Exercise

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You need to sample from the conditional distribution of $X \mid X < c$, where $X \sim \mathcal{N}(0,1)$ and $c \in \mathbb{R}$. Assume:

- you can generate Uniform$(0,1)$ random variables, and
- you can evaluate both the c.d.f. $F(x)$ and the inverse c.d.f. $F^{-1}(u)$ of the $\mathcal{N}(0,1)$ distribution.

How would you draw samples from $X \mid X < c$?

Solution

Approach 1 (Simple, but not great)

To draw a sample Z from the distribution of $X \mid X < c$,

1. sample $U \sim \text{Uniform}(0, 1)$,

2. set $X = F^{-1}(U)$,

3. if $X \geq c$ then return to step 1 (reject), otherwise, output $Z = X$ as a sample (accept).

Why does it work? By the inverse c.d.f. method, we know $X = F^{-1}(U) \sim \mathcal{N}(0, 1)$. By the rejection principle, if we reject any samples X such that $X \geq c$, then what remains has the conditional distribution given $X < c$. This approach is not ideal, however, since the rejection rate may be very high, especially when $c \ll 0$.

Approach 2 (Better)

To draw a sample Z from the distribution of $X \mid X < c$,

1. sample $U \sim \text{Uniform}(0, 1)$,

2. set $V = F(c)U$, and

3. set $Z = F^{-1}(V)$.

Why does this work? Note that in Approach 1, rejecting when $X \geq c$ is identical to rejecting when $U \geq F(c)$, and by the rejection principle, we know that the distribution of the U's that remain after rejection is $U \mid U < F(c)$, in other words, $\text{Uniform}(0, F(c))$. But that means that the rejection step can be bypassed completely by just sampling $V \sim \text{Uniform}(0, F(c))$ and setting $Z = F^{-1}(V)$! And we can directly sample $V \sim \text{Uniform}(0, F(c))$, by drawing $U \sim \text{Uniform}(0, 1)$ and setting $V = F(c)U$.