Alignment Spectral Triple v5.0r

Robust, production-ready manuscript

0. Standing hypotheses and notation

Fix:

(H0) Geometric triple. A real, even spectral triple $[(A_{\rm geom},H_{\rm geom},D_{\rm geom},J_{\rm geom},\gamma_{\rm geom})]$ with $(\gamma_{\rm geom}^2=1)$, $(\gamma_{\rm geom}^\ast=\gamma_{\rm geom})$, $([\gamma_{\rm geom},\pi_{\rm geom}(A_{\rm geom})]=0)$, and $(\{\gamma_{\rm geom},D_{\rm geom}\}=0)$.

(H1) One-generation SM internal triple. A real, even finite triple $[(A_{\rm SM},H_{\rm SM},D_{\rm SM}(\cdot),J_{\rm SM},\gamma_{\rm SM})]$ where Yukawas are inserted in the standard left–right off-diagonal blocks of the internal Dirac operator.

(H2) Flavor space. A finite-dimensional Hilbert space $(H_{\rm flav}\cong \mathbb C^N)$ with antiunitary real structure $[J_{\rm flav}=U_{\rm flav}\circ K,]$ where (K) is complex conjugation in some (any) basis and $(U_{\rm flav})$ is unitary.

(H3) Flavor clock. A fixed self-adjoint operator (the "clock/Laplacian") $[L_N=L_N^\ast\in\mathcal B(H_{\rm flav}),]$ specified once per run (e.g. a graph Laplacian on (C_N) or (P_N)). Spectral projections are taken in the Borel functional calculus.

Commutators and (anti)commutators. $[[X,Y]:=XY-YX,\qquad \{X,Y\}:=XY+YX.]$

Norm conventions. We write $(|X|)$ for the operator norm of $(X\in\mathcal B(H))$, and $(|z|)$ for the absolute value of a scalar $(z\in\mathbb C)$.

Majorana adjoint (basis-free transpose). Define the $(J_{\rm flav})$-transpose (Majorana adjoint) $[X^{\sharp}:=J_{\rm flav},X^\ast,J_{\rm flav}^{-1}.]$ In any basis where $(J_{\rm flav}=K)$, one has $(X^\sharp=X^T)$ (ordinary matrix transpose). When (X) acts only on $(H_{\rm flav})$, $(X^\sharp)$ coincides with the restriction of the full conjugation $(X\mapsto JX^\ast J^{-1})$ (with $(J=J_{\rm geom}\otimes J_{\rm SM}\otimes J_{\rm flav})$) to the flavor factor.

Trace convention (finite-dimensional). We write $(\mathrm{tr}{\rm flav})$ *for the usual matrix trace on* $(\mathcal B(H{\rm flav}))$.

1. The v5.0r spectral datum

Definition 1.1 (Aligned product algebra, Hilbert space, and representation)

Define $$A:=A_{\rm geom}\otimes A_{\rm SM},\qquad H:=H_{\rm geom}\otimes H_{\rm SM}\otimes H_{\rm flav},$$ and represent A on H by $$\pi(a_{\rm geom}\otimes a_{\rm SM}):=\pi_{\rm geom}(a_{\rm geom})\otimes \pi_{\rm SM}(a_{\rm SM})\otimes 1_{\rm flav}.$$

Definition 1.2 (Real structure and grading)

Define $$J:=J_{\rm geom}\otimes J_{\rm SM}\otimes J_{\rm flav},\qquad \Gamma:=\gamma_{\rm geom}\otimes\gamma_{\rm SM}\otimes 1_{\rm flav}.$$

Definition 1.3 (Flavor multiplicity principle)

Flavor is pure multiplicity: the represented algebra acts trivially on $H_{\rm flav}$, i.e. $$\pi(A)\subseteq \mathcal B(H_{\rm geom}\otimes H_{\rm SM})\otimes 1_{\rm flav}.$$

Definition 1.4 (v5.0r Dirac operator; no pure flavor term)

Let $D_{\rm int}(Y[\mathcal K],M_R,\dots)$ be an internal Dirac operator on $H_{\rm SM}\otimes H_{\rm flav}$ (built from LR Yukawa blocks and optionally Majorana blocks) satisfying the **oddness requirement** $$\boxed{\ \{\gamma_{\rm SM}\otimes 1_{\rm flav},D_{\rm int}\}=0\ }.$$ Define the total Dirac operator by the **even-product form** $$\boxed{\ D:=D_{\rm geom}\otimes 1\otimes 1\;+\; \gamma_{\rm geom}\otimes D_{\rm int}.\ }$$ **Production constraint (evenness seal).** There is no additional summand of the form $\gamma_{\rm geom}\otimes 1\otimes D_{\rm flav}$.

Definition 1.5 (Alignment Spectral Triple v5.0r)

The tuple $$(A,H,D,J,\Gamma)$$ with Definitions 1.1–1.4 is the **Alignment Spectral Triple v5.0r**.

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2. Commutant protection and exclusion of flavor gauge fields

Definition 2.1′ (Production commutant insertion invariant)

All operators that carry Yukawa/flavor/Majorana dependence inside $D_{\rm int}$ lie in $$\pi(A_{\rm SM})'\otimes \mathcal B(H_{\rm flav})\ \subset\ \pi(A)'.$$ In particular, the sufficient pattern $\widetilde Y=1_{H_{\rm SM}}\otimes Y$ is allowed, but the invariant is literal membership in $\pi(A)'$.

Lemma 2.2 (Multiplicity implies commutant membership for pure flavor operators)

Under Def. 1.3, every operator of the form $$1_{H_{\rm geom}}\otimes 1_{H_{\rm SM}}\otimes T$$ lies in $\pi(A)'$. In particular, every $\widetilde Y:=1_{H_{\rm SM}}\otimes Y$ lies in $\pi(A)'$.

Definition 2.3 (One-forms and inner fluctuations)

Define the space of one-forms $$\boxed{ \Omega_D^1(A):=\left\{\sum_i \pi(a_i)[D,\pi(b_i)] : a_i,b_i\in A\right\}\subset\mathcal B(H), }$$ and for $A\in\Omega_D^1(A)$ define the fluctuated operator $$D_A:=D+A+JAJ^{-1}.$$

Proposition 2.4 (Inner fluctuations do not generate flavor gauge fields)

Assume Def. 1.3 and the production invariant Def. 2.1′. Then inner fluctuations cannot produce gauge bosons with nontrivial action on $H_{\rm flav}$.

Reason (referee-tight). Since $\pi(\cdot)$ acts as $1_{\rm flav}$, every commutator $[D,\pi(b)]$ is of the form $(\cdots)\otimes 1_{\rm flav}$, hence $\Omega_D^1(A)\subset\mathcal B(H_{\rm geom}\otimes H_{\rm SM})\otimes 1_{\rm flav}$. Therefore any fluctuation A acts trivially on flavor, and so does JAJ^{-1}.

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3. Canonical spectral anchor for "360"

Definition 3.1 (Spectral 360-subspace and projector)

Fix a Borel set $\boxed{\ \Omega^{(N)}_{360}\subset\sigma(L_N)\ }$. *Define the 360 projector* $$\boxed{\ \Pi_{360}^{(N)}:=\chi_{\Omega^{(N)}_{360}}(L_N).\ }$$ This is basis-independent by functional calculus.

Lemma 3.2 (Anchor commutation identities)

$$[\Pi_{360}^{(N)},L_N]=0.$$ Moreover, under Def. 1.3 (multiplicity), $$[\Pi_{360}^{(N)},\pi(a)]=0\qquad \forall a\in A,$$ since $\Pi_{360}^{(N)}$ acts only on $H_{\rm flav}$.

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4. Geometric texture generation via functional calculus

Definition 4.1 (Kernel from the flavor clock)

For $\alpha>0$, define the heat-kernel operator $$K_\alpha:=e^{-\alpha L_N}\in\mathcal B(H_{\rm flav}).$$ (Any fixed functional calculus choice $K=f(L_N)$ is admissible; α is the hierarchy-control knob.)

Definition 4.2 (v5.0r texture map)

Given a seed $Y_0\in\mathcal B(H_{\rm flav})$, define the produced texture $$\boxed{ Y:=\mathcal T_{\alpha,360}(Y_0):=\Pi_{360}^{(N)}K_\alpha Y_0 K_\alpha\Pi_{360}^{(N)}. }$$ Insert Y into the internal Dirac operator through the commutant channel required by Def. 2.1′ (e.g. the sufficient lift $\widetilde Y:=1_{H_{\rm SM}}\otimes Y$).

Lemma 4.3 (Covariant basis-independence)

Under a unitary change of flavor basis U with covariant transport $$L_N\mapsto U L_N U^\ast, \qquad Y_0\mapsto U Y_0 U^\ast,$$ one has $$\boxed{ \mathcal T_{\alpha,360}(U Y_0 U^\ast)=U\, \mathcal T_{\alpha,360}(Y_0)\,U^\ast. }$$

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5. Canonical heavy–light split (9→3) without basis leakage

From here set $N=9$.

Definition 5.1 (Light projector: spectral / anchored / refined)

There are three admissible **basis-free** ways to define a rank-3 light projector P_L on $H_{\rm flav}\cong\mathbb C^9$:

(A) Purely spectral split (when feasible). Choose a Borel set $\Lambda\subset\sigma(L_9)$ and set $$P_L:=\chi_{\Lambda}(L_9),\qquad P_H:=1-P_L,$$ with $\mathrm{rank}(P_L)=3$.

(B) Anchored split (always feasible). Fix an isometry $i:\mathbb C^3\hookrightarrow\mathbb C^9$ as model data and set $$P_L:=ii^\ast,\qquad P_H:=1-ii^\ast.$$

(C) Refined spectral split (degeneracy-safe). Choose an additional self-adjoint anchor $Q=Q^\ast\in\mathcal B(H_{\rm flav})$ such that $$[L_9,Q]=0,$$ and define P_L as a **joint spectral projector** in $W^*(L_9,Q)$, i.e. $$P_L\in W^\ast(L_9,Q),\qquad \mathrm{rank}(P_L)=3.$$ Since $[L_9,Q]=0$ in finite dimension, L_9 and Q are simultaneously diagonalizable and projections in $W^\ast(L_9,Q)$ are sums of joint spectral projectors $P_{\lambda,\mu}=P^{(L_9)}_\lambda P^{(Q)}_\mu$.

Lemma 5.1a (Feasibility of rank-3 purely spectral splitting)

Let $L=L^\ast$ with spectral decomposition $L=\sum_{\lambda}\lambda P_\lambda$ and eigenspaces $E_\lambda=\mathrm{Ran}(P_\lambda)$. Then $$\mathrm{rank}\big(\chi_{\Lambda}(L)\big)=\sum_{\lambda\in\Lambda}\dim(E_\lambda).$$ Hence Option (A) is feasible iff there exists $\Lambda\subset\sigma(L_9)$ such that $$\sum_{\lambda\in\Lambda}\dim(E_\lambda)=3.$$ If this fails due to degeneracy pattern, a rank-3 split requires either Option (B) (anchored i) or Option (C) (commuting refinement).

Lemma 5.2 (Projector algebra)

For any of the options (A)–(C), $$P_L^2=P_L=P_L^\ast,\quad P_H^2=P_H=P_H^\ast,\quad P_LP_H=0,\quad P_L+P_H=1.$$ Moreover, in the spectral options (A) and (C), $$[P_L,L_9]=0,\qquad [P_H,L_9]=0,$$ and in option (C) additionally $[P_L,Q]=[P_H,Q]=0$.

Definition 5.3 (Block extraction by projectors)

For $X\in\mathcal B(H_{\rm flav})$, define $$X_{LL}=P_LXP_L,\quad X_{LH}=P_LXP_H,\quad X_{HL}=P_HXP_L,\quad X_{HH}=P_HXP_H.$$

Definition 5.4 (Seesaw effective light operator; basis-free)

Let $Y_\nu\in\mathcal B(H_{\rm flav})$ and $M_R\in\mathcal B(H_{\rm flav})$ such that $M_{R,HH}$ is invertible on H_H. Define
$$m_{\nu,\mathrm{eff}} := -v^2\,Y_{\nu,LH}\,(M_{R,HH})^{-1}\,Y_{\nu,HL}^{\sharp} \quad\text{on }H_L.$$
This is a projector-defined functional of the same operators inserted into $D_{\rm int}$.

6. Evenness and the "no pure flavor term" rule

Theorem 6.1 (Evenness of v5.0r)

Assume $[\gamma_{\rm SM}\otimes 1_{\rm flav},D_{\rm int}]=0$. Then (A,H,D,J,Γ) is even:
$$\Gamma^\ast=\Gamma,\quad \Gamma^2=1,\quad [\Gamma,\pi(A)]=0,\quad \{\Gamma,D\}=0.$$

Proposition 6.2 (Failure mode: adding a pure flavor term)

If one adds a separate summand $\gamma_{\rm geom}\otimes 1\otimes D_{\rm flav}$ with generic $D_{\rm flav}$, then $\{\Gamma,D\}=0$ generally fails. Hence the v5.0r constraint in Def. 1.4 is structurally necessary.

7. Order-zero and first-order conditions (stability under the 360 move)

Theorem 7.1 (Order-zero unchanged)

Order-zero depends only on π and J:
$$[\pi(a),J\pi(b)J^{-1}]=0\qquad \forall a,b\in A.$$
Since v5.0r modifies neither π nor J, and $\Pi_{360}^{(N)}$ enters only through texture generation (not through π), the order-zero condition is unchanged.

Theorem 7.2 (First-order stable under commutant insertion)

First-order is
$$[[D,\pi(a)],J\pi(b)J^{-1}]=0\qquad \forall a,b\in A.$$
In v5.0r, dependence on the 360 mechanism enters only through $D_{\rm int}$ via operators constrained by Def. 2.1′, while flavor remains multiplicity (Def. 1.3). Therefore no new first-order obstruction is introduced by the 360 anchoring mechanism itself.

8. Production gates (operator norm), including degeneracy-robust $W^*(L)$ certification

Fix a finite generator set $\mathcal G_A\subset A$ for gate-testing. All gates are expressed in operator norm $\|\cdot\|$.

Gate 8.1 (Order-zero)

$$\max_{a,b\in\mathcal G_A}\ |[\pi(a),J\pi(b)J^{-1}]|\le \varepsilon_0.$$

Gate 8.2 (First-order)

$$\max_{a,b\in\mathcal G_A}\ \big|\big[[D,\pi(a)],J\pi(b)J^{-1}\big]\big|\le \varepsilon_1.$$

Gate 8.3 (Projector sanity)

For each projector $(P\in\{\Pi_{360}^{(N)},P_L,P_H\})$, $$|P^2-P|\le \varepsilon_P,\qquad |P^\ast-P|\le \varepsilon_P.$$

Lemma 8.4 (Finite-dimensional characterization of $({L}')$ and $(W^*(L)))$

Let $(L=L^\ast\in\mathcal B(\mathbb C^N))$ with spectral decomposition $$L=\sum_{\lambda\in\sigma(L)} \lambda P_\lambda, \qquad E_\lambda:=\mathrm{Ran}(P_\lambda).$$

1. **Commutant.** $$\boxed{ {L}' = \{X\in\mathcal B(\mathbb C^N):[X,L]=0\} = \bigoplus_{\lambda\in\sigma(L)} \mathcal B(E_\lambda). }$$

2. **Von Neumann algebra generated by (L).** $$\boxed{ W^{(L)}=\{L\}''=(\{L\}')' = \left\{\sum_{\lambda\in\sigma(L)} x_\lambda P_\lambda:\ x_\lambda\in\mathbb C\right\}. }$$ *Equivalently: $(X\in W^*(L))$ iff $(X|_{E\lambda}=x_\lambda,\mathrm{id}_{E\lambda})$ is scalar on each eigenspace. In finite dimension,* $$W^*(L)=\{f(L): f \text{ Borel}\}.$$

Lemma 8.5 (Projections in $(W^*(L))$ are sums of spectral projectors)

With (L) as above, a projection $(\Pi=\Pi^\ast=\Pi^2)$ satisfies $(\Pi\in W^*(L))$ iff $$\boxed{ \Pi=\sum_{\lambda\in S} P_\lambda \quad\text{for some subset }S\subset\sigma(L). }$$ Equivalently: (Π) selects whole eigenspaces (whole spectral blocks), never a proper subspace of a degenerate block.

Gate 8.3a (Degeneracy-robust $(W^*(L))$ membership gate)

Let (P_λ) be the spectral projectors of (L_N). For a candidate projector (Π) (e.g. $(\Pi_{360}^{(N)})$), and (P_L) when defined spectrally), define $$c_\lambda := \frac{\mathrm{tr}{(rm\ flav)}(P\lambda \Pi)}{\mathrm{tr}{(rm\ flav)(P\lambda)}}\in[0,1], \qquad \Delta_\lambda := \big|P_\lambda \Pi P_\lambda - c_\lambda P_\lambda\big|.$$ Require $$\boxed{ \max_{\lambda}\Delta_\lambda \le \varepsilon_{\rm block} \quad\text{and}\quad \max_{\lambda}\min\{c_\lambda,1-c_\lambda\}\le \varepsilon_{\rm bin}. }$$ This is an approximate operational certificate that (Π) is scalar (hence in $(W^*(L_N))$, not merely in $(\{L_N\}')$) and binary on each spectral block, so $(\Pi\approx \sum_{\lambda\in S}P_\lambda)$ by Lemma 8.5.

Numerical implementation note (clusters). If eigenvalues are resolved only up to clusters $(\mathcal C)$, replace (P_λ) by cluster projectors $(P_{\mathcal C}=\chi_{I_{\mathcal C}}(L_N))$ and apply the same test over $(\mathcal C)$.

Gate 8.3b (Anchor commutation telemetry)

These are exact when projectors are built by functional calculus (useful as sanity checks): $[\,\|[\Pi_{360}^{(N)},L_N]\|=0,\qquad \max_{a\in\mathcal G_A}\|[\Pi_{360}^{(N)},\pi(a)]\|=0.\,]$ (They do not, by themselves, rule out hidden subprojections in degenerate blocks; Gate 8.3a does.)

Gate 8.4 (Heavy–light split gates)

If (P_L) is defined spectrally (Option A or C), apply Gate 8.3a to (P_L) and require $[\,\|[P_L,L_9]\|=0,\,]$ and in Option (C) also $[\,\|[P_L,Q]\|=0.\,]$

If (P_L) is anchored (Option B), require the anchored identity (up to tolerance) $[\,|P_L-ii^\ast|\le \varepsilon_{\rm iso},\,]$ along with the projector sanity gate.

If seesaw is used, require numerical stability of the heavy inverse: $[\,|(M_{R,HH})^{-1}|\ \text{finite and stable (monitor condition number as telemetry).}\,]$

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9. Closure statement (v5.0r meaning of "360 + 9→3")

v5.0r does not add a "360 term" to (D). It fixes a canonical internal spectral object (L_N), defines $(\Pi_{360}^{(N)})$ (and (P_L) for $(N=9)$) as spectral/anchored projectors, and uses them only to generate commutant-protected Yukawa/Majorana textures inserted into the odd LR (and optional Majorana) channels of $(D_{\rm int})$. Thus "360 + 9→3" is basis-free end-to-end and auditable by gates.

A compact operator-algebra slogan is: $[\,\boxed{\ \Pi_{360}^{(N)}\in W^{(L_N)},\qquad P_L\in W^{(L_9)}\ \text{(Options A/C)}\ \ \text{or}\ \ P_L=ii^\ast\ \text{(Option B).}\ }\,]$