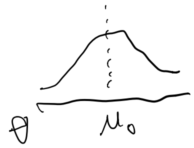


$X_1, \dots, X_n \mid \theta \overset{iid}{\sim} N(\theta, \lambda^{-1})$ $\quad \lambda = \frac{1}{\sigma^2}$ fixed + known.

$\theta \sim N(\mu_0, \lambda_0^{-1})$, μ_0, λ_0 are fixed + known.

Goal: Find $p(\theta \mid x_{1:n})$.

(1) $N(x \mid \theta, \ell^{-1}) = \boxed{\sqrt{\frac{\ell}{2\pi}}} \exp\left\{-\frac{\ell}{2}(x-\theta)^2\right\}$

$$\underset{\theta}{\propto} \exp\left\{-\frac{\ell}{2}\left(\boxed{x^2} - 2x\theta + \theta^2\right)\right\}$$

$$\underset{\theta}{\propto} \exp\left\{\ell x \theta - \frac{\ell}{2}\theta^2\right\} \qquad (5)$$

$\uparrow \sum x_i \qquad \uparrow n$

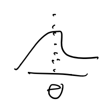
$x = \mu_0$

$\ell = \lambda_0$

(2) Work with the prior

$$N(\theta \mid \mu_0, \lambda_0^{-1}) \overset{\boxed{\text{symm}}}{=} N(\mu_0 \mid \theta, \lambda_0^{-1})$$

$\mu_0 = x$



$$\underset{\theta}{\propto} \exp\left\{\lambda_0 \mu_0 \theta - \frac{1}{2}\lambda_0 \theta^2\right\} \quad \text{by } 5 \quad (6)$$

(3) Posterior update.

$$p(\theta \mid x_{1:n}) \underset{\theta}{\propto} \underline{p(x_{1:n} \mid \theta)}\; p(\theta)$$

$$\underset{\theta}{\propto} \underbrace{N(\theta \mid \mu_0, \lambda_0^{-1})} \times \underbrace{\prod_{i=1}^{n} N(x_i \mid \theta, \lambda^{-1})}$$

by eqns (5,6)

$$\underset{\theta}{\propto} \underbrace{\exp\left\{ \lambda_0 \mu_0 \theta - \frac{\lambda_0}{2}\theta^2 \right\}}_{\text{by 6}}^{\text{prior}} \underbrace{\exp\left\{ \lambda \sum_{i=1}^{n} x_i \cdot \theta - \frac{n\lambda}{2}\theta^2 \right\}}_{\text{by 5}}^{\text{likelihood}}$$

$$\underset{\theta}{\propto} \exp\left\{ (\lambda_0 \mu_0 + \lambda \Sigma x_i)\,\theta - \frac{1}{2}\Big(\underbrace{\lambda_0 + n\lambda}_{\text{define as } L}\Big)\theta^2 \right\}$$

define $M = \dfrac{\lambda_0 \mu_0 + \lambda \Sigma x_i}{\lambda_0 + n\lambda}$

$ML = \lambda_0 \mu_0 + \lambda \Sigma x_i$

$$\underset{\theta}{\propto} \exp\left\{ ML\theta - \frac{1}{2} L \theta^2 \right\} \qquad x = M,\ \ell = L$$

$$\underset{\theta}{\propto} N(M \mid \theta, L^{-1}) \overset{\text{symm}}{=} N(\theta \mid M, L^{-1})$$

$$\theta \mid x_{1:n} \sim N(\underset{\text{post mean}}{M}, \underset{\text{post precision.}}{L^{-1}}).$$

$\boxed{\text{Why does symmetry hold!!}}$

$$\mathcal{N}(\theta \mid \mu_0, \underline{\underline{\lambda_0^{-1}}}) \overset{??}{=} \mathcal{N}(\mu_0 \mid \theta, \underline{\underline{\lambda_0^{-1}}})$$

LHS :

$$p(\theta \mid \mu_0, \lambda_0^{-1}) = \sqrt{\frac{\lambda_0}{2\pi}} \exp\{(\theta - \mu_0)^2\}$$

$$= \sqrt{\frac{\lambda_0}{2\pi}} \exp\{(\mu_0 - \theta)^2\}.$$

$$= \mathcal{N}(\mu_0 \mid \theta, \lambda_0^{-1}).$$ RHS

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