STA360 Practice Exam I Fall 2020

Instructions

- This exam is closed note and closed book.
- Write your name, NetID, and signature below.
- **Only what is on the exam will be graded (or written work submitted as one pdf file).**
- **Show all work and back up all your results for full credit.**
- **Only what's on the exam will be graded. It must be legible.**
- **You must label/assign pages when submitting to Gradescope to avoid losing points.**
- **The exam should be submitted via Gradescope. You will have 15 minutes to submit your exam to Gradescope after the exam as a single pdf file.**

Community Standard

To uphold the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

I have adhered to the Duke Community Standard in completing this exam.

Name: ______________________________

NetID: ______________________________

Signature: ______________________________

Score

(For TA use only — leave this section blank.)

1. ______________/2.5

2. ______________/10

Overall: ______________________/12.5

List of common distributions

$$\text{Geometric}(x|\theta) = \theta(1-\theta)^x\, \mathbb{1}(x \in \{0,1,2,\ldots\}) \text{ for } 0 < \theta < 1$$

$$\text{Bernoulli}(x|\theta) = \theta^x(1-\theta)^{1-x}\, \mathbb{1}(x \in \{0,1\}) \text{ for } 0 < \theta < 1$$

$$\text{Binomial}(x|n,\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}\, \mathbb{1}(x \in \{0,1,\ldots,n\}) \text{ for } 0 < \theta < 1$$

$$\text{Poisson}(x|\theta) = \frac{e^{-\theta}\theta^x}{x!}\, \mathbb{1}(x \in \{0,1,2,\ldots\}) \text{ for } \theta > 0$$

$$\text{Exp}(x|\theta) = \theta e^{-\theta x}\, \mathbb{1}(x > 0) \text{ for } \theta > 0$$

$$\text{Uniform}(x|a,b) = \frac{1}{b-a}\, \mathbb{1}(a < x < b) \text{ for } a < b$$

$$\text{Gamma}(x|a, b = \text{rate}) = \frac{b^a}{\Gamma(a)}x^{a-1}e^{-bx}\, \mathbb{1}(x > 0) \text{ for } a, b > 0,$$

$$\text{Gamma}(x|a, b = \text{scale}) = \frac{1}{b^a\Gamma(a)}x^{a-1}e^{-x/b}\, \mathbb{1}(x > 0) \text{ for } a, b > 0,$$

$$\text{Pareto}(x|\alpha, c) = \frac{\alpha c^\alpha}{x^{\alpha+1}}\, \mathbb{1}(x > c) \text{ for } \alpha, c > 0$$

$$\text{Beta}(x|a,b) = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}\, \mathbb{1}(0 < x < 1) \text{ for } a, b > 0$$

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\big(-\tfrac{1}{2\sigma^2}(x-\mu)^2\big) \text{ for } \mu \in \mathbb{R},\ \sigma^2 > 0$$

$$\mathcal{N}(x|\mu,\lambda^{-1}) = \sqrt{\frac{\lambda}{2\pi}}\exp\big(-\tfrac{1}{2}\lambda(x-\mu)^2\big) \text{ for } \mu \in \mathbb{R},\ \lambda > 0$$

1. (2.5 points) Assume the data is $x_{1:n} = (x_1, \ldots, x_n)$ and θ is the unknown parameter of interest. Let x_{n+1} be a new data point. **Circle the most correct answer (or correct answers).**

   (a) (2.5 points) Assume that you have the likelihood $p(x_{1:n} \mid \theta)$, the prior $p(\theta)$, and marginal distribution $p(x_{1:n})$ available to you. The posterior distribution $p(\theta \mid x_{1:n})$ can be derived (written) as the following:

      i.
      $$p(\theta \mid x_{1:n}) = \frac{p(x_{1:n} \mid \theta)p(\theta)}{p(x_{1:n})}$$

      ii.
      $$p(\theta \mid x_{1:n}) = \frac{p(x_{1:n}, \theta)}{p(x_{1:n})}$$

      iii.
      $$p(\theta \mid x_{1:n}) = \frac{p(\theta, x_{1:n})}{p(x_{1:n})}$$

      iv.
      $$p(\theta \mid x_{1:n}) \propto p(x_{1:n} \mid \theta)p(\theta)$$

2. (Normality and Bayes, 10 points) Suppose that the random variables $Y_1, \ldots, Y_n$ satisfy

$$Y_i = x_i\beta + \epsilon_i, i = 1, \ldots, n$$

where $x_1, \ldots, x_n$ are fixed constants, $\epsilon_1, \ldots, \epsilon_n$ are iid $\mathrm{N}(0, \sigma^2)$. Assume that σ^2 is known.

(a) (2 points) Show that the maximum likelihood (ML) estimator of β is

$$\hat{\beta} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2}.$$

(b) (1.5 points) Show that (step by step) that

$$\hat{\beta} \sim N(\beta, \sigma^2 / \sum_i x_i^2).$$

(c) (1.5 point) Find the distribution of an alternative estimator of β given by

$$\tilde{\beta} = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n x_i}.$$

Specifically, show (step by step) that

$$\tilde{\beta} \sim N(\beta, n\sigma^2 / (\sum_i x_i)^2).$$

(d) (2.5 points) Find the posterior distribution of β under a normal prior with mean 0 and variance $\frac{\tau^2}{\sum_{i=1}^n x_i^2}$.

(e) (2.5 points) Show that the posterior expectation of β can be written as a function of $\hat{\beta}$. Specifically, show that

$$\beta \mid y, x \sim N(\Sigma^{-1}a, \Sigma^{-1}) = N(\frac{\tau^2}{\tau^2 + \sigma^2} \frac{\sum_i x_i y_i}{\sum_i x_i^2}, (\frac{\sigma^2 \tau^2}{\tau^2 + \sigma^2}) / \sum_i x_i^2).$$

Continued Workpage for Problem 2.

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