Review Exam II, Fall 2021

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Problem 4 (Exam 2, 2020), part a

$$
\begin{align}
& p(y_{1:n} \mid z_i, \theta_i, \sigma_i^2) \tag{1} \\
& \propto \prod_i \{N(y_i \mid \theta_1, \sigma_1^2) I(Z_i = 0) + N(y_i \mid \theta_2, \sigma_2^2) I(Z_i = 1)\} \tag{2} \\
& \propto \prod_i \{N(y_i \mid \theta_1, \sigma_1^2) p + N(y_i \mid \theta_2, \sigma_2^2)(1 - p)\} \tag{3} \\
& \propto \prod_{i=1}^{n} \{N(y_i \mid \theta_{Z_i}, \sigma_{Z_i}^2) P(Z_i \mid p)\} \tag{4}
\end{align}
$$

Problem 4 (Exam 2, 2020), part b

Using a latent variable approach, as illustrated in class, this will be easier to sample from as we should be able to sample from the full conditional distributions.

Problem 4 (Exam 2, 2020), part c

Derive the joint posterior distribution.

Problem 4 (Exam 2, 2020), part c

$$
\begin{aligned}
&p(p, \theta_1, \theta_2, \sigma_1^2, \sigma_2^2, Z, p \mid y_{1:n}) && (5)\\
\propto &p(p, \theta_1, \theta_2, \sigma_1^2, \sigma_2^2, Z, y_{1:n}) && (6)\\
&\propto p(y_{1:n} \mid z_i, \theta_i, \sigma_i^2) && (7)\\
&p(\theta_1 \mid \mu, \tau^2) p(\theta_2 \mid \mu, \tau^2) && (8)\\
&p(\sigma_1^2 \mid v/2, v\gamma^2/2) p(\sigma_2^2 \mid v/2, v\gamma^2/2) && (9)\\
&p(p \mid a, b) \times p(Z_i \mid p) && (10)
\end{aligned}
$$

Problem 4 (Exam 2, 2020), part c (alternative way)

$$
\begin{aligned}
& p(p, \theta_1, \theta_2, \sigma_1^2, \sigma_2^2, Z, p \mid y_{1:n}) && (11) \\
\propto & \prod_{i=1}^{n} \{ N(y_i \mid \theta_{Z_i}, \sigma_{Z_i^2}) P(Z_i \mid p) \} && (12) \\
& \prod_{j=1}^{2} \{ N(\theta_j \mid \mu, \tau^2) \times IG(\sigma_j^2 \mid v/2, v\gamma^2/2) \} && (13) \\
& \times \text{Beta}(p \mid a, b) && (14) \\
& \times \text{Bernoulli}(Z_i \mid p) && (15)
\end{aligned}
$$

Problem 4, part d (Full conditionals)

$$
\begin{aligned}
p(p \mid z) \propto & p(p, \theta_1, \theta_2, \sigma_1^2, \sigma_2^2, Z, p \mid y_{1:n}) && (16)\\
\propto & p(p, \theta_1, \theta_2, \sigma_1^2, \sigma_2^2, Z, y_{1:n}) && (17)\\
& \propto p(y_{1:n} \mid z_i, \theta_i, \sigma_i^2) && (18)\\
& p(\theta_1 \mid \mu, \tau^2) p(\theta_2 \mid \mu, \tau^2) && (19)\\
& p(\sigma_1^2 \mid v/2, v\gamma^2/2) p(\sigma_2^2 \mid v/2, v\gamma^2/2) && (20)\\
& p(p \mid a, b) && (21)\\
& \propto \prod_{i=1}^{n} \{N(y_i \mid \theta_{Z_i}, \sigma_{Z_i^2}) P(Z_i \mid p)\} && (22)\\
& \prod_{j=1}^{2} \{N(\theta_j \mid \mu, \tau^2) \times IG(\sigma_j^2 \mid v/2, v\gamma^2/2)\} && (23)\\
& \times \text{Beta}(p \mid a, b) \times \times \text{Bernoulli}(Z_i \mid p) && (24)
\end{aligned}
$$

Problem 4, part d (Full conditionals)

What is relevant here and what is not relevant?

Goal: We want to identify terms that we can drop that do not depend on our random variable. We repeat this process again and again!

Problem 4, part d (Full conditionals)

$$
\begin{aligned}
p(p \mid z_i) \propto \prod_{i=1}^{n} \{N(y_i \mid \theta_{Z_i}, \sigma_{Z_i^2}) P(Z_i \mid p)\} \quad & (25) \\
\prod_{j=1}^{2} \{N(\theta_j \mid \mu, \tau^2) \times IG(\sigma_j^2 \mid v/2, v\gamma^2/2)\} \quad & (26) \\
\times \, \mathrm{Beta}(p \mid a, b) \times Bernoulli(Z_i \mid p) \quad & (27) \\
\propto \mathrm{Bernoulli}(Z_i \mid p) \times \mathrm{Beta}(p \mid a, b) \quad & (28)
\end{aligned}
$$

This will be an updated Beta distribution just like in Module 7, part III of the in class notes (slide 18).

Problem 4, part d (Full conditionals)

Calculating the other full conditionals is similar. Let's why this is true.

Problem 4, part d (Full conditionals)

$$
\begin{aligned}
p(\theta_j \mid \theta_{-j}, y_{1:n}, z, p) &\propto \prod_{i=1}^{n} \{N(y_i \mid \theta_{Z_i}, \sigma_{Z_i^2}) P(Z_i \mid p)\} && (29) \\
&\prod_{j=1}^{2} \{N(\theta_j \mid \mu, \tau^2) \times IG(\sigma_j^2 \mid v/2, v\gamma^2/2)\} && (30) \\
&\times \text{Beta}(p \mid a, b) \times \text{Bernoulli}(Z_i \mid p) && (31) \\
&\propto \prod_{i:Z_i=j} \{N(y_i \mid \theta_j, \sigma_j^2) N(\theta_j \mid \mu, \tau^2)\} && (32) \\
& && (33)
\end{aligned}
$$

This leads to an updated normal-normal as we did in Module 3, expect we have two of them.