Homework 4

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1 Change of Variables

a

From the question, we obtain the following equations:

$$\psi = \log(\theta/(1-\theta)) \implies \theta = h(\psi) = \frac{e^\psi}{1+e^\psi}$$

$$\begin{aligned}\left|\frac{dh}{d\psi}\right| &= \frac{dh}{d\psi} \\ &= \frac{e^\psi}{(1+e^\psi)^2} \because \frac{dh}{d\psi} > 0\end{aligned}$$

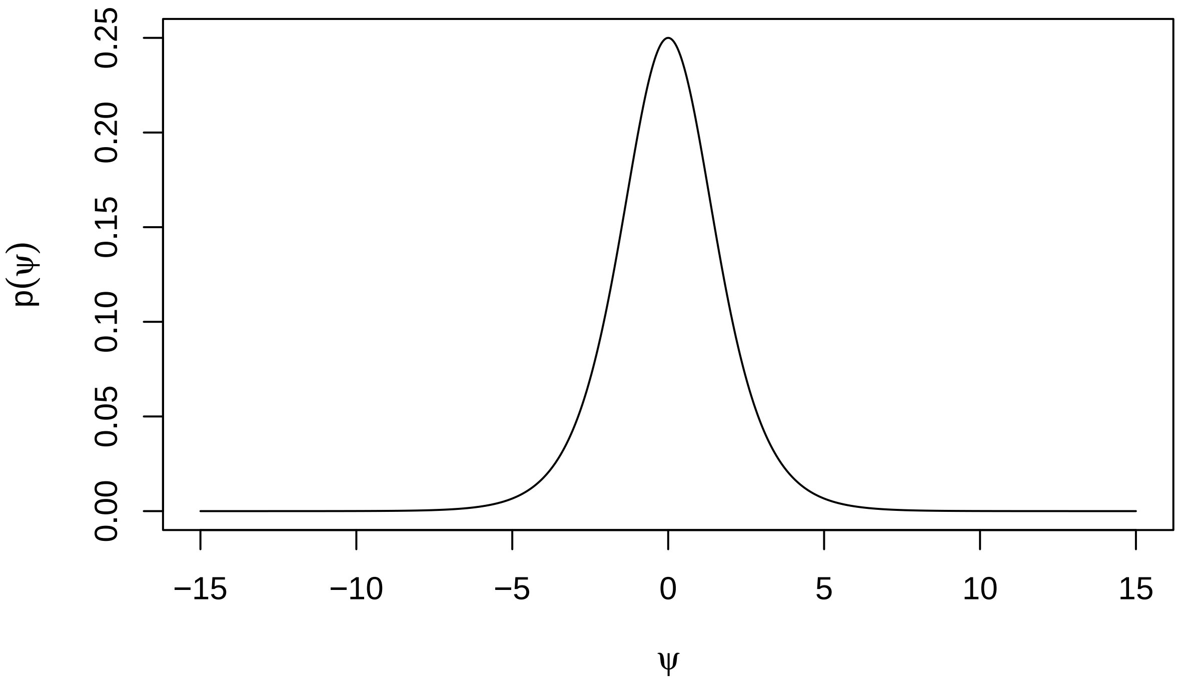
We also know that $\theta = h(\psi) \sim Beta(a,b)$. Given this we can obtain the form of p_ψ:

$$\begin{aligned}p_\psi(\psi) &= p_\theta(h(\psi)) \times \left|\frac{dh}{d\psi}\right| \\ &= \frac{1}{B(a,b)}\left(\frac{e^\psi}{1+e^\psi}\right)^{a-1}\left(1-\frac{e^\psi}{1+e^\psi}\right)^{b-1}\left(\frac{e^\psi}{(1+e^\psi)^2}\right) \\ &= \frac{1}{B(a,b)}\left(\frac{e^\psi}{1+e^\psi}\right)^{a}\left(\frac{1+e^\psi}{e^\psi}\right)\left(1-\frac{e^\psi}{1+e^\psi}\right)^{b}(1+e^\psi)\left(\frac{e^\psi}{(1+e^\psi)^2}\right) \\ &= \boxed{\frac{1}{B(a,b)}\left(\frac{e^\psi}{1+e^\psi}\right)^{a}\left(1-\frac{e^\psi}{1+e^\psi}\right)^{b}}\end{aligned}$$

Now we plot this for the case that $a = b = 1$:

Now we plot this for the case that $a = b = 1$:

```r
### Set seed for reproducibility of code
set.seed(123)

### Generate psi sequence
beta_psis <- seq(-15,15, by = 0.01)

### Generate distribution
beta_thetas <- exp(beta_psis)/(1+exp(beta_psis))
p_beta_theta <- dbeta(beta_thetas, 1, 1)
p_beta_psi <- p_beta_theta * exp(beta_psis)/(1+exp(beta_psis)) ^ 2

plot(beta_psis, p_beta_psi, type = "l", ylab = expression(p(psi)), xlab = expression(psi))
```



b

From the question, we obtain the following equations:

$$\psi = \log(\theta) \implies \theta = h(\psi) = e^{\psi}$$

$$\begin{aligned} \left|\frac{dh}{d\psi}\right| &= \frac{dh}{d\psi} \\ &= e^{\psi} \because \frac{dh}{d\psi} > 0 \end{aligned}$$

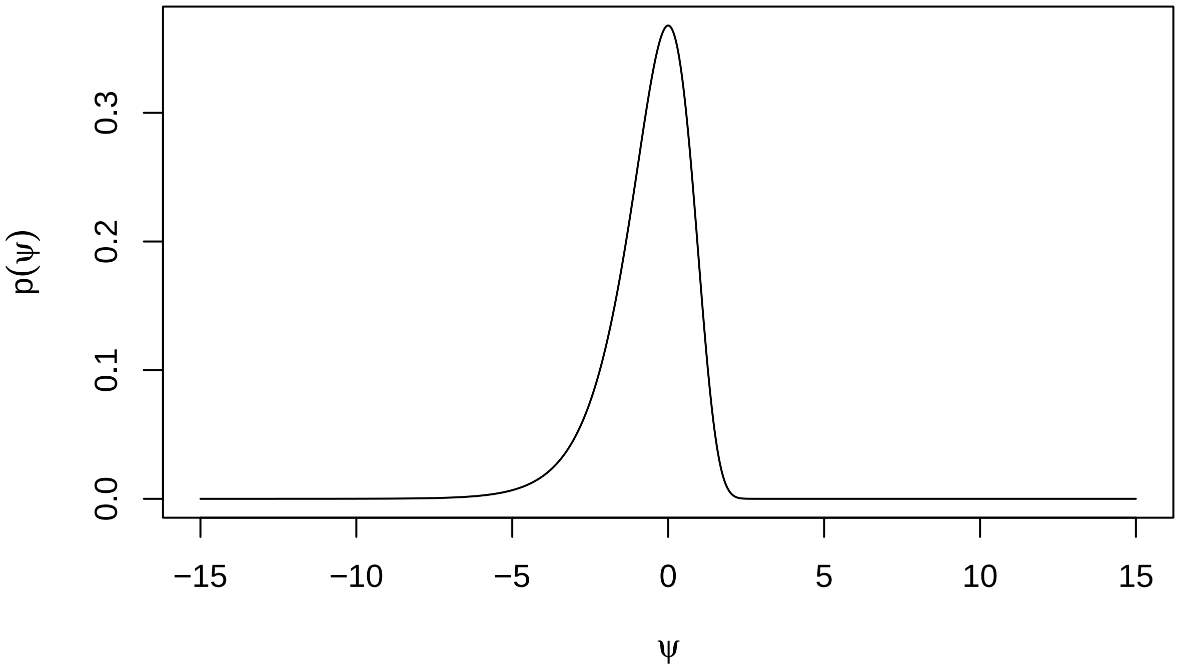
We also know that $\theta = h(\psi) \sim Gamma(a, b)$. Given this we can obtain the form of p_{ψ}:

$$\begin{aligned} p_{\psi}(\psi) &= p_{\theta}(h(\psi)) \times \left|\frac{dh}{d\psi}\right| \\ &= \frac{b^a}{\Gamma(a)}(e^{\psi})^{a-1} e^{-be^{\psi}} e^{\psi} \\ &= \boxed{\frac{b^a}{\Gamma(a)}(e^{a\psi - be^{\psi}})} \end{aligned}$$

Now we plot this for the case that $a = b = 1$:

```r
### Generate psi sequence
gamma_psis <- seq(-15,15, by = 0.01)

### Generate distribution
gamma_thetas <- exp(gamma_psis)
```

```r
p_gamma_theta <- dgamma(gamma_thetas, 1, 1)
p_gamma_psi <- p_gamma_theta * exp(gamma_psis)

plot(gamma_psis, p_gamma_psi, type = "l", ylab = expression(p(psi)), xlab = expression(psi))
```



2 Lab Component

```r
set.seed(123)
# input data
# spurters
x = c(18, 40, 15, 17, 20, 44, 38)
# control group
y = c(-4, 0, -19, 24, 19, 10, 5, 10,
      29, 13, -9, -8, 20, -1, 12, 21,
      -7, 14, 13, 20, 11, 16, 15, 27,
      23, 36, -33, 34, 13, 11, -19, 21,
      6, 25, 30,22, -28, 15, 26, -1, -2,
      43, 23, 22, 25, 16, 10, 29)
# store data in data frame
iqData = data.frame(Treatment = c(rep("Spurters", length(x)),
                                  rep("Controls", length(y))),
                                  Gain = c(x, y))
```

```r
prior = data.frame(m = 0, c = 1, a = 0.5, b = 50)
findParam = function(prior, data){
```

```
  postParam = NULL
  c = prior$c
  m = prior$m
  a = prior$a
  b = prior$b
  n = length(data)
  postParam = data.frame(m = (c*m + n*mean(data))/(c + n),
                c = c + n,
                a = a + n/2,
                b =  b + 0.5*(sum((data - mean(data))^2)) +
                  (n*c *(mean(data)- m)^2)/(2*(c+n)))
  return(postParam)
}
postS = findParam(prior, x)
postC = findParam(prior, y)
```

Task 4

```
# sampling from two posteriors

# Number of posterior simulations
sim = 1000000

# initialize vectors to store samples
mus = NULL
lambdas = NULL
muc = NULL
lambdac = NULL

# Following formula from the NormalGamma with
# the update paramaters accounted accounted for below

lambdas = rgamma(sim, shape = postS$a, rate = postS$b)
lambdac = rgamma(sim, shape = postC$a, rate = postC$b)


mus = sapply(sqrt(1/(postS$c*lambdas)),rnorm, n = 1, mean = postS$m)
muc = sapply(sqrt(1/(postC$c*lambdac)),rnorm, n = 1, mean = postC$m)
```

```
### Calculate probability mus>muc
sum(mus>muc)/length(mus)
```

```
## [1] 0.970673
```

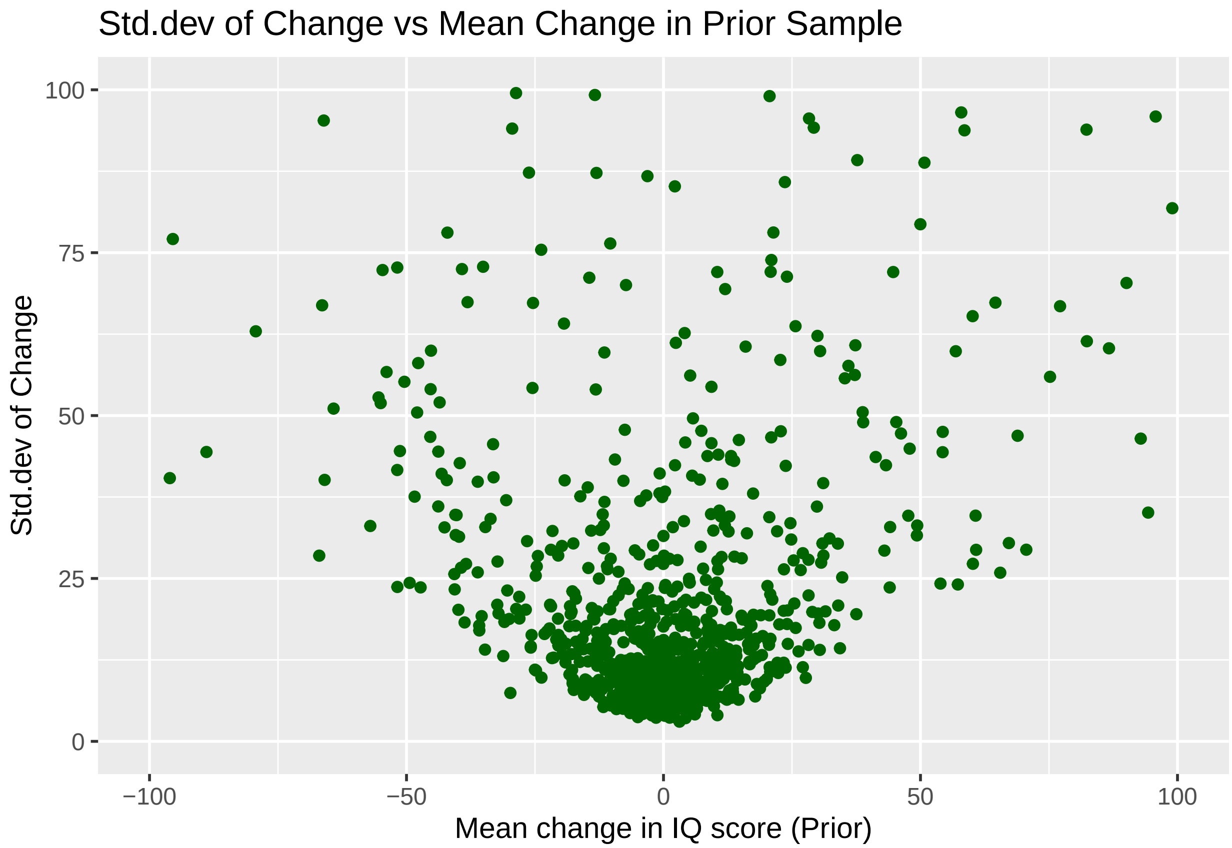
Given the data collected and our chosen prior distribution, Monte Carlo Approximation, $P(\mu_s > \mu_c) \approx \boxed{0.970673}$. Thus, there is strong evidence that the mean increase in IQ in spurters is higher than that of the control group.

Task 5

```
# sampling from prior

# Number of prior simulations
prior_sim = 1000
```

```r
# initialize vectors to store samples
prior_mus = NULL
prior_lambdas = NULL

# Following formula from the NormalGamma with
# the prior paramaters accounted accounted for below

prior_lambdas = rgamma(prior_sim, shape = prior$a, rate = prior$b)
prior_mus = sapply(sqrt(1/(prior$c*prior_lambdas)),rnorm, n = 1, mean = prior$m)

# Store simulations
prior_simDF = data.frame(lambda = prior_lambdas,
                         mu = prior_mus)

prior_simDF$sdev = prior_simDF$lambda^{-0.5}

# Plot Mean against Standard Deviation

ggplot(data = prior_simDF, aes(x = mu, y = sdev)) +
  geom_point(color = "dark green") +
  xlim(-100,100) +
  ylim(0,100) +
  xlab("Mean change in IQ score (Prior)") +
  ylab("Std.dev of Change") +
  labs(title="Std.dev of Change vs Mean Change in Prior Sample")
```

```
## Warning: Removed 81 rows containing missing values (geom_point).
```



From the plot, firstly we can see that the large majority of samples are centered around a mean IQ change of about 0 and a standard deviation of about 10. This conforms well with our chosen priors, with a mean μ IQ change of 0 and choosing the hyperparameters of the gamma distribution such that the expected value of the standard deviation $E(\lambda^{-1/2})$ is 10. The samples also have a fan-shaped pattern, where the means are centered around 0 and become more sparse the further away from the center we get, which is consistent with our priors where the μs have a normal distribution with mean. The standard deviations are clustered much closer to the expected value of 10, and are strictly positive, which is consistent with our chosen prior gamma distribution of $\lambda^{-1/2}$ such that $a = 1/2$, a weak prior.