Lecture 5

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Summary

- Last week we learned:
  - Runge Kutta.
  - Second Order v first order method.
  - log log plots.
- Today we will:
  - Error analysis.
    - Local truncation error.
    - Global error.
  - Stability analysis.

Introduction

We want to be able to evaluate the error involved in approximating equations of the form:

$$\frac{dy}{dt} = f(t, y)$$
$$y(t_0) = y_0$$

Here t_0 is initial time, and y_0 is the initial value. How do we analyze our results?

Approximation with Taylor Series

We want to approximate $y(t+\Delta t)$ using talyor series. We don't know the solution to the ODE but this will give us insight into our approximation.

$$y(t+\Delta t) = y(t) + \Delta t \dot{y}(t) + \ldots \Delta t^n y^{(n)}(\tau)$$
$$t \leq \tau \leq t + \Delta t$$

This is the truncated Taylor series for the function $y(t+\Delta t)$. Note: all of this is analytical and continuous.

Continued

Now we want to use the taylor series to solve for (t)

$$\dot{y}(t) = \frac{y(t+\Delta t) - y(t)}{\Delta t} + \mathcal{O}(\Delta t)$$

Note that there is equality here and that all this is continuous. Now we plug this into our ODE equation and solve for $y(t+\Delta t)$.

$$y(t+\Delta t) = y(t) + \Delta t f(t, y(t)) + \mathcal{O}(\Delta t)$$

Local Truncation Error

We can now take a difference of the actual solution and our approximation and it becomes clear to see:

$$\frac{y(t + \Delta t) - y(t)}{\Delta t} - f(t, y(t)) = \mathcal{O}(\Delta t)$$

We can now say that at time t_{n+1} our exact solution to our equation can be written as:

$$y(t_{n+1}) = y(t_n) + \Delta t f(t_n, y(t_n)) + \Delta t LTE$$

Compare to Numerical Approximation

We now want to compare this to Euler method (now for the first time we are discussing numerical approximations: $y(t_n) \approx y_n$
Recall:

$$y_{n+1} = y_n + \Delta t f(t_n, y_n)$$

Note: $y(t_n) \approx y_n$. We can now try and evaluate (global error):

$$\begin{aligned} |y(t_{n+1} - y_{n+t}| &= |y(t_n) - y_n + \Delta t(f(t_n, y(t_n)) - f(t_n, y_n)) + \Delta t LTE \\ &\leq |y(t_n) - y_n| + \Delta t|f(t_n, y(t_n)) - f(t_n, y_n)| + \Delta t LTE \\ &\leq |y(t_n) - y_n| + \Delta t \mathcal{L}|y(t_n) - y_n| + \Delta t LET \end{aligned}$$

Expression for Global Error

We can now define an expression for global error:

$e_n = |y(t_n) - y_n|$

$$\begin{aligned} e_{n+1} &= e_n(1 + \Delta t\mathcal{L}) + \Delta tLET \\ &= e^{\mathcal{L}T}(\frac{\mathcal{M}}{s\mathcal{L}}\Delta t) = \mathcal{O}(\Delta t) \end{aligned}$$

This means that forward Euler is a first order global method!
But what does this say about the local error?

Local Truncation Error

We can evaluate the error generated by our method in one step as follows, take the difference:

$$
\begin{aligned}
|y_1 - y(t + \Delta t)| &= ? \\
&= |y_0 + \Delta t f(t_0, y_0) - (y(t_0) + \Delta t(t_0) + \mathcal{O}(\Delta t^2) \\
&= \mathcal{O}(\Delta t^2)
\end{aligned}
$$

Locally the error is second order, but then what am I defining in my example code as error?

Norms

A norm is a function, f which acts on a vector space $\mathcal{V}$ and has the following properties: For $\mathbf{u},\mathbf{v} \epsilon \mathcal{V}$

$$f(a\mathbf{u}) = |a| f(\mathbf{u})$$
$$f(\mathbf{u}+\mathbf{v}) \leq f(\mathbf{u}) + f(\mathbf{v})$$
$$f(\mathbf{v}) = 0 \rightarrow \mathbf{v} = 0$$

Now what I'm having you do is the Infinity Norm:

$$\| \mathbf{x} \|_{inf} = max(|x_1|, |x_2|, \ldots, |x_n|)$$