Proof of Normal Gamma update.

1. Write out prior $\rightarrow$ Normal Gamma

$$\mathrm{NormalGamma}(\boxed{\mu, \lambda} \mid m, c, a, b)$$

$$= N(\mu \mid m, (c\lambda)^{-1}) \; \mathrm{Gamma}(\lambda \mid a, b)$$

$$= \sqrt{\frac{c\lambda}{2\pi}} \exp\left\{-\frac{c\lambda}{2}(\mu - m)^2\right\} \times \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}$$

$$\underset{\mu,\lambda}{\propto} \lambda^{1/2 + a - 1} \exp\left\{-\frac{c\lambda}{2}(\mu - m)^2\right\} e^{-b\lambda} \qquad (6)$$

$$\underset{\mu,\lambda}{\propto} \lambda^{a - \frac{1}{2}} \exp\left\{-\frac{\lambda}{2}\left[c\mu^2 - 2cm\mu + cm^2 + 2b\right]\right\}$$

2. Write out the Normal likelihood

$$N(x \mid \mu, \lambda^{-1}) = \sqrt{\frac{\lambda}{2\pi}} \exp\left\{-\frac{\lambda}{2}(x - \mu)^2\right\}$$

$$\underset{\mu,\lambda}{\propto} \underset{\lambda^{n/2}}{\lambda^{1/2}} \exp\left\{-\frac{\lambda}{2}\left(\underset{n}{\mu^2} - \underset{\sum x_i}{2x\mu} + \underset{\sum x_i^2}{x^2}\right)\right\} \qquad (7)$$

3. Put 1. + 2. together (remember to use all ~~the~~ data pts)

$\nearrow$ eqn(6) $\qquad\qquad$ $\nearrow$ eqn(7) for $x_{1:n}$

$$p(\mu, \lambda \mid x_{1:n}) \underset{\mu,\lambda}{\propto} \underline{p(\mu,\lambda)}\ \underline{p(x_{1:n} \mid \mu, \lambda)}$$

$$\underset{\mu,\lambda}{\propto} \quad \lambda^{a-\frac{1}{2}} \exp\left\{-\frac{\lambda}{2}\left[c\mu^2 - 2cm\mu + cm^2 + 2b\right]\right\}$$

$$\times \quad \lambda^{n/2} \exp\left\{-\frac{\lambda}{2}\left[n\mu^2 - 2\sum x_i \mu + \sum x_i^2\right]\right\}$$

$$= \lambda^{\boxed{a+n/2}-1/2} \exp\left\{-\frac{\lambda}{2}\left[\mu^2 \overbrace{\boxed{(c+n)}}^{C} - 2\overbrace{\boxed{(cm+\sum x_i)}}^{CM}\mu + \underbrace{\boxed{cm^2 + 2b + \sum x_i^2}}_{CM^2 + 2B}\right]\right\}$$

$A = a + n/2$

$A = a + \frac{n}{2}$ ✓

$C = c + n$ ✓

1. $CM = cm + \sum x_i$
2. $CM^2 + 2B = cm^2 + 2b + \sum x_i^2$

$\Big\}$ solve for M and B

(exercise)

$$M = \frac{cm + \sum x_i}{c+n}$$

$$B = b + \frac{1}{2}\left(cm^2 + CM^2 + \sum x_i^2\right).$$

$$\Rightarrow (\mu, \lambda \mid x_{1:n}) \sim \text{NormalGamma}(M, C, A, B).$$