3 (\mu_0 - \mu_j)^2 \right) \right\} \\ & \Rightarrow \phi_0 \mid - \sim \text{Gamma} \left(7/2, 2 + \frac{1}{2} \sum_{j=1}^3 (\mu_0 - \mu_j)^2 \right) \end{aligned}$$

Task 2

$$\begin{aligned} & p(\phi_0 \mid Y_{1:n}, \mu_{1:3}, \mu_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\phi_0) \\ & \propto \left(\prod_{j=1}^3 \sqrt{\frac{\phi_0}{2\pi}} \exp \left\{ -\frac{\phi_0}{2} (\mu_0 - \mu_j)^2 \right\} \right) \phi_0^{2-1} \exp \{-2\phi_0\} \\ & \propto \phi_0^{7/2-1} \exp \left\{ -\phi_0 \left(2 + \frac{1}{2} \sum_{j=1}^3 (\mu_0 - \mu_j)^2 \right) \right\} \\ & \Rightarrow \phi_0 \mid - \sim \text{Gamma} \left(7/2, 2 + \frac{1}{2} \sum_{j=1}^3 (\mu_0 - \mu_j)^2 \right) \end{aligned}$$

Task 2

$$\begin{aligned} & p(\phi_0 \mid Y_{1:n}, \mu_{1:3}, \mu_0, w_{1:3}, \tau) \\ & \propto \left(\prod_{i=1}^n p(Y_i \mid \mu_{1:3}, w_{1:3}, \tau) \right) \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \left(\prod_{j=1}^3 p(\mu_j \mid \mu_0, \phi_0) \right) p(\phi_0) \\ & \propto \left(\prod_{j=1}^3 \sqrt{\frac{\phi_0}{2\pi}} \exp \left\{ -\frac{\phi_0}{2} (\mu_0 - \mu_j)^2 \right\} \right) \phi_0^{2-1} \exp \{-2\phi_0\} \\ & \propto \phi_0^{7/2-1} \exp \left\{ -\phi_0 \left(2 + \frac{1}{2} \sum_{j=1}^3 (\mu_0 - \mu_j)^2 \right) \right\} \\ & \Rightarrow \phi_0 \mid - \sim \text{Gamma} \left(7/2, 2 + \frac{1}{2} \sum_{j=1}^3 (\mu_0 - \mu_j)^2 \right) \end{aligned}$$

Task 3

Where necessary, (re)-derive the full conditionals under the data augmentation scheme.

Data augmentation scheme:

- ▶ Same priors as before, but we decompose the likelihood:

$$Z_i \mid w_1, w_2, w_3 \sim \text{Cat}(3, \mathbf{w})$$

$$Y_i \mid Z_i, \mu_{1:3}, \tau \sim N(\mu_{Z_i}, 1/\tau)$$

- ▶ The marginal distribution of Y_i is **unchanged**.
- ▶ Using the variables Z_i helps derive full conditional distributions.

In other words

$$p(Y_i \mid Z_i, \mu_1, \mu_2, \mu_3, \varepsilon^2) = N(Y_i; \mu_{Z_i}, \varepsilon^2)$$

$$P(Z_i = j \mid w_{1:3}) = w_j.$$

Task 3

Where necessary, (re)-derive the full conditionals under the data augmentation scheme.

Data augmentation scheme:

- ▶ Same priors as before, but we decompose the likelihood:

$$Z_i \mid w_1, w_2, w_3 \sim \text{Cat}(3, \mathbf{w})$$

$$Y_i \mid Z_i, \mu_{1:3}, \tau \sim N(\mu_{Z_i}, 1/\tau)$$

- ▶ The marginal distribution of Y_i is **unchanged**.
- ▶ Using the variables Z_i helps derive full conditional distributions.

In other words

$$p(Y_i \mid Z_i, \mu_1, \mu_2, \mu_3, \varepsilon^2) = N(Y_i; \mu_{Z_i}, \varepsilon^2)$$

$$P(Z_i = j \mid w_{1:3}) = w_j.$$

Task 3

Where necessary, (re)-derive the full conditionals under the data augmentation scheme.

Data augmentation scheme:

- ▶ Same priors as before, but we decompose the likelihood:

$$Z_i \mid w_1, w_2, w_3 \sim \text{Cat}(3, \mathbf{w})$$

$$Y_i \mid Z_i, \mu_{1:3}, \tau \sim N(\mu_{Z_i}, 1/\tau)$$

- ▶ The marginal distribution of Y_i is **unchanged**.
- ▶ Using the variables Z_i helps derive full conditional distributions.

In other words

$$p(Y_i \mid Z_i, \mu_1, \mu_2, \mu_3, \varepsilon^2) = N(Y_i; \mu_{Z_i}, \varepsilon^2)$$

$$P(Z_i = j \mid w_{1:3}) = w_j.$$

Task 3

Where necessary, (re)-derive the full conditionals under the data augmentation scheme.

Data augmentation scheme:

- ▶ Same priors as before, but we decompose the likelihood:

$$Z_i \mid w_1, w_2, w_3 \sim \text{Cat}(3, \mathbf{w})$$

$$Y_i \mid Z_i, \mu_{1:3}, \tau \sim N(\mu_{Z_i}, 1/\tau)$$

- ▶ The marginal distribution of Y_i is **unchanged**.
- ▶ Using the variables Z_i helps derive full conditional distributions.

In other words

$$p(Y_i \mid Z_i, \mu_1, \mu_2, \mu_3, \varepsilon^2) = N(Y_i; \mu_{Z_i}, \varepsilon^2)$$

$$P(Z_i = j \mid w_{1:3}) = w_j.$$

Task 3

Where necessary, (re)-derive the full conditionals under the data augmentation scheme.

Data augmentation scheme:

- ▶ Same priors as before, but we decompose the likelihood:

$$Z_i \mid w_1, w_2, w_3 \sim \text{Cat}(3, \mathbf{w})$$

$$Y_i \mid Z_i, \mu_{1:3}, \tau \sim N(\mu_{Z_i}, 1/\tau)$$

- ▶ The marginal distribution of Y_i is **unchanged**.
- ▶ Using the variables Z_i helps derive full conditional distributions.

In other words

$$p(Y_i \mid Z_i, \mu_1, \mu_2, \mu_3, \epsilon^2) = N(Y_i; \mu_{Z_i}, \epsilon^2)$$

$$P(Z_i = j \mid w_{1:3}) = w_j.$$

Task 3

Where necessary, (re)-derive the full conditionals under the data augmentation scheme.

Data augmentation scheme:

- ▶ Same priors as before, but we decompose the likelihood:

$$Z_i \mid w_1, w_2, w_3 \sim \text{Cat}(3, \mathbf{w})$$

$$Y_i \mid Z_i, \mu_{1:3}, \tau \sim N(\mu_{Z_i}, 1/\tau)$$

- ▶ The marginal distribution of Y_i is **unchanged**.
- ▶ Using the variables Z_i helps derive full conditional distributions.

In other words

$$p(Y_i \mid Z_i, \mu_1, \mu_2, \mu_3, \epsilon^2) = N(Y_i; \mu_{Z_i}, \epsilon^2)$$

$$P(Z_i = j \mid w_{1:3}) = w_j.$$

Task 3

Where necessary, (re)-derive the full conditionals under the data augmentation scheme.

Data augmentation scheme:

- ▶ Same priors as before, but we decompose the likelihood:

$$\begin{aligned}Z_i \mid w_1, w_2, w_3 &\sim \text{Cat}(3, \mathbf{w}) \\ Y_i \mid Z_i, \mu_{1:3}, \tau &\sim N(\mu_{Z_i}, 1/\tau)\end{aligned}$$

- ▶ The marginal distribution of Y_i is **unchanged**.
- ▶ Using the variables Z_i helps derive full conditional distributions.

In other words

$$\begin{aligned}p(Y_i \mid Z_i, \mu_1, \mu_2, \mu_3, \varepsilon^2) &= N(Y_i; \mu_{Z_i}, \varepsilon^2) \\ P(Z_i = j \mid w_{1:3}) &= w_j.\end{aligned}$$

Task 3

Now the full joint becomes:

$$\begin{aligned} & p(Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid Z_{1:n}, \mu_{1:3}, \tau) p(Z_{1:n}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ &= \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})^{w_{Z_i}}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \end{aligned}$$

Task 3

Now the full joint becomes:

$$\begin{aligned} & p(Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid Z_{1:n}, \mu_{1:3}, \tau) p(Z_{1:n}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ &= \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})^{w_{Z_i}}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \end{aligned}$$

Task 3

Now the full joint becomes:

$$\begin{aligned} & p(Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ &= p(Y_{1:n} \mid Z_{1:n}, \mu_{1:3}, \tau) p(Z_{1:n}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ &= \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})^{w_{Z_i}}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \end{aligned}$$

Task 3

Full conditional of $w_{1:3}$:

$$\begin{aligned} & p(w_{1:3} \mid Z_{1:n}, Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})^{w_{Z_i}}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n w_{Z_i} \\ & \propto w_1^{N_1} w_2^{N_2} w_3^{N_3} \\ & \Rightarrow w_{1:3} \sim \text{Dirichlet}(N_1 + 1, N_2 + 1, N_3 + 1) \end{aligned}$$

where $N_j = \sum_{i=1}^n \mathbb{I}(Z_i = j)$.

Task 3

Full conditional of $w_{1:3}$:

$$\begin{aligned} & p(w_{1:3} \mid Z_{1:n}, Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n w_{Z_i} \\ & \propto w_1^{N_1} w_2^{N_2} w_3^{N_3} \\ & \Rightarrow w_{1:3} \sim \text{Dirichlet}(N_1 + 1, N_2 + 1, N_3 + 1) \end{aligned}$$

where $N_j = \sum_{i=1}^n \mathbb{I}(Z_i = j)$.

Task 3

Full conditional of $w_{1:3}$:

$$\begin{aligned} & p(w_{1:3} \mid Z_{1:n}, Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n w_{Z_i} \\ & \propto w_1^{N_1} w_2^{N_2} w_3^{N_3} \\ & \Rightarrow w_{1:3} \sim \text{Dirichlet}(N_1 + 1, N_2 + 1, N_3 + 1) \end{aligned}$$

where $N_j = \sum_{i=1}^n \mathbb{I}(Z_i = j)$.

Task 3

Full conditional of $w_{1:3}$:

$$\begin{aligned} & p(w_{1:3} \mid Z_{1:n}, Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n w_{Z_i} \\ & \propto w_1^{N_1} w_2^{N_2} w_3^{N_3} \\ & \Rightarrow w_{1:3} \sim \text{Dirichlet}(N_1 + 1, N_2 + 1, N_3 + 1) \end{aligned}$$

where $N_j = \sum_{i=1}^n \mathbb{I}(Z_i = j)$.

Task 3

Full conditional of $w_{1:3}$:

$$\begin{aligned} & p(w_{1:3} \mid Z_{1:n}, Y_{1:n}, \mu_{1:3}, \mu_0, \phi_0, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n w_{Z_i} \\ & \propto w_1^{N_1} w_2^{N_2} w_3^{N_3} \\ & \Rightarrow w_{1:3} \sim \text{Dirichlet}(N_1 + 1, N_2 + 1, N_3 + 1) \end{aligned}$$

where $N_j = \sum_{i=1}^n \mathbb{I}(Z_i = j)$.

Task 3

Full conditional of μ_j :

$$\begin{aligned} & p(\mu_j \mid Y_{1:n}, Z_{1:n}, \mu_{(-j)}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})) p(\mu_j \mid \mu_0, \phi_0) \\ & \propto \exp \left\{ \frac{-\tau}{2} \sum_{i:Z_i=j} (Y_i - \mu_j)^2 \right\} \exp \left\{ \frac{-\phi_0}{2} (\mu_j - \mu_0)^2 \right\} \\ & \propto \exp \left\{ \frac{-1}{2} \left[\mu_j^2 (\tau N_j + \phi_0) - 2\mu_j \left(\tau \sum_{i:Z_i=j} Y_i + \phi_0 \mu_0 \right) \mu_j \right] \right\} \\ & \Rightarrow \mu_j \mid - \sim N((\tau N_j + \phi_0)^{-1} (\tau \sum_{i:Z_i=j} Y_i + \phi_0 \mu_0), (\tau N_j + \phi_0)^{-1}) \end{aligned}$$

Task 3

Full conditional of μ_j :

$$\begin{aligned} & p(\mu_j \mid Y_{1:n}, Z_{1:n}, \mu_{(-j)}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})) p(\mu_j \mid \mu_0, \phi_0) \\ & \propto \exp \left\{ \frac{-\tau}{2} \sum_{i:Z_i=j} (Y_i - \mu_j)^2 \right\} \exp \left\{ \frac{-\phi_0}{2} (\mu_j - \mu_0)^2 \right\} \\ & \propto \exp \left\{ \frac{-1}{2} \left[\mu_j^2 (\tau N_j + \phi_0) - 2\mu_j \left(\tau \sum_{i:Z_i=j} Y_i + \phi_0 \mu_0 \right) \mu_j \right] \right\} \\ & \Rightarrow \mu_j \mid - \sim N((\tau N_j + \phi_0)^{-1} (\tau \sum_{i:Z_i=j} Y_i + \phi_0 \mu_0), (\tau N_j + \phi_0)^{-1}) \end{aligned}$$

Task 3

Full conditional of μ_j :

$$\begin{aligned} & p(\mu_j \mid Y_{1:n}, Z_{1:n}, \mu_{(-j)}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})) p(\mu_j \mid \mu_0, \phi_0) \\ & \propto \exp \left\{ \frac{-\tau}{2} \sum_{i:Z_i=j} (Y_i - \mu_j)^2 \right\} \exp \left\{ \frac{-\phi_0}{2} (\mu_j - \mu_0)^2 \right\} \\ & \propto \exp \left\{ \frac{-1}{2} \left[\mu_j^2 (\tau N_j + \phi_0) - 2\mu_j \left(\tau \sum_{i:Z_i=j} Y_i + \phi_0 \mu_0 \right) \mu_j \right] \right\} \\ & \Rightarrow \mu_j \mid - \sim N((\tau N_j + \phi_0)^{-1} (\tau \sum_{i:Z_i=j} Y_i + \phi_0 \mu_0), (\tau N_j + \phi_0)^{-1}) \end{aligned}$$

Task 3

Full conditional of μ_j :

$$\begin{aligned} & p(\mu_j \mid Y_{1:n}, Z_{1:n}, \mu_{(-j)}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})) p(\mu_j \mid \mu_0, \phi_0) \\ & \propto \exp \left\{ \frac{-\tau}{2} \sum_{i:Z_i=j} (Y_i - \mu_j)^2 \right\} \exp \left\{ \frac{-\phi_0}{2} (\mu_j - \mu_0)^2 \right\} \\ & \propto \exp \left\{ \frac{-1}{2} \left[\mu_j^2 (\tau N_j + \phi_0) - 2\mu_j (\tau \sum_{i:Z_i=j} Y_i + \phi_0 \mu_0) \mu_j \right] \right\} \\ & \Rightarrow \mu_j \mid - \sim N((\tau N_j + \phi_0)^{-1} (\tau \sum_{i:Z_i=j} Y_i + \phi_0 \mu_0), (\tau N_j + \phi_0)^{-1}) \end{aligned}$$

Task 3

Full conditional of μ_j :

$$\begin{aligned} & p(\mu_j \mid Y_{1:n}, Z_{1:n}, \mu_{(-j)}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})) p(\mu_j \mid \mu_0, \phi_0) \\ & \propto \exp \left\{ \frac{-\tau}{2} \sum_{i: Z_i=j} (Y_i - \mu_j)^2 \right\} \exp \left\{ \frac{-\phi_0}{2} (\mu_j - \mu_0)^2 \right\} \\ & \propto \exp \left\{ \frac{-1}{2} \left[\mu_j^2 (\tau N_j + \phi_0) - 2\mu_j \left(\tau \sum_{i: Z_i=j} Y_i + \phi_0 \mu_0 \right) \mu_j \right] \right\} \\ & \Rightarrow \mu_j \mid - \sim N((\tau N_j + \phi_0)^{-1} (\tau \sum_{i: Z_i=j} Y_i + \phi_0 \mu_0), (\tau N_j + \phi_0)^{-1}) \end{aligned}$$

Task 3

Full conditional of μ_j :

$$\begin{aligned} & p(\mu_j \mid Y_{1:n}, Z_{1:n}, \mu_{(-j)}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})) p(\mu_j \mid \mu_0, \phi_0) \\ & \propto \exp \left\{ \frac{-\tau}{2} \sum_{i:Z_i=j} (Y_i - \mu_j)^2 \right\} \exp \left\{ \frac{-\phi_0}{2} (\mu_j - \mu_0)^2 \right\} \\ & \propto \exp \left\{ \frac{-1}{2} \left[\mu_j^2 (\tau N_j + \phi_0) - 2\mu_j \left(\tau \sum_{i:Z_i=j} Y_i + \phi_0 \mu_0 \right) \mu_j \right] \right\} \\ & \Rightarrow \mu_j \mid - \sim N((\tau N_j + \phi_0)^{-1} (\tau \sum_{i:Z_i=j} Y_i + \phi_0 \mu_0), (\tau N_j + \phi_0)^{-1}) \end{aligned}$$

Task 3

Full conditional for τ :

$$\begin{aligned} & p(\tau \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})) p(\tau) \\ & \propto \tau^{n/2} \exp \left\{ \frac{-\tau}{2} \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 \right\} \tau^{2-1} \exp \{-2\tau\} \\ & \propto \tau^{2+n/2-1} \exp \left\{ -\tau \left(2 + \frac{1}{2} \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 \right) \right\} \\ & \Rightarrow \tau \mid - \sim \text{Gamma} \left(2 + n/2, 2 + \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 / 2 \right) \end{aligned}$$

Task 3

Full conditional for τ :

$$\begin{aligned} & p(\tau \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})) p(\tau) \\ & \propto \tau^{n/2} \exp \left\{ \frac{-\tau}{2} \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 \right\} \tau^{2-1} \exp \{-2\tau\} \\ & \propto \tau^{2+n/2-1} \exp \left\{ -\tau \left(2 + \frac{1}{2} \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 \right) \right\} \\ & \Rightarrow \tau \mid - \sim \text{Gamma} \left(2 + n/2, 2 + \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 / 2 \right) \end{aligned}$$

Task 3

Full conditional for τ :

$$\begin{aligned} & p(\tau \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})) p(\tau) \\ & \propto \tau^{n/2} \exp \left\{ \frac{-\tau}{2} \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 \right\} \tau^{2-1} \exp\{-2\tau\} \\ & \propto \tau^{2+n/2-1} \exp\left\{-\tau\left(2 + \frac{1}{2} \sum_{i=1}^n (Y_i - \mu_{Z_i})^2\right)\right\} \\ & \Rightarrow \tau \mid - \sim \text{Gamma}\left(2 + n/2, 2 + \sum_{i=1}^n (Y_i - \mu_{Z_i})^2/2\right) \end{aligned}$$

Task 3

Full conditional for τ :

$$\begin{aligned} & p(\tau \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})) p(\tau) \\ & \propto \tau^{n/2} \exp \left\{ \frac{-\tau}{2} \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 \right\} \tau^{2-1} \exp \{-2\tau\} \\ & \propto \tau^{2+n/2-1} \exp \left\{ -\tau \left(2 + \frac{1}{2} \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 \right) \right\} \\ & \Rightarrow \tau \mid - \sim \text{Gamma} \left(2 + n/2, 2 + \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 / 2 \right) \end{aligned}$$

Task 3

Full conditional for τ :

$$\begin{aligned} & p(\tau \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})) p(\tau) \\ & \propto \tau^{n/2} \exp \left\{ \frac{-\tau}{2} \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 \right\} \tau^{2-1} \exp \{-2\tau\} \\ & \propto \tau^{2+n/2-1} \exp \left\{ -\tau \left(2 + \frac{1}{2} \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 \right) \right\} \\ & \Rightarrow \tau \mid - \sim \text{Gamma}(2 + n/2, 2 + \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 / 2) \end{aligned}$$

Task 3

Full conditional for τ :

$$\begin{aligned} & p(\tau \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})) p(\tau) \\ & \propto \tau^{n/2} \exp \left\{ \frac{-\tau}{2} \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 \right\} \tau^{2-1} \exp \{-2\tau\} \\ & \propto \tau^{2+n/2-1} \exp \left\{ -\tau \left(2 + \frac{1}{2} \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 \right) \right\} \\ & \Rightarrow \tau \mid - \sim \text{Gamma}(2 + n/2, 2 + \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 / 2) \end{aligned}$$

Task 3

Full conditional for Z_i :

$$\begin{aligned} & p(Z_i \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1})^{w_{Z_i}}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto N(Y_i; \mu_{Z_i}, \tau^{-1})^{w_{Z_i}} \\ & \Rightarrow P(Z_i = j \mid -) \propto w_j N(Y_i; \mu_j, \tau^{-1}). \end{aligned}$$

Task 3

Full conditional for Z_i :

$$\begin{aligned} & p(Z_i \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i} \\ & \Rightarrow P(Z_i = j \mid -) \propto w_j N(Y_i; \mu_j, \tau^{-1}). \end{aligned}$$

Task 3

Full conditional for Z_i :

$$\begin{aligned} & p(Z_i \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i} \\ & \Rightarrow P(Z_i = j \mid -) \propto w_j N(Y_i; \mu_j, \tau^{-1}). \end{aligned}$$

Task 3

Full conditional for Z_i :

$$\begin{aligned} & p(Z_i \mid Y_{1:n}, Z_{1:n}, \mu_{1:3}, \mu_0, \phi_0, w_{1:3}, \tau) \\ & \propto \prod_{i=1}^n (N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i}) p(\mu_{1:3} \mid \mu_0, \phi_0) p(\mu_0) p(\phi_0) p(\tau) \\ & \propto N(Y_i; \mu_{Z_i}, \tau^{-1}) w_{Z_i} \\ & \Rightarrow P(Z_i = j \mid -) \propto w_j N(Y_i; \mu_j, \tau^{-1}). \end{aligned}$$

Task 3

Summary:

$$p(\mu_0 \mid \dots) = N \left(\frac{\sigma_0^2 \sum_{i=1}^3 \mu_i}{1/3 + 3\sigma_0^{-2}}, (1/3 + 3\sigma_0^{-2})^{-1} \right),$$

$$p(\sigma_0^2 \mid \dots) = IG \left(2 + 3/2, 2 + (1/2) \sum_{i=1}^3 (\mu_i - \mu_0)^2 \right),$$

$$p(\epsilon^2 \mid \dots) = IG \left(2 + n/2, 2 + (1/2) \sum_{i=1}^n (Y_i - \mu_{Z_i})^2 \right),$$

$$p(\mathbf{w} \mid \dots) = Dir(3, (1 + N_1, 1 + N_2, 1 + N_3)),$$

$$p(\mu_j \mid \dots) = N \left(\left(\mu_0 \sigma_0^{-2} + \epsilon^{-2} \sum_{i:Z_i=j} y_i \right) (\sigma_0^{-2} + N_j \epsilon^{-2})^{-1}, (\sigma_0^{-2} + N_j \epsilon^{-2})^{-1} \right),$$

$$P(Z_i = j) = \frac{w_j N(y_i \mid \mu_j, \epsilon^2)}{\sum_{k=1}^3 w_k N(y_i \mid \mu_k, \epsilon^2)}.$$

Load Libraries

```
# load libraries #
```

```
library(gtools)
```

```
library(pscl)
```

```
## Classes and Methods for R developed in the
```

```
## Political Science Computational Laboratory
```

```
## Department of Political Science
```

```
## Stanford University
```

```
## Simon Jackman
```

```
## hurdle and zeroinfl functions by Achim Zeileis
```

Load Data

```
# set seed for reproducibility #  
set.seed(666)  
# set number of simulations #  
nsims = 10  
# read in the data #  
y = read.csv("Lab8Mixture.csv", header=FALSE)
```

Prepare Data

```
# set matrix for w values #  
w = matrix(0, nrow = nsims, ncol=3)  
colnames(w) = c("w1", "w2", "w3")
```

Initialization

```
# set initial values for w #  
w[1,] = c(1/3,1/3,1/3)  
# set matrix for w values #  
mu_j = matrix(0, nrow = nsims, ncol=3)  
colnames(mu_j) = c("mu1", "mu2", "mu3")  
# set initial values for w #  
mu_j[1,] = c(1,1,1)
```

Initialization

```
# create vectors #  
sigma0 = c()  
e2 = c()  
mu0 = c()  
# set initial values #  
sigma0[1] = 1  
e2[1] = 1  
mu0[1] = 1
```

Initialization

```
# create matrix for z values #  
z = matrix(0, ncol = nrow(y), nrow = nsims)  
# set initial values for z #  
z[1,] = sample(c(1,2,3), size = nrow(y), replace = TRUE)
```

Initialize Gibbs

```
N_vals = c()  
S_vals = c()  
Y_vals = list()
```


Gibbs Sampler (not elegant)

```
# gibbs sampling #
for (sim in 2:nsims) {

  for (i in 1:3) { # find values for each category
    N_vals[i] = sum(z[sim-1, ] == i)
    Y_vals[[i]] = which(z[sim-1, ] == i)
    S_vals[i] = sum((y[Y_vals[[i]], ] - mu_j[sim-1, i])^2)
  }

  w[sim,] = rdirichlet(n = 1, alpha = N_vals+1) # sample w values

  e2[sim] = rigamma(n = 1, alpha = (4+sum(N_vals))/2, # sample e2 values
    beta = sum(S_vals)/2)

  # sample mu0 values
  mu0[sim] = rnorm(n = 1,
    mean = ((1/((1/3) + (3 / sigma0[sim-1])))*
      (sum(mu_j[sim-1, ] / sigma0[sim-1])),
    sd = sqrt((1/((1/3) + (3 / sigma0[sim-1])))))
  sigma0[sim] = rigamma(n = 1, alpha = 3.5, # sample sigma0 values
    beta = (2 + ((sum((mu_j[sim-1, ] - mu0[sim])^2))/2)))

  for (g in 1:3) { # sample mu values
    mu_j[sim, g] = rnorm(n = 1, mean = (((mu0[sim]/sigma0[sim]) +
      (sum(y[Y_vals[[g]], ])/e2[sim])) *
      (1 / ((1/sigma0[sim]) + (N_vals[g]/e2[sim])))),
    sd = sqrt((1 / ((1/sigma0[sim]) + (N_vals[g]/e2[sim])))))
  }

  # sample z values
  for(h in 1:nrow(y)) {
    prob_vals = w[sim, ]*dnorm(y[h,], mean = mu_j[sim, ], sd = sqrt(e2[sim]))
    z[sim, h] = sample(x = c(1,2,3), size = 1, replace = TRUE,
      prob = prob_vals/sum(prob_vals))
  }
}

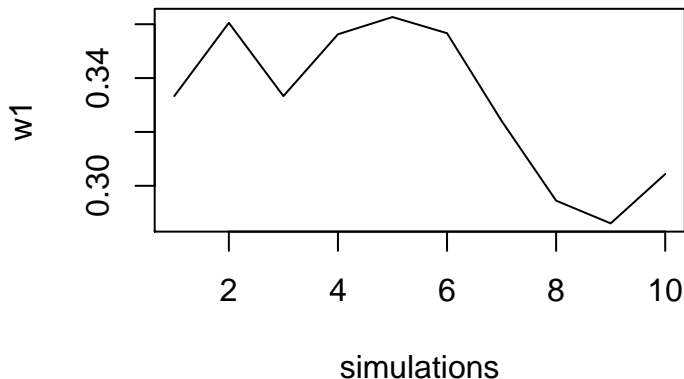
mixture_results = cbind(w, mu_j, e2, mu0, sigma0, z)
```

Task 5

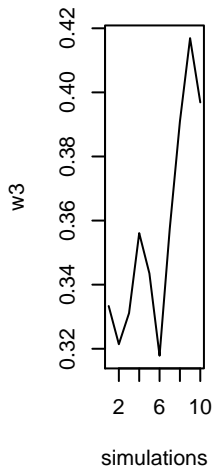
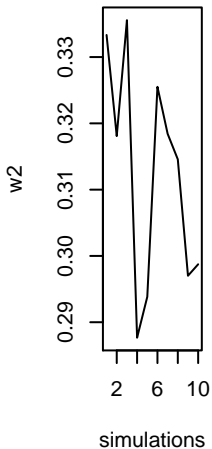
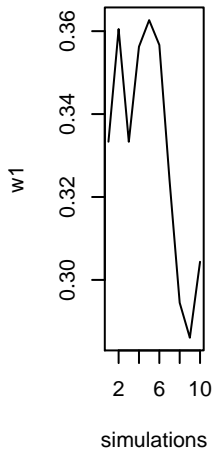
Now, we want to create trace plots, averages, and 95% confidence intervals for the marginal posterior distributions of all the parameters.

Traceplot of w_1

```
plot(1:nrow(mixture_results), mixture_results[,1],  
     type = "l", xlab = "simulations",  
     ylab = colnames(mixture_results)[1])
```



Traceplots of w_1 , w_2 , w_3



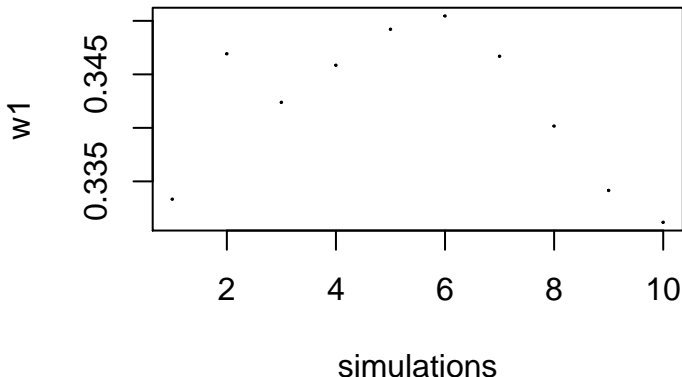
Traceplots of weights

What can you conclude regarding the traceplot of the weights?

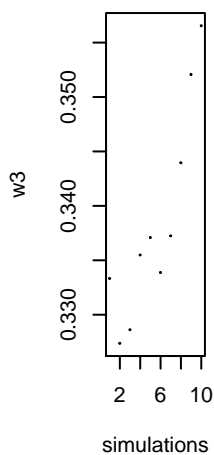
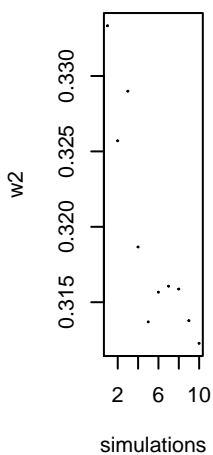
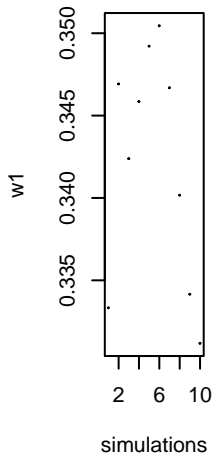
What should you do to fix the situation?

Running average plots for w_1

```
# plot moving average trace plots  
mov_avg = cumsum(mixture_results[,1])/(1:nrow(mixture_results))  
plot(1:length(mov_avg), mov_avg, cex = 0.1,  
     xlab = "simulations", ylab = colnames(mixture_results)[1])
```



Running average plots of w_1 , w_2 , w_3



Calculate our 95% credible intervals

```
# calculate 95% credible intervals #
for (i in 1:9) {
  upper = quantile(mixture_results[,i], 0.975)
  lower = quantile(mixture_results[,i], 0.25)
  cat("The 95% credible interval for",
      colnames(mixture_results)[i], "is between",
      lower, "and", upper, "\n")
}
```

```
## The 95% credible interval for w1 is between 0.3093064 and 0.3621755
## The 95% credible interval for w2 is between 0.2974454 and 0.3350666
## The 95% credible interval for w3 is between 0.3316713 and 0.4124205
## The 95% credible interval for mu1 is between 2.448104 and 3.403709
## The 95% credible interval for mu2 is between 3.008058 and 3.573371
## The 95% credible interval for mu3 is between 2.156434 and 2.767639
## The 95% credible interval for e2 is between 4.059747 and 6.904096
## The 95% credible interval for mu0 is between 1.875007 and 3.454596
## The 95% credible interval for sigma0 is between 0.5745476 and 2.452617
```