Deep-learning based numerical BSDE method for barrier options

Bing Yu[*], Xiaojing Xing[†], Agus Sudjianto[‡]

April 15, 2019

Abstract

As is known, an option price is a solution to a certain partial differential equation (PDE) with terminal conditions (payoff functions). There is a close association between the solution of PDE and the solution of a backward stochastic differential equation (BSDE). We can either solve the PDE to obtain option prices or solve its associated BSDE. Recently a deep learning technique has been applied to solve option prices using the BSDE approach. In this approach, deep learning is used to learn some deterministic functions, which are used in solving the BSDE with terminal conditions. In this paper, we extend the deep-learning technique to solve a PDE with both terminal and boundary conditions. In particular, we will employ the technique to solve barrier options using Brownian motion bridges.

1   Introduction

A barrier option is a type of derivative where the payoff depends on whether the underlying asset has breached a predetermined barrier price. For a simple barrier case, an analytical pricing formula is available (see [1]). Because barrier options have additional conditions built in, they tend to have cheaper premiums than comparable options without barriers. Therefore, if a trader believes the barrier is unlikely to be reached, they may prefer to buy a knock-out barrier option for a lower premium. There are different methods to solve option prices, ranging from an analytical solution, solving PDE numerically, and Monte Carlo simulations. Recently, a different approach using machine learning has been proposed.

Using machine learning to solve PDE was studied in [2]. In this work, a new method was proposed for solving parabolic partial differential equations with terminal conditions, which we will call the standard framework hereafter. In this

[*]Corporate Model Risk, Wells Fargo, bing.yu@wellsfargo.com

[†]Corporate Model Risk, Wells Fargo

[‡]Corporate Model Risk, Wells Fargo

1

new method, the PDE is formulated as a stochastic control problem through a Feymann-Kac formula. In this formulation, a connection between PDE for option prices and BSDE is made. The option price is obtained by solving the BSDE rather than solving PDE. The solution to the BSDE is represented by two deterministic functions. One innovation (shown in [2]) is the use of a neural network and deep-learning technique to learn these deterministic functions. The mathematical foundation of this approach is based on a Kolmogorov-Arnold representation theorem. This theorem states that any continuous function can be approximated by a finite composition of continuous functions of a single variable. Cybenko (see[3]) found that a feed-forward neural network is a natural realization of the theorem and he provided a concrete implementation using a sigmoid function.

In addition to [2], [4] extended the method to solve fully non-linear PDE and second-order backward stochastic differential equation. Other works related to this deep-learning method include [5] and [6]. In [5], a different way of simulating the processes in the forward-backward stochastic differential equation (FBSDE) is proposed. Rather than using a neural network to approximate the derivative of a PDE solution, the network is used to directly approximate the PDE solution and the derivative is calculated using automatic differentiation. A number of different choices for building the neural network and learning structure and two new types of structures are proposed in [6] . These problems are in the framework of a PDE with some terminal conditions. These PDE can be solved by an equivalent BSDE.

In the aforementioned standard framework, the PDE solved has no boundary conditions. There are some works on PDE with free boundary conditions. In these works, a BSDE is replaced by a reflected BSDE (RBSDE). A penalty term is added to the loss function to take into account the free boundary condition in order to solve the RBSDE. Again, machine learning can be used in solving these problems. This approach is used in [7] to solve American options. Bermudan Swaptions is solved by exercising the option at a boundary in [8]. In our work, we consider barrier options. We treat boundary conditions of barrier options differently. Rather than using RBSDE with a penalty function or exercise options at a boundary, we incorporated the boundary conditions as terminal conditions. To our best knowledge, this approach has not previously been done.

In this paper, we organize as follows. In section 2, we present the standard framework, which is designed for Cauchy problem. In section 3, we describe how we extend the standard framework to handle barrier options, which corresponds to a Cauchy-Dirichlet problem. In section 4, we present numerical considerations and the results we obtained from our experiments. Finally, we make some concluding remarks in section 5.

2 Basic method to solve BSDE by machine learning

We briefly introduced the deep-learning-based numerical BSDE algorithm proposed in [2]. We start from an FBSDE, which is first proposed in [9].

$$X_t = X_0 + \int_0^t b_s(X_s)ds + \int_0^t \sigma_s(X_s)dW_s$$

$$Y_t = h(X_T) + \int_t^T f_s(X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

Here, $\{W_s\}_{0<s<T}$ is a Brownian motion and $h(X_T)$ is the terminal condition. The pair $(Y, Z)_{0<t<T}$ solves the BSDE. It is known that there exists a deterministic function $u = u(t, x)$ such that $Y_t = u(t, X_t)$, $Z_t = \nabla u(t, X_t)\sigma_t(X_t)$ and $u(t, x)$ solves a quasi-linear PDE. For both the forward and backward process, we can use Euler scheme to approximate:

$$X_{t_{i+1}} \approx X_{t_i} + b_{t_i}(X_{t_i})(t_{i+1} - t_i) + \sigma_{t_i}(X_{t_i})(W_{t_{i+1}} - W_{t_i}) \tag{1}$$

$$Y_{t_{i+1}} \approx Y_{t_i} - f_{t_i}(X_{t_i}, Y_{t_i}, Z_{t_i})(t_{i+1} - t_i) + Z_{t_i}(W_{t_{i+1}} - W_{t_i}) \tag{2}$$

Note that we have made the backward process to be forward; this is a commonly used technique in treating FBSDEs. This set of equations has the following interpretation on a given path: X_{t_i} is the underlying price; Y_{t_i} is the option price and Z_{t_i} is related to the delta at time t_i.

In the deep-learning-based numerical BSDE algorithm, a neural network structure is used to approximate the term Z_{t_i} at each time step with parameter θ. Starting from an underlying price X_0 at time 0 and an initial guess Y_0, Z_0, we use equations (1) and (2) to calculate $X_{t_{i+1}}$ and $Y_{t_{i+1}}$ at every time step until the terminal time T. At terminal time, the loss is given by $l(\theta, Y_0, Z_0) = E[(Y_T - h(X_T))^2]$. A stochastic gradient descent method is used to minimize the loss function by iterating to the optimal value of Y_0, Z_0 and θ. Note that the success of this idea relies on the fact that the neural network can approximate non-linear function $Z_t(X_t)$ well; this is guaranteed by the work of Cybenko (see [3]).

This framework solves a PDE with Cauchy conditions (standard framework). However, there are certain types of options that will correspond to a PDE with Cauchy-Dirichlet conditions. For example, a barrier option is a Cauchy-Dirichlet PDE problem. In this paper, we would consider extending the standard framework to handle this case.

3   Extension of basic method to solve barrier option

Barrier options are options where the payoff depends on whether the underlying asset's price reaches a certain level during a certain period of time. These barrier options can be classified as either knock-out options or knock-in options. A knock-out option ceases to exist when the underlying asset price reaches a certain barrier. A knock-in option comes into existence only when the underlying asset price reaches a barrier. An up-and-out call is a regular call option that ceases to exist if the asset price reaches a barrier level, B, that is higher than the current asset price. An up-and-in call is a regular call option that comes into existence only if the barrier is reached. The down-in and down-out option are similarly defined. Under the Black-Scholes Framework, assuming constant coefficient, it is not hard to derive an analytical solution for these kinds of barrier options. Therefore, we will use an analytical solution as the benchmark.

We would like to start from the most general form of a Cauchy-Dirichlet problem. As is well known, the Feynman-Kac formula has provided a way of translating the problem of a partial differential equation into a probabilistic problem. A Dirichlet condition needs to be translated in a probabilistic way by stopping the underlying diffusion process as it exits from a domain.

Theorem 1. *(See [10] Chapter 4). Let W be a Brownian motion. Assume process X_t satisfies:*

$$X_t = X_0 + \int_0^t b_s(X_s)ds + \int_0^t \sigma_s(X_s)dW_s$$

where b and σ satisfy some usual regularity and boundedness conditions. Let D be a bounded domain in real space and define $\tau^{t,x} = inf\{s > t, X_s^{t,x} \notin D\}$ as the first exit time from domain D by process X started from (t,x). Assume the boundary ∂D is smooth. Assume functions $r, g : [0,T] \times \bar{D} \to R$ are continuous. Then the solution $u(t,x)$ of class $C^{1,2}$ of the following PDE

$$\begin{cases} \partial_t u(t,x) + b(t,x)\partial_x u(t,x) + \frac{1}{2}\sigma(t,x)^2 \partial_{xx}u(t,x) - r(t,x)u(t,x) = 0, & t < T, x \in D \\ u(T,x) = g(T,x) & x \in \bar{D} \\ u(t,x) = g(t,x) & (t,x) \in [0,T] \times \partial D \end{cases}$$
$$(3)$$

can be expressed by the following probabilistic representation

$$u(t,x) = E[g(\tau^{t,x} \wedge T, X_{\tau^{t,x} \wedge T}^{t,x})e^{-\int_t^{\tau^{t,x} \wedge T} r(s, X_s^{t,x})ds}].$$

Remark 1. The domain D is assumed to be bounded. In fact, it is enough for the boundary to be compact.

$u(t,x)$ is an average over all paths start at (t,x). If the path never exits the domain D (i.e., $\tau^{t,x} \geq T$), we use the value at terminal $g(T, X_T^{t,x})$. If the path

exits the domain (i.e., $\tau^{t,x} < T$), we use the boundary value at the point the path exits, (i.e., $g(\tau^{t,x}, X_{\tau^{t,x}}^{t,x})$). We can write it this way:

$$
\begin{aligned}
u(t,x) &= E[g(T, X_T^{t,x})e^{-\int_t^T r(s,X_s^{t,x})ds}|\tau^{t,x} \geq T]P(\tau^{t,x} \geq T) \\
&+ E[g(\tau^{t,x}, X_{\tau^{t,x}}^{t,x})e^{-\int_t^{\tau^{t,x}} r(s,X_s^{t,x})ds}|\tau^{t,x} < T]P(\tau^{t,x} < T). \quad (4)
\end{aligned}
$$

An up-out call barrier option is a special case of the above general case with domain, terminal condition, and boundary condition specifically defined. In up-out call option pricing, domain is $D = \{x < B\}$, terminal condition is $g(T,x) = (x-K)^+ 1_{\{x<B\}}$, and boundary condition is $g(t,x) = 0$ when $x \geq B$. We can also assume constant drift b, volatility σ, and interest rate r for simplicity. Then, the second term in equation (4) can be dropped since the boundary condition is zero; the probabilistic representation then becomes:

$$
u(t,x) = E[g(T, X_T^{t,x})e^{-r(T-t)}|\tau^{t,x} \geq T]P(\tau^{t,x} \geq T) \quad (5)
$$

To calculate this probabilistic representation, we write it into conditional expectation of the terminal value $X_T^{t,x}$. By using Lemma 2 shown in the Appendix, we have:

$$
u(t,x) = E\{g(T, X_T^{t,x})e^{-r(T-t)}P(\tau^{t,x} \geq T|X_T^{t,x})\} \quad (6)
$$

The term $P(\tau^{t,x} \geq T|X_T^{t,x})$ is the probability that, given the underlying value at time t and T, the underlying process never crosses the barrier in between. We can apply a commonly used technique in dealing with the simulation of a stopped process - a Brownian motion bridge. Given the start point and end point of a geometric Brownian motion, the probability that the process never exceeds a certain level in between can be explicitly given as stated in the following Lemma.

Lemma 1. *(Brownian Motion Bridge). Assume X_t follows $dX_t/X_t = bdt + \sigma dW_t$ and define*

$$
\xi(y) = exp[-\frac{2*ln(y/X_t)*ln(y/X_{t+\Delta t})}{\sigma^2 \Delta t}],
$$

then
 (1) If $B > max(X_t, X_{t+\Delta t})$, then $P[max_{t<s<t+\Delta t}X_s < B] = 1 - \xi(B)$
 (2)If $B < min(X_t, X_{t+\Delta t})$, then $P[min_{t<s<t+\Delta t}X_s > B] = 1 - \xi(B)$

Taking the use of Lemma 1 and plugging in the terminal condition for barrier option, the probabilistic representation becomes:

$$
u(t,x) = E\{[(X_T^{t,x} - K)^+ 1_{\{X_T^{t,x}<B\}}e^{-r(T-t)}][1 - e^{-\frac{2ln(B/x)*ln(B/X_T^{t,x})}{\sigma^2(T-t)}}]\} \quad (7)
$$

The only random variable in the expectation is the underlying at terminal $X_T^{t,x}$ and, therefore, we can view this as a vanilla option with a relatively complicated payoff. Then, we can use the standard framework to solve this problem. The BSDE we use in our algorithm is:

$$X_T = x + \int_t^T bX_s ds + \int_t^T \sigma X_s dW_s$$

$$Y_t = h(X_T) - \int_t^T rY_s ds - \int_t^T Z_s dW_s$$

where $h(X) = [(X-K)^+ 1_{\{X<B\}}][1 - e^{-\frac{2ln(B/x)ln(B/X)}{\sigma^2(T-t)}}]$. The Euler scheme and loss function described in the standard framework (Section 2) can be applied. For completeness, the PDE corresponding to this BSDE is:

$$\begin{cases} \partial_t u(t,x,x_0) + b\partial_x u(t,x,x_0) + \frac{1}{2}\sigma^2 \partial_{xx} u(t,x,x_0) - ru(t,x,x_0) = 0 \\ \qquad\qquad u(T,x,x_0) = h(x,x_0) \end{cases}$$

where $h(x,x_0) = [(x-K)^+ 1_{\{x<B\}}][1 - e^{-\frac{2ln(B/x_0)ln(B/x)}{\sigma^2(T-t)}}]$. Note that the value $u(t,x_0,x_0)$ is the price we want.

4   Numerical consideration

4.1   Choosing basic neural network structure

Similar to other works mentioned before, a neural network is used to approximate the term Z_{t_i} at each time step. The structure of the neural net can significantly influence the convergence. In the original work in [2], at each time step, the parameter set θ_{t_i} are different (i.e., building N different neural networks for the N time steps):

$$Z_{t_i} = net^{\theta_{t_i}}(X_{t_i}), i = 0...N. \tag{8}$$

In our work, we choose to use a type of merged neural network, as proposed by [6]. Rather than building N neural networks, we use one neural network for all the N time steps, so θ does not need subscript t_i. We also need to add one additional dimension to the underlying; that is, the neural network also needs to take the time to maturity as an input for the merged neural network structure to work:

$$Z_{t_i} = net^{\theta}(X_{t_i}, T - t_i), i = 0...N. \tag{9}$$

Finally, we choose to use Elu as the base function and use one hidden layer with 20 neurons. Using more hidden layers or using more neurons in a hidden layer does not apparently improve the convergence. Learning rates are chosen differently in each test. All weights in the network are initialized using normal or uniform random variable without pre-training. We can achieve quite accurate

results by using this structure in most cases. However, for some isolated cases, a batch normalization technique is needed to improve the convergence of the algorithm. We discuss details on improving the convergence later.

To improve computational efficiency, we use variance discretization rather than the commonly used time discretization. The number of time steps needed will roughly depend on the total variance of the model; in our case, it is the square of volatility times maturity time, $\sigma^2 T$. For cases where volatility is small, we can take a larger time step; in cases where volatility is large, we take a smaller time step to improve accuracy. As a rule of thumb, we use the following formula to determine time step $timestep\# = max(80, \frac{\sigma^2 T}{0.025})$.

To improve efficiency further, we use different learning rates. At the beginning of the training, we use larger learning rates to search for solutions that are close to the optimal point. When solutions become relatively stable, we decrease the learning rate to zoom in on the optimal solution.

Finally, we used a stopping criteria to stop the training when a change of the last 50 iteration average price is less than a certain small amount. We also insist on a minimum iteration number that training needs to run and a maximum iteration number where the program should stop.

4.2 Test result summary

4.2.1 Test overview

We ran the numerical experiment for a wide range of inputs for an up-out call option. Namely, we vary the length of maturity, underlying price, volatility, and barrier value. For simplicity, we set the interest rate and drift to both be zero. In total, we tested 72 cases (see Table 1). For all tests, we used the tensorflow Adam optimizer for the stochastic gradient descent method. We ran these 72 cases in three different settings, as shown in Table 2. In the first setting, we used brutal force of 8,000 iterations to ensure convergence. However, some isolated results are still not converged. In the second setting, we introduced batch normalization at every layer to improve convergence. In the last setting, we only use batch normalization at the input layer. In the following section, we will discuss the results in detail.

Table 1: Barrier Option Information

Option type	Strike	Underlying	Maturity	Rate	Drift	Volatility	Barrier
Up-out call	23	17,22,27,32	0.5, 2	0	0	0.4,0.8,1.2	40,60,100

Table 2: Setting of tests

	Test 1 setting	Test 2 setting	Test 3 setting
Layers	3		
Neurons	d+20		
Base function	Elu		
Initial learning rate	0.01	0.02	0.02
LR decay factor	0.5	0.5	0.5
LR decay frequency	1,500	500	1,000
Maximum iteration	8,000	1,500	3,000
Stopping criteria	No	Price change<0.002	Price change<0.005
Minimum iteration	8,000	750	1,500
Time step	max(80,Var/0.025)		
Batch size	512		
Batch normalization	No	At every layer	At input layer

4.2.2 Test result

In this section, we present the results for 72 cases tested. In test 1, we ran 8,000 iterations. As Table 3 indicates, on average the results differ from analytical solutions by 4.5 cents and relative differences of 1.21%. In approximately 50% of cases, the pricing difference is less than a penny, while there is a relative error of 0.4%. Some isolated cases, shown in Table 4, have large differences. To improve convergence, we applied batch normalization technique.

Batch normalization technique is proposed in [11]. When we have a neural network, the change in the input distribution at each layer presents a problem because parameters need to continuously adapt to a new distribution. This is a phenomenon known as covariate shift. Eliminating internal covariate shift provides faster training and batch normalization is the mechanism to do so. Batch normalization accomplishes this via a normalization step that fixes the means and variances of input distribution. We would like to note that, when adding a batch normalization technique in tests to improve performance in our test, we add additional trainable variables; these trainable parameters within batch normalization are different among different time steps (i.e., the network will still depend on t_i).

$$Z_{t_i} = net^{\theta,\beta_{t_i}}(X_{t_i}, T - t_i), i = 0...N \tag{10}$$

Here β_{t_i} are the parameters introduced in batch normalization.

Test 2, shown in Table 3, are results of applying batch normalization at every layer. Comparing the results from test 1 and those from test 2, we can see significant improvement in both absolute differences and relative differences when batch normalization is applied. In addition, those isolated points where they failed to converge in test 1 now converge nicely, with less than 1% relative differences, as shown in Table 4.

When doing batch normalization, we can choose to apply it at every layer or we can apply it only at the input layer. Whether we apply batch normalization

at each layer or just the input layer, the results are similar, as shown in Table 3. However, computation efficiency is very different as we will show later.

Table 3: Grid test result - error statistics

Statistics	Test 1 rel	Test 2 rel	Test 3 rel	Test 1 abs	Test 2 abs	Test 3 abs
Average	1.21%	0.57%	0.56%	0.0452	0.0099	0.0086
STD	2.56%	0.54%	0.53%	0.1399	0.0116	0.0093
25% quantile	0.24%	0.21%	0.13%	0.0017	0.0017	0.0013
Median	0.41%	0.39%	0.48%	0.0043	0.0059	0.0047
75% quantile	1.04%	0.70%	0.76%	0.0174	0.0133	0.0132

Table 4: Result of isolated test cases

Maturity	Underlying	Volatility	Barrier	Test 1 rel error	Test 2 rel error
0.5	17	1.2	100	18%	0.65%
0.5	22	0.8	100	4.09%	0.34%
0.5	27	0.8	100	6.76%	0.28%
0.5	32	0.8	100	9.03%	0.39%
2	22	0.4	100	4.02%	0.18%
2	27	0.4	100	6.42%	0.15%

4.2.3 Improving the efficiency

As we have mentioned before, we can apply batch normalization at every layer or apply it only at the input layer. When we apply it at each layer, we increase the trainable variable but less iterations are required to achieve convergence. On the other hand, if we apply it only at the input layer, we have less trainable parameters but more iterations are needed to achieve convergence. However, the overall time needed to achieve convergence is less in the latter case, as shown in Table 5. The overall running time for applying batch normalization at only the input layer is almost three times faster.

Table 5: Efficiency Results

Indicator	Test 2 result	Test 3 result
Average iteration step needed	1000	1670
Time consumed per 200 training iterations	15-20s	4-5s
Approximate running time per case (including building time)	200s	75s

5 Conclusion

In this work, we solved a PDE with boundary conditions, using barrier options as a concrete example. In this problem, the diffusion domain is restricted by a barrier. By viewing the terminal condition probabilistically (i.e., including a breaching probability of barrier), we are able to recast this problem into the standard framework, namely a PDE with terminal conditions. This PDE is solved by its equivalent BSDE using a machine learning technique. We have completed extensive testing using a wide range of market conditions and achieved good results when comparing with known analytical results. In some isolated cases, the batch normalization technique is needed to improve learning.

6 Appendix

For completeness, we present a technique Lemma that was used in section 3 for transition from equation (5) to (6).

Lemma 2. *Assume X is a random variable, A is an event. Then $E[f(X)|A]P(A) = E[f(X)P(A|X)]$ for any function $f(\cdot)$.*

Proof. First starting from the left side, we have $E[f(X)|A]P(A) = E[1_A E[f(X)|A]] = E[E[f(X)1_A|A]] = E[f(X)1_A]$. Then, starting from the right side, $E[f(X)P(A|X)] = E[f(X)E[1_A|X]] = E[E[f(X)1_A|X]] = E[f(X)1_A]$. We arrive at same quantity from both side; thus, the statement is proved.

References

[1] John C. Hull. *Options, Futures and other Derivatives, 9th Edition.* 2015.

[2] Weinan E, Jiequn Han, Arnulf Jantzen. Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations. *Communications in Mathematics and Statistics*, 2017.

[3] G. Cybenko. Approximation by superposition of a sigmoidal function. *Mathematics of Control, Signal and System*, 1989.

[4] Christian Beck, Weinan E, Arnulf Jantzen. Machine learning approximation algorithms for high dimensional fully nonlinear partial differential equations and second order backward stochastic differential equations. *Archive 1709.05963*, 2017.

[5] Maziar Raissi. Forward backward stochastic neural networks: Deep learning of high-dimensional partial differential equations. *Arxiv1804.07010*, 2018.

[6] Quentin Chan-Wai-Nam, Joseph Mikael, Xavier Warin. Machine learning for semi linear PDEs. *arXiv:1809.07609v1*, 2018.

[7] Masaaki Fujii, Akihiko Takahashi, Masayuki Takahashi. Asymptotic expansion as prior knowledge in deep learning method for high dimensional BSDEs. *Arxiv 1710.07030*, 2017.

[8] Haojie Wang, Han Chen, Agus Sudjianto, Richard Liu, Qi Shen. Deep learning-based BSDE solver for LIBOR market model with application to Bermudan swaption pricing and hedging. *Arxiv 1807.06622*, 2018.

[9] Etienne Pardous, Shige Peng. Adapted solution of a backward stochastic differential equation. *System & Control Letters*, 1990.

[10] Emmanuel Gobet. *Monte Carlo Method and Stochastic Processes: from Linear to non-Linear*. Chapman and Hall/CRC; 1 edition, 2016.

[11] Sergey Ioffe, Christian Szegedy. Batch normalization: Accelerating deep network training by reducing internal covariate shift. *Proceedings of Machine Learning Research*, 2015.

□