Functional Analysis 1

LECTURE SCRIPT

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1 Metric spaces

In this chapter we recall the basic notions of metric spaces and prove Baire's theorem and the theorem of Arzela-Ascoli. Throughout this lecture $\mathbb{K}$ will always denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. Let X be a set. Then a map $d\colon X \times X \to [0, \infty)$ is called a *metric on* X, if for all $x, y, z \in X$

$$\begin{array}{lll} (i) & d(x,y) = 0 \Leftrightarrow x = y. & \\ (ii) & d(x,y) = d(y,x). & \\ (iii) & d(x,z) \leq d(x,y) + d(y,z). & \text{(Triangle inequality)} \end{array}$$

(X, d) is then called a *metric space*, and $d(x, y)$ is referred to as the *distance* between x and y. If $Y \subseteq X$, then $d|_{Y\times Y}$ is the *induced metric on* Y.

Notice that the non-negativity of a metric already follows from

$$0 = d(x,x) \leq d(x,y) + d(y,x) = 2d(x,y).$$

Next we will give some important examples of metrics on function spaces, sequence spaces and $\mathbb{K}^n$. Also, we can define a metric on every set as the first example will show.

Example 1.2.

(1) Let X be a set and let $d\colon X \times X \to [0, \infty)$ be defined by

$$d(x,y) := \begin{cases} 1 : x \neq y \\ 0 : \text{else} \end{cases}$$

This is the so-called *discrete metric.* Hence this always defines a metric.

(2) Let X be a set and define

$$B(X) = \{f\colon X \to \mathbb{K} : f \text{ is bounded}\}.$$

Then

$$d(f,g) := \sup_{x\in X} |f(x) - g(x)|$$

is a metric on $B(X)$, the so-called *supremum metric.* Let now $X = [a, b]$ and set

$$C[a,b] = \{f\colon [a,b] \to \mathbb{K} : f \text{ continuous}\}.$$

Then

$$C[a,b] \subseteq B[a,b]$$

and hence d induces a metric on $C[a, b]$.

(3) For $1 \leq p < \infty$, let $d_p\colon \mathbb{K}^n \times \mathbb{K}^n \to [0, \infty)$ be defined by

$$d_p(x,y) := \left(\sum_{j=1}^{n} |x_j - y_j|^p \right)^{\frac{1}{p}}, \; x = (x_j)_{j\in\mathbb{N}}, \; y = (y_j)_{j\in\mathbb{N}},$$

and let $d_\infty\colon \mathbb{K}^n \times \mathbb{K}^n \to [0,\infty)$ be defined by

$$d_\infty(x,y) := \max_{1\le j\le n} |x_j - y_j|.$$

$p = 1, \infty$: This is obviously a metric
$1 < p < \infty$: Theorem 1.4 will imply that this is a metric.

(4) The spaces $(\mathbb{K}^n, d_p)$ can be generalized to "infinite-dimensional sequence spaces". For this, for $1 \le p < \infty$, set

$$\ell_p := \left\{ x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{K},\ \sum_{n=1}^\infty |x_n|^p < \infty \right\},$$

and define $d_p\colon \ell_p \times \ell_p \to [0,\infty)$ by

$$d_p(x,y) = \left(\sum_{n=1}^\infty |x_n - y_n|^p \right)^{\frac{1}{p}}.$$

This is well-defined, since by Theorem 1.4 ℓ_p is a linear space. Let further ℓ_∞ be defined by

$$\ell_\infty := \{x = (x_n)_{n\in\mathbb{N}} : (x_n)_{n\in\mathbb{N}} \text{ bounded}\}$$

and define $d_\infty\colon \ell_\infty \times \ell_\infty \to [0,\infty)$ by

$$d_\infty(x,y) := \sup_{n\in\mathbb{N}} |x_n - y_n|.$$

Then (ℓ_p, d_p), $1 \le p \le \infty$ are metric spaces, again partly proven by Theorem 1.4.

To show the triangle inequality for the ℓ_p-spaces we need another inequality, which is important in its own right. Hölder's inequality gives upper bounds on a series of products in terms of products of series.

Theorem 1.3 (Hölder's inequality)**.** *Let $1 < p < \infty$, and let $1 < q < \infty$ be defined by $q := \frac{p}{p-1}$ (hence $\frac{1}{p} + \frac{1}{q} = 1$). Then, for $x \in \ell_p$ and $y \in \ell_q$, we have*

$$\sum_{n=1}^\infty |x_n y_n| \le \left(\sum_{n=1}^\infty |x_n|^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty |y_n|^q \right)^{\frac{1}{q}}.$$

(Case $p = q = 2$: Schwarz' inequality)

Proof. Let $c = \frac{1}{p}$ and define $\varphi\colon [0,\infty) \to \mathbb{R}$ by $\varphi(t) = t^c - ct$. Then

$$\varphi'(t) = ct^{c-1} - c \text{ and } \varphi''(t) = c(c-1)t^{c-2}.$$

Thus φ has a global maximum value in $t = 1$. This implies

$$1 - c \ge t^c - ct \text{ for all } t > 0,$$

hence

$$t^c - 1 \le c(t-1). \tag{$*$}$$

Let now $a, b > 0$, and set $t = \frac{a^p}{b^q}$. Then, by $(*)$, we obtain

$$\frac{a}{b^{\frac{q}{p}}} - 1 \leq \frac{1}{p}\left(\frac{a^p}{b^q} - 1\right) \Rightarrow \frac{a}{b^{q\left(\frac{1}{p}-1\right)}} - b^q \leq \frac{1}{p}\left(a^p - b^q\right).$$

Since $1 = \frac{1}{p} + \frac{1}{q}$, this implies

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{$**$}$$

We now set

$$A := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} \text{ and } B := \left(\sum_{n=1}^{\infty} |y_n|^q\right)^{\frac{1}{q}}$$

as well as $\widetilde{x}_n := \frac{x_n}{A}$ and $\widetilde{y}_n := \frac{y_n}{B}$. WLOG, we assume $A, B > 0$. By $(**)$, we obtain

$$|\widetilde{x}_n \widetilde{y}_n| \leq \frac{1}{p}|\widetilde{x}_n|^p + \frac{1}{q}|\widetilde{y}_n|^q.$$

Hence

$$\sum_{n=1}^{\infty} |\widetilde{x}_n \widetilde{y}_n| \leq \frac{1}{p}\sum_{n=1}^{\infty} |\widetilde{x}_n|^p + \frac{1}{q}\sum_{n=1}^{\infty} |\widetilde{y}_n|^q = \frac{1}{p} + \frac{1}{q} = 1.$$

And, finally:

$$\sum_{n=1}^{\infty} |x_n y_n| \leq AB,$$

which is the assertion. □

The following Minkowski's inequality sets the ground for the triangle inequality of the metric d_p.

Theorem 1.4 (Minkowski's inequality)**.** *For $1 < p < \infty$ and $x, y \in \ell_p$,*

$$\left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}.$$

Proof. With $z_n := x_n + y_n$, we first obtain

$$|z_n|^p \leq \sum_{n=1}^{m} |x_n + y_n| \cdot |z_n|^{p-1} \leq \left(|x_n| + |y_n|\right)|z_n|^{p-1}.$$

This implies

$$\sum_{n=1}^{m} |z_n|^p \leq \sum_{n=1}^{m} |x_n| \cdot |z_n|^{p-1} + \sum_{n=1}^{m} |y_n| \cdot |z_n|^{p-1} \text{ for all } m \in \mathbb{N}.$$

By Theorem 1.3,

$$\sum_{n=1}^{m} |z_n|^p \leq \left(\sum_{n=1}^{m} |x_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{m} |z_n|^{(p-1)q}\right)^{\frac{1}{q}} + \left(\sum_{n=1}^{m} |y_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{m} |z_n|^{(p-1)q}\right)^{\frac{1}{q}}.$$

Since $(p-1)q = p$, we conclude that

$$\left(\sum_{n=1}^{m} |z_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{m} |z_n|^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{n=1}^{m} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{m} |y_n|^p\right)^{\frac{1}{p}}.$$

We now consider $m \to \infty$, which we are allowed to do since the right-hand-side converges. This proves the theorem. $\square$

The triangle inequality $d_p(u,w) \leq d_p(u,v) + d_p(v,w)$ for all $u, v, w \in \ell_p$ for the metric d_p can now directly be concluded from Theorem 1.4 by setting $x_n = u_n - v_n$ and $y_n = v_n - w_n$.

Definition 1.5. Let (X, d) be a metric space.

(1) For $x \in X$ and $r > 0$, the set $U_r(x)$ defined by

$$U_r(x) := \{y \in X : d(x,y) < r\}$$

is called the *open ball* of radius r and center x. $U \subseteq X$ is called *open*, if for each $x \in U$ there exists some $\varepsilon > 0$ such that $U_\varepsilon(x) \subseteq U$.

(2) A set $A \subseteq X$ is *closed*, if $X \backslash A$ is open. The set

$$K_r(x) := \{y \in X : d(x,y) \leq r\}, \ x \in X, \ r > 0,$$

is called the *closed ball* of radius r and center x.

(3) If $E \subseteq X$, then $x \in E$ is an *interior point of* E, if there exists some open set $U \subseteq X$ with $x \in U \subseteq E$. E is then called a *neighbourhood of* x. The set of all interior points is refered to as the *interior of* E and is denoted by $\mathring{E}$.

(4) A point $x \in X$ is called *limit point of* E if $U \cap E \neq \varnothing$ for each neighbourhood U of x. The set of all limit points of E is the *closure of* E, which is denoted by $\overline{E}$. E is *dense* in X if $\overline{E} = X$.

The openness of a set and all properties that can be defined with reference only to open sets are called topological. In particular, all terms just defined are topological. The open sets in a metric space form a system of sets called topology. This terminology will be generalized to the notion of a topological space in Chapter X. Reference!

Lemma 1.6. *Let (X, d) be a metric space.*

(i) We have

(a) $\varnothing$, X are open.

(b) $U_1, \ldots U_r \subseteq X$ open $\Rightarrow \bigcap_{i=1}^{r} U_i$ is open.

(c) $U_i \subseteq X$, $i \in I$ open $\Rightarrow \bigcup_{i \in I} U_i$ is open.

Hence d defines a topology *on X with $U_\varepsilon(x)$, $x \in X$, $\varepsilon > 0$, as basis.*

(ii) We have

(a) $\varnothing$, X are closed.

(b) $A_i \subseteq X$, $i \in I$ closed $\Rightarrow \bigcap_{i\in I} A_i$ is closed.

(c) $A_1, \dots A_r \subseteq X$ open $\Rightarrow \bigcup_{i=1}^{r} A_i$ is closed.

(iii) For each $x \in X$, $r > 0$, the set $K_r(x)$ is closed.

(iv) For $E \subseteq X$, $\overline{E}$ is the smallest closed set containing E.

(v) For $E \subseteq X$, $\mathring{E}$ is the biggest open set contained in E.

Proof. Tutorials □

The next definition generalizes the notion of convergence from $\mathbb{K}^n$ (with the Euclidian metric) to general metric spaces.

Definition 1.7. Let (X, d) be a metric space.

(1) A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ *converges* to $x \in X$ if for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ with $d(x_n, x) < \varepsilon$ for all $n \geq N_\varepsilon$. We then write $x_n \to x$, as $n \to \infty$ or $x = \lim_{n\to\infty} x_n$. x is called the *limit* of $(x_n)_{n\in\mathbb{N}}$.

(2) A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is a *Cauchy-sequence*, if for each $\varepsilon > 0$ there exists some $N_\varepsilon \in \mathbb{N}$ with

$$d(x_n, x_m) < \varepsilon \quad \text{for all } n, m \geq N_\varepsilon.$$

(3) (X, d) is *complete*, if each Cauchy-sequence in X converges.

Convergence of a sequence is a topological property. The sequence $(x_n)_{n\in\mathbb{N}}$ converges to x, if and only if every neighborhood of x contains all but finitely many elements of the sequence. In particular, the limit of a sequence is independent of the ordering of the sequence's terms. Which sequences are Cauchy-sequences does not only depend on the open sets but also on the chosen metric (see Remark 1.11).

In general, topological properties in metric spaces can be tested by sequences. We note that there is a characterization of closedness of a set by convergent sequences.

Lemma 1.8. *Let (X, d) be a metric space.*

(i) A sequence can have at most one limit.

(ii) Let $E \subseteq X$. Then $x \in \overline{E}$ if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq E$ with $x_n \to x$ as $n \to \infty$.

(iii) If $(x_n)_{n\in\mathbb{N}} \subseteq X$ is convergent, then $(x_n)_{n\in\mathbb{N}} \subseteq X$ is a Cauchy-sequence. The converse is not always true[1]. A Cauchy-sequence is convergent, if it contains a convergent subsequence.

[1] For example, consider $X = (0, 1]$, $x_n = \frac{1}{n}$.

(iv) *If X is complete and $E \subseteq X$ closed, then E is complete. If $E \subseteq X$ is complete, then E is closed in X.*

Proof. Tutorials $\square$

The following example provides the reader with some complete metric spaces.

Example 1.9.

(1) $B(X)$ is complete.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy-sequence in $B(X)$, and for $\varepsilon > 0$ let $N_\varepsilon \in \mathbb{N}$ be such that

$$d(f_n, f_m) < \varepsilon \text{ for all } n, m \geq N_\varepsilon.$$

This implies $|f_n(x) - f_m(x)| < \varepsilon$ for all $x \in X$, $n, m \geq N_\varepsilon$. Hence, for all $x \in X$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy-sequence in $\mathbb{K}$. Setting $f(x) := \lim\limits_{n\to\infty} f_n(x)$, we obtain

$$|f(x) - f_m(x)| = \lim_{n\to\infty} |f_n(x) - f_m(x)| < \varepsilon \quad \forall m \geq N_\varepsilon.$$

Hence $|f(x)| \leq |f_n(x)| + \varepsilon$, which implies $f \in B(X)$. Further, for $m \geq N_\varepsilon$,

$$d(f, f_m) = \sup_{x\in X} |f(x) - f_m(x)| < \varepsilon,$$

and thus $f = \lim\limits_{n\to\infty} f_n$. $\square$

(2) $C[a, b]$ is complete, since it is closed in $B[a, b]$ (see lemma 1.8), the reason being that a uniform limit of continuous functions is again continuous.

(3) $(\mathbb{K}^n, d_p)$, $n \in \mathbb{N}$, $1 \leq p \leq \infty$ is complete, since convergence in $\mathbb{K}^n$ w.r.t. d_p is the same as convergence in $\mathbb{K}^n$ w.r.t. the component sequences.

(4) The spaces ℓ_p, $1 \leq p \leq \infty$ are complete.

Proof. Let $(x_k)_{k\in\mathbb{N}}$ be a Cauchy-sequence in ℓ_p, $x_k = (x_{k,n})_{n\in\mathbb{N}}$, and for $\varepsilon > 0$ let $N_\varepsilon \in \mathbb{N}$ be with

$$d_p(x_k, x_l) = \left(\sum_{n=1}^{\infty} |x_{k,n} - x_{l,m}|^p\right)^{\frac{1}{p}} < \varepsilon \quad \text{and} \qquad (*)$$

$$d_\infty(x_k, x_l) = \sup_{n\in\mathbb{N}} |x_{k,n} - x_{l,n}| < \varepsilon \qquad \text{for all } k, l > N_\varepsilon \qquad (**)$$

Thus, for fixed $n \in N_\varepsilon$, $(x_{k,n})_{k\in\mathbb{N}}$ is a Cauchy-sequence in $\mathbb{K}$. Now set $y_n := \lim\limits_{k\to\infty} x_{k,n}$ and $y := (y_n)_{n\in\mathbb{N}}$. Then $y \in \ell_p$ and $y = \lim\limits_{k\to\infty} x_n$.
<u>Reason:</u> Consider $l \to \infty$, which implies (by $(*)$,$(**)$)

$$\sum_{n=1}^{m} |x_{k,n} - y_n|^p < \varepsilon^p \ \forall m \in \mathbb{N} \Rightarrow \sum_{n=1}^{\infty} |x_{k,n} - y_n|^p < \varepsilon^p \ \forall k \geq N_\varepsilon$$

and $|x_{k,n} - y_n| < \varepsilon \ \forall k \geq N_\varepsilon,\ n \in \mathbb{N}$.

Hence $x_n - y \in \ell_p$, and thus $y \in \ell_p$ and $y = \lim\limits_{k\to\infty} x_k$. $\square$

The following theorem of Baire only holds in complete metric spaces. It is a key ingredient in the proofs of the fundamental theorems of functional analysis. Thus, they will only hold under some completeness assumption.

Theorem 1.10 (Baire's theorem)**.** *Let (X, d) be a complete metric space, and let D_n, $n \in \mathbb{N}$ be open, dense subsets of X. Then also $\bigcap_{n\in\mathbb{N}} D_n$ is dense in X.*

Proof. We need to prove that for all $x \in X$ and $r > 0$ we have

$$U_r(x) \cap \bigcap_{n=1}^{\infty} D_n \neq \varnothing.$$

For this, let $x \in X$ and $r > 0$ be arbitrary, but fixed. By induction define a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $(r_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ by

a) $K_{r_{n+1}}(x_{n+1}) \subseteq D_n \cap U_{r_n}(x_n)$

b) $r_n \leq \frac{1}{n}$

This can be done as follows: First, set $x_1 = x$ and $r_1 = \min\{1, r\}$. Second, assume that $x_1, \ldots, x_n, r_1, \ldots, r_n$ be already chosen ($n \geq 1$). Since D_n is open and dense, also $D_n \cap U_{r_n}(x_n) \neq \varnothing$ is open. Hence there exists $x_{n+1} \in X$ and $r_{n+1} > 0$ with

$$U_{2r_{n+1}}(x_{n+1}) \subseteq D_n \cap U_{r_n}(x) \text{ and } r_{n+1} \leq \frac{1}{n+1}.$$

This implies a) and b), since $K_{r_{n+1}}(x_{n+1}) \subseteq U_{2r_{n+1}}(x_{n+1})$.

Having constructed sequences (x_n) and (r_n) satisfying a) and b), we obtain

$$x_n \in K_{r_n} \subseteq D_{n-1} \cap U_{r_{n-1}}(x_{n-1}) \subseteq U_{r_{n-1}} \subseteq \ldots \subseteq U_{r_m}(x_m)$$

for all $n > m$. Thus $d(x_n, x_m) < r_m \leq \frac{1}{m}$ for all $n > m$. This implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in X.

Now set $x_0 := \lim_{n\to\infty} x_n$ (remember that X is complete). Since $d(x_n, x_m) \leq r_m$ for all $n > m$, we obtain $d(x_0, x_m) \leq r_m$ for all $m \in \mathbb{N}$. Thus, finally,

$$x_0 \in \bigcap_{m=1}^{\infty} K_{r_{m+1}}(x_{m+1}) \subseteq \bigcap_{m=1}^{\infty} D_m \cap U_{r_m}(x_m) \subseteq U_{r_1}(x_1) \cap \bigcap_{m=1}^{\infty} D_m \subseteq U_r(x) \cap \bigcap_{m=1}^{\infty} D_m,$$

and the theorem is proved. □

Remark 1.11.

(a) Theorem 1.10 is in general false if X is not complete. As an example choose $X = \mathbb{Q} = \{q_1, q_2, \ldots\}$ and $D_n = X \backslash \{q_n\}$, $n \in \mathbb{N}$, which are open and dense. We immediately see that however

$$\bigcap_{n=1}^{\infty} D_n = \varnothing.$$

(b) Let (X, d) be complete and $A_n \subseteq X$, $n \in \mathbb{N}$ closed with $X = \bigcup_{n=1}^{\infty} A_n$. Then there exists at least one $n \in \mathbb{N}$ with

$$\mathring{A}_n \neq \varnothing.$$

Proof. Towards a contradiction, assume that

$$\mathring{A}_n = \varnothing \text{ for all } n \in \mathbb{N}.$$

Then $X \backslash A_n$ are open and dense for all $n \in \mathbb{N}$. By Baire's theorem 1.10, $\bigcap\limits_{n=1}^{\infty} (X \backslash A_n)$ dense in X. But $\bigcap\limits_{n=1}^{\infty} (X \backslash A_n) = X \backslash \bigcup\limits_{n=1}^{\infty} A_n = \varnothing$. $\lightning$ ☐

(c) Completeness is a property of the particular metric and <u>not</u> the convergence in X. For example, consider $X = (0,1]$, $d_1(x,y) := \left|\frac{1}{x} - \frac{1}{y}\right|$ and $d_2(x,y) = |x-y|$. Then we have

$$x_n \to x \text{ in } (X, d_1) \Leftrightarrow x_n \to x \text{ in } (X, d_2),$$

but (X, d_1) is complete and (X, d_2) is not (see tutorials).

Definition 1.12. Let (X, d) be a metric space.

(1) Let $\varepsilon > 0$. Then $M \subseteq X$ is called *ε-net*, if $X = \bigcup\limits_{x \in M} U_\varepsilon(x)$. X is called *totally bounded*, if for each $\varepsilon > 0$ there exists a finite ε-net. $A \subseteq X$ is *totally bounded*, if $(A, d|_{A \times A})$ is totally bounded.

(2) X is *compact*, if every open cover of X (that is, a family of open sets U_i, $i \in I$, such that $X = \bigcup_{i \in I} U_i$) has a finite subcover. $(A, d|_{A \times A})$ is *compact* if and only if every open cover of A (of open sets in X) has a finite subcover.

Compactness and total boundedness are intrinsic properties, that is a subset $A \subseteq (X, d)$ of some metric space is compact (totally bounded) if the metric space $(A, d|_{A \times A})$ is compact (totally bounded).

It is easy to see that every compact metric space is totally bounded. The following theorem shows that the two notions coincide for complete metric spaces. Note that this does not imply that these two properties coincide for all subsets of a complete metric space (see Corollary 1.15).

Theorem 1.13. *Let (X, d) be a metric space. Then the following are equivalent:*

(i) (X, d) is complete and totally bounded.

(ii) (X, d) is compact.

(iii) Each finite sequence in X has a convergent subsequence.

Proof. (i)$\Rightarrow$(ii). Towards a contradiction, assume that X is not compact. Let $\mathfrak{A}$ be an open cover of X which does not contain a finite subcover. By induction, we now define a sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ satisfying

a) $U_{2^{-n}}(x_n)$ is not covered by finitely many $U \subseteq \mathfrak{A}$.

b) $U_{2^{-n}}(x_n) \cap U_{2^{-(n+1)}}(x_{n-1}) \neq \varnothing$.

First, for $n = 1$, notice that X is totally bounded. Hence $X = \bigcup_{y \in M} U_{\frac{1}{2}}(y)$, $|M| < \infty$, which implies that there exists $y_{i_0} =: x_1 \in X$ such that $U_{\frac{1}{2}}(x_1)$ is not covered by finitely many $U \subseteq \mathfrak{A}$. Second $(n \to n+1)$, again by totally boundedness, there exists a finite M such that $X = \bigcap_{y \in M} U_{2^{-(n+1)}}(y)$. Assume $x_1, \dots, x_n$ are chosen such that a) and b) are satisfied. Towards a contradiction assume that for each $y \in M$ with $U_{2^{-(n+1)}}(y) \cap U_{2^{-n}}(x_n) \neq \varnothing$, the set $U_{2^{-(n+1)}}(y)$ is covered by finitely many $U \in \mathfrak{A}$. Then this is also true for $U_{2^{-n}}(x_n)$ $\lightning$. Hence there exists $x_{n+1} \in X$ with $U_{2^{-(n+1)}}(x_{n+1})$ is not covered by finitely many $U \in \mathfrak{A}$ and $U_{2^{-n}}(x_n) \cap U_{2^{-(n+1)}}(x_{n+1}) \neq \varnothing$.

For each $n \in \mathbb{N}$, let $z_n \in U_{2^{-n}}(x_n) \cap U_{2^{-(n+1)}}(x_{n+1})$. Then, for $m > n$,

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{\nu=n}^{m-1} d(x_{\nu+1}, x_\nu) \leq \sum_{\nu=n}^{m-1} \left(d(x_{\nu+1}, z_\nu) + d(z_\nu, x_\nu)\right) \\ &\leq \sum_{\nu=n}^{m-1} \left(2^{-(\nu+1)} + 2^{-\nu}\right) \leq 2 \sum_{\nu=n}^{m-1} 2^{-\nu} \leq \frac{1}{2^{n-2}}. \end{aligned}$$

This implies that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy-sequence. Since X is complete, there exists $x = \lim_{n \to \infty} x_n$.

Now choose $U \subseteq \mathfrak{A}$ with $x \in U$ and choose $\varepsilon > 0$ such that $U_\varepsilon(x) \subseteq U$. Then $(x_n) \in U_{\frac{\varepsilon}{2}}(x)$ for all $n \geq N$, hence $U_{2^{-n}}(x_n) \subseteq U$ for all $n \geq N$ with $2^{-n} < \frac{\varepsilon}{2}$ $\lightning$ to choice of $U_{2^{-n}}(x_n)$.

(ii)$\Rightarrow$(iii). Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X and set $A_n := \overline{\{x_\nu : \nu > n\}} \subseteq X$. Towards a contradiction assume that

$$\bigcap_{n \in \mathbb{N}} A_n = \varnothing.$$

This implies $\bigcup_{n \in \mathbb{N}} (X \backslash A_n) = X$. Since X is compact, the open cover $\{X \backslash A_n\}_{n \in \mathbb{N}}$ contains an open subcover $\{X \backslash A_{n_j} : 1 \leq j \leq r\}$. Since $A_{n+1} \subseteq A_n$, hence $X \backslash A_n \subseteq X \backslash A_{n+1}$, for $N := \max\{n_j : 1 \leq j \leq r\}$ we have

$$X = \bigcup_{j=1}^{r} X \backslash A_{n_j} = X \backslash A_N.$$

Thus $A_N = \varnothing$ $\lightning$.
This proves $\bigcap_{n \in \mathbb{N}} A_n \neq \varnothing$. Choosing $x \in \bigcap_{n \in \mathbb{N}} A_n$, there exists a sequence $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ with $n_{k+1} > n_k$ and $d(x_{n_k}, x) \leq \frac{1}{k}$ [if n_k is chosen, then $x \in A_{n_{k+1}}$]. This shows (iii), since $(x_{n_k})_{k \in \mathbb{N}}$ is a convergent subsequence of X.

(iii)$\Rightarrow$(i). Each Cauchy-sequence in X contains by hypothesis a convergent susequence, is hence itself convergent. This implies that X is complete.
Towards a contradiction, we now assume that X is not totally bounded. Then there exists $\varepsilon > 0$ such that X is not covered by finitely many $U_\varepsilon(x)$, $x \in X$. By induction, we define a sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ with

$$x_n \notin U_\varepsilon(x_j),\ 1 \leq j \leq n-1.$$

This can be achieved in the following way: Let $x_1 \in X$ be arbitrary. Then assume $x_1, \dots, x_n$ are already constructed. Since

$$X \backslash \bigcup_{j=1}^{n} U_\varepsilon(x_j) \neq \varnothing,$$

choose $x_{n+1} \in X \setminus \bigcup_{j=1}^{n} U_\varepsilon(x_j)$. Then, for $n \neq m$, we have

$$d(x_n, x_m) \geq \varepsilon.$$

By (iii), $(x_n)_{n\in\mathbb{N}}$ contains a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$. Let $x := \lim_{k\to\infty} x_{n_k}$. Then $d(x_{n_k}, x) < \frac{\varepsilon}{2}$ for all $k > k_0$, hence $d(x_{n_k}, x_{n_l}) < \varepsilon$ for all $k, l > k_0$ $\lightning$. □

Lemma 1.14. *Let (X, d) be a metric space, and let $A \subseteq X$, $A \neq \varnothing$.*

(i) If X is totally bounded, then also A is totally bounded.

(ii) If A is totally bounded, then also $\overline{A}$ is totally bounded.

Proof. (i). Let $\varepsilon > 0$. By hypothesis, there exists an $\frac{\varepsilon}{2}$-net $\{x_1, \ldots x_n\}$ of X. WLOG, let $A \cap U_{\frac{\varepsilon}{2}}(x_j) \neq \varnothing$ if and only if $1 \leq j \leq m$, $m \leq n$. For each $1 \leq j \leq m$, choose $y_j \in A \cap U_{\frac{\varepsilon}{2}}(x_j)$. Let $y \in A$. Then there exists $1 \leq j \leq m$ with $y \in U_{\frac{\varepsilon}{2}}(x_j)$, and hence

$$d(y, y_j) \leq d(y, x_j) + d(x_j, y_j) < \varepsilon.$$

This implies that $\{y_1, \ldots, y_n\}$ is an $\varepsilon-$net for A.

(ii). Let $\varepsilon > 0$. By hypothesis, there exists an $\frac{\varepsilon}{2}-$net $\{y_1, \ldots, y_n\}$ for A. Let $x \in \overline{A}$. Then there exists $y \in A$ with $d(x, y) < \frac{\varepsilon}{2}$. Let y_j be such that $d(y, y_j) < \frac{\varepsilon}{2}$. This yields

$$d(x, y_j) \leq d(x, y) + d(y, y_j) < \varepsilon,$$

hence $\{y_1, \ldots, y_n\}$ is an $\varepsilon-$net for $\overline{A}$. □

Corollary 1.15. *Let (X, d) be a complete metric space, and let $A \subseteq X$. Then the following are equivalent.*

(i) $\overline{A}$ is compact.

(ii) A is totally bounded.

Proof. (i)$\Rightarrow$(ii). Since $\overline{A}$ is compact, by 1.13, $\overline{A}$ is totally bounded. By Lemma 1.14, A is totally bounded.

(ii)$\Rightarrow$(i). Since A is totally bounded, by Lemma 1.14, $\overline{A}$ is totally bounded. Since X is compact, $\overline{A}$ is also complete. Hence Theorem 1.13 implies that $\overline{A}$ is compact. □

Definition 1.16. Let (X, d) and (X', d') be metric spaces, and let $f \colon X \to X'$.

(1) f is *continuous* in $x \in X$, if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon$ for all $y \in X$.

(2) f is a *homeomorphism*, if f is bijective and f and f^{-1} are both continuous. f is an *isometry*, if f is bijective and $d(x, y) = d(f(x), f(y))$ for all $x, y \in X$.

(3) f is *uniformly continuous*, if for each $\varepsilon > 0$ there exists $\delta > 0$ with $d(x, y) < \delta \Rightarrow d'(f(x), f(y)) < \varepsilon$ for all $x, y \in X$.

Two metric spaces being isometric is a strong notion of equivalence for metric spaces, being homeomorphic is the properly weaker topological equivalence of metric spaces.

Lemma 1.17. *Let (X, d) and (X', d') be metric spaces, and let $f \colon X \to X'$.*

(i) f is continous $\Leftrightarrow$ $f^{-1}(U)$ is open in X for all $U \subseteq X'$ open $\Leftrightarrow$ $f(x_n) \to f(x)$ for all $x_n \to x$ in X.

(ii) Let X be compact and f continuous. Then f is automatically uniformly continuous.

Proof. Exercises. □

We want to relate the relative compactness, that is the compactness of the closure, of a set of continuous real functions to the pointwise relative compactness of these functions. The relatively compact sets in $\mathbb{R}$ are the bounded sets by Heine-Borel theorem. If a set of continuous real functions is relatively compact, we obtain pointwise relative compactness, by continuity of $C(X) \to \mathbb{R}, f \mapsto f(x)$ for every $x \in X$. However, to prove the converse a second condition is needed: the equicontinuity of the functions.

Definition 1.18. $F \subseteq C(X)$ is *equicontinuous in* $x \in X$, if for each $\varepsilon > 0$ there exists a neighbourhood U of x with $|f(x) - f(y)| < \varepsilon$ for all $y \in U$ and $f \in F$. F is called *equicontinuous*, if it is equicontinuous in each $x \in X$.

Theorem 1.19 (Arzela-Ascoli)**.** *Let X be a compact metric space and $F \subseteq C(X)$. Then the following are equivalent.*

(i) $\overline{F}$ is compact.

(ii) F is equicontinuous and pointwise bounded.

Proof. (i)$\Rightarrow$(ii). Exercise.

(ii)$\Rightarrow$(i). Let F be equicontinuous and $F(x) \in \mathbb{K}$ bounded for all $x \in X$. Since $C(X)$ is complete, by 1.15 it remains to prove that F is totally bounded. For this, let $\varepsilon > 0$, and, for each $x \in X$, let U_x be an open neighbourhood of x with

$$|f(y) - f(x)| < \frac{\varepsilon}{2} \text{ for all } f \in F \text{ and } y \in U_x.$$

Let now $x_1, \dots, x_n \in X$ be chosen such that $X = \bigcup_{i=1}^{n} U_{x_i}$ and set

$$K := \bigcup_{i=1}^{n} f(x_i) \subseteq \mathbb{K}.$$

Since K is bounded, there exists $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ with

$$K \subseteq \bigcup_{j=1}^{n} U_{\frac{\varepsilon}{2}}(\lambda_j).$$

Define Φ to be the set of maps $\varphi\colon \{1, \dots, n\} \to \{1, \dots, m\}$. Also, for $\varphi \in \Phi$, set

$$F_\varphi := \{f \in F : |f(x_i) - \lambda_{\varphi(i)}| < \frac{\varepsilon}{3} \text{ for } 1 \le i \le n\}.$$

Then

$$F = \bigcup_{\varphi \in \Phi} F_\varphi.$$

To see this, note that for $f \in F$ and each $1 \le i \le n$, there exists $\varphi(i) \in \{1, \dots, m\}$ with $f(x_i) \in U_{\frac{\varepsilon}{6}}(\lambda_{\varphi(i)})$. Hence $f \in F_\varphi$. For $f, g \in F_\varphi$ and $y \in U_{x_i}$, we then obtain

$$|f(y) - g(y)| \le |f(y) - f(x_i)| + |f(x_i) - \lambda_{\varphi(i)}| + |\lambda_{\varphi(i)} - g(y)| \le \varepsilon.$$

Thus $d(f, g) \le \varepsilon$ for all $f, g \in F_\varphi$, and hence a finite $\varepsilon-$net does exist. □

2 Normed Spaces

Definition 2.1. Let E be a linear space over $\mathbb{K}$.

(1) Then a map $\|\cdot\|: E \to [0,\infty)$ is called a *norm an* E, and $(E, \|\cdot\|)$ a *normed space*, if for all $x, y \in E$, $\lambda \in \mathbb{K}$

  (i) $\|x\| = 0 \Leftrightarrow x = 0$.

  (ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$.

  (iii) $\|x+y\| \leq \|x\| + \|y\|$.

  E is called a *Banach space*, if $(E, d_{\|\cdot\|})$ is complete.

(2) Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent* if there exist $\alpha, \beta > 0$ such that

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1 \text{ for all } x \in E.$$

Remark 2.2. Let E be a linear space.

(1) If $\|\cdot\|$ is a norm on E, then $d_{\|\cdot\|}(x,y) = \|x-y\|$ defines a metric on E.

(2) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be equivalent. Then $(E, \|\cdot\|_1)$ is complete if and only if $(E, \|\cdot\|_2)$ is complete.

(3) $|\|x\| - \|y\|| \leq \|x-y\|$. In particular $\|\cdot\| : E \mapsto \mathbb{R}$ is Lipschitz-continous.

(4) The algebraic operations

$$\begin{aligned} &+ : E \times E \mapsto E \qquad (x,y) \mapsto x+y \\ &\cdot : \mathbb{K} \times E \mapsto E \qquad (\lambda, y) \mapsto \lambda \cdot y \end{aligned}$$

are continous:

$$\begin{aligned} &\|(x+y) - (x_0 + y_0)\| \leq \|x - x_0\| + \|y - y_0\| \\ &\|\lambda x - \lambda_0 x_0\| \leq |\lambda|\|x - x_0\| + |\lambda - \lambda_0|\|x_0\| \end{aligned}$$

(5) If $F \subset E$ is a subspace, so is $\overline{F}$

Definition 2.3 (Quotientspace)**.** Let E be a linear space, $F \subset E$. Then

$$x \sim y :\Leftrightarrow x - y \in F \qquad (x, y \in E)$$

is an equivalence relation.

$$[x]_\sim = \{y \in E : y - x \in F\} = \{y \in E : y \in x + F\} = x + F$$

So $[x]_\sim$ is an *affine subspace.*

The *quotient space* E/F is defined by

$$E/F := \{x + F : x \in E\}$$

via

$$[x]_\sim + [y]_\sim := [x+y]_\sim \qquad (x+F) + (y+F) := ((x+y)+F)$$

and

$$\lambda[x]_\sim := [\lambda x]_\sim \qquad \lambda(x+F) := ((\lambda x) + F)$$

the space E/F becomes a linear space.

Lemma 2.4. *Let $(E, \|\cdot\|)$ be a normed space and let $F \subset E$ be a* closed *subspace. Then*

$$\|x + F\| := \inf\{\|x + y\| : y \in F\}$$

defines a norm on E/F. Moreover, if E is Banach space so is E/F.

Proof. (i) Let $\|x + F\| = 0$, this implies it exists $(y_n)_{n\in\mathbb{N}} \subset F$, such that

$$\|x - y_n\| \xrightarrow{n\to\infty} 0$$

Since $y_n \in F$, F closed and $x \in F$

$$x + F = F + F = F = [0]_\sim = 0 + F = 0$$

(ii) $\|\lambda(x + F)\| = \|(\lambda x) + F\| = \inf\{\|\lambda x + y : y \in F\|\}$
For $\lambda = 0$ we have:

$$\|\lambda(x + F)\| = 0 = |\lambda|\|x + F\|$$

And for $\lambda \neq 0$:

$$\begin{aligned}\|\lambda(x + F)\| &= \inf\{\|\lambda x + y\| : y \in F\} \\ &= |\lambda| \inf\{\|x + y\| : y \in F\} \\ &= |\lambda|\|x + F\|\end{aligned}$$

(iii) Let $x, y \in E, \varepsilon > 0$. Choose z_1, z_2 such that,

$$\begin{aligned}\|x + F\| &\geq \|x + z_1\| - \frac{\varepsilon}{2} \\ \|y + F\| &\geq \|y + z_2\| - \frac{\varepsilon}{2}\end{aligned}$$

which gives us

$$\begin{aligned}\|(x + F)(y + F)\| &= \|(x + y) + F\| \\ &\leq \|x + z_1 + y + y_2\| \\ &\leq \|x + F\| + \|y + F\| + \varepsilon\end{aligned}$$

Let E be complete and let $(x_n + F)_{n\in\mathbb{N}}$ be a Cauchy-sequence in E/F, i.e.

$$\forall\varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n, m \geq N : \|(x_n - x_m) + F\| \leq \varepsilon$$

So for all $i \in \mathbb{N}$ we can find n_i, such that:

$$\|x_{n_{i+i}} - x_{n_i} + F\| \leq 2^{-i}$$

in particulat it exists $y_i \in F$ such that

$$\|x_{n_{i+1}} - x_{n_i} + y_i\| \leq 2^i$$

We may assume $n_i < n_{i+1}$. Now define

$$\begin{aligned}z_1 &:= 0 \\ z_{i+1} &:= y_i - z_i \qquad i \geq 1\end{aligned}$$

In particular we have $y_i = z_{i+1} - z_i$:

$$\|(x_{n_{i+1}} + z_{i+1}) - (x_{n_i} + z_i)\| < 2^{-i}$$

Now we define $\eta_i := x_{n_i} + z_i$, which gives us

$$\begin{aligned}
& \|\eta_{i+1} - \eta_i\| < 2^{-1} \\
\Rightarrow \quad & \|\eta_{m+k} - \eta_m\| \leq \sum_{i=0}^{k-1} \|\eta_{m+i+1} - \eta_{m+i}\| < \sum_{i=0}^{k+1} 2^{-m-1} \leq 2^{1-m} \\
\Rightarrow \quad & (\eta_n)_{n\in\mathbb{N}} \text{ is a Cauchy-sequence in } E \\
\Rightarrow \quad & (\eta_n)_{n\in\mathbb{N}} \text{ converges}
\end{aligned}$$

Now we set $\lim_{n\to\infty} \eta_n =: x$. We obtain:

$$\begin{aligned}
& \|(x_n + F) - (x + F)\| \\
= & \|(x_n - x) + F\| \\
\overset{z_i \in F}{\leq} & \|x_{n_i} + z_i - x\| \\
= & \|\eta_i - x\| \to 0
\end{aligned}$$

Which gives us a convergent subsequence, so the Cauchy-sequence is covergent itself.

□

Lemma 2.5. *Let E be a normed space, $F \subset E$ closed subspace. If F and E/F are Banach spaces so is E.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset E$ be a Cauchy-sequence in E. Which gives us.

$$\|(x_n + F) - (x_m + F)\| = \|(x_n + x_M) + F\| \leq \|x_n + x_m\|.$$

So $(x_n + F)_n \subset E/F$ is a Cauchy-sequence in E/F. With $x + F := \lim_{n\to\infty} x_n + F$. we obtain:

$$\begin{aligned}
\Rightarrow & \inf\{\|x_n - x + y\| : x \in F\} = \|(x_n - x) + F\| \to 0 \\
\Rightarrow & \exists (y_n)_{n\in\mathbb{N}} \subset F : \|x_n - x + y_n\| \to 0 \\
\Rightarrow & \|y_n - y_m\| = \|y_n + x_n - x - x_n + x_m - y_m - x_m + \\
& \|y_n + x_n - x\| + \|x_n - x_m\| + \|y_m + x_m - x\| \xrightarrow{m,n\to\infty} \\
\Rightarrow & (y_n)_{n\in\mathbb{N}} \text{ is a Cauchy-sequence in } F. \\
\Rightarrow & y := \lim_{n\to\infty} y_n \in F \text{ exists} \\
\Rightarrow & \|x_n - x + y\| \leq \|x_n + y_n - x\| + \|y - y_n\| \xrightarrow{n\to\infty} 0 \\
\Rightarrow & x_n \to (x - y), n \to \infty
\end{aligned}$$

□

Corollary 2.6. *A finite-dimensional normed space E is always a Banach-space.*

Proof. Proof by induction.

$n = 1$ Let $\dim E = 1$. Choose $x \in E$ such that $\|x\| = 1$. Then $q : \mathbb{R} \mapsto E, q(\lambda) := \lambda x$ is isometric. So $E \sim \mathbb{R}$ and E is a Banach-space.

$n \to n+1$ Let $\dim E = n+1$. Choose $x \in E \setminus \{0\}$ and set $F := \operatorname{span}\{\mathrm{x}\}$. Because $\dim F = 1$ we know that F is complete, so closed. Now consider:

$$\dim(E/F) = \dim E - \dim F = n$$

By assumption we get that E/F is complete and by lemma 2.5 we see that E is a Banach space.

$\square$

Lemma 2.7. *Let F be a closed subspace of a normed space E. Then for each $x \in E \setminus F$ there exists M, M' such that,*

$$\begin{aligned} \forall_{y \in F, \lambda \in \mathbb{K}} : |\lambda| &\leq M|\ \lambda x + y\| \\ \|y\| &\leq M'\ \|\lambda x + y\| \end{aligned}$$

Proof. For $xn \in \mathbb{N}F$ we have $\|x + F\| \neq 0$. We set

$$\begin{aligned} M &:= \|x + F\|^{-1} \\ M' &:= 1 + M\|x\| \end{aligned}$$

Then for $y \in F$ and $\lambda \in \mathbb{K}$ we have

$$\begin{aligned} \|\lambda\| &\leq M|\lambda|\|x + F\| \\ &= M\|\lambda x + y\| \\ &\leq M\|\lambda x + y\| \end{aligned}$$

For $\lambda \neq 0$ we get

$$\begin{aligned} \|y\| &\leq \|y + \lambda x\| + |\lambda|\|x\| \\ &\leq \|y + \lambda x\| + M\|\lambda x + y\|\|x\| \\ &= \|y + \lambda x\|(1 + M\|x\|) \end{aligned}$$

$\square$

3 Linear Operators, Dual Space

Lemma 3.1. *Let E, F be normed spaces over $\mathbb{K}$ and $T : E \to F$ be a linear operator (a linear map). Then the following are equivalent:*

(i) T is continuous on E.

(ii) T is continuous in one point $x_0 \in E$.

(iii) T is bounded i.e. $\|Tx\| \leq c\|x\|$ for all $x \in E$ and some $c \in \mathbb{R}_{\geq 0}$.

Proof. (i) $\Rightarrow$ (ii): This is trivial.

(ii) $\Rightarrow$ (iii): Let T be continuous in x_0. Then there exists $\delta \in \mathbb{R}_{>0}$ such that $\|x - x_0\| \leq \delta \Rightarrow \|Tx - Tx_0\| \leq 1$ for all $x \in E$. Let $y = \delta^{-1} \cdot (x_0 - x)$ then $\|\frac{\delta \cdot y}{\|y\|}\| \leq \delta$. We obtain $\|Tx_0 - Tx\| = \|T(\frac{\delta \cdot y}{\|y\|})\| \leq 1. \Leftrightarrow \|Ty\| \leq \frac{\|y\|}{\delta}$.

(ii) $\Rightarrow$ (i): Let $\varepsilon > 0$ then $\|Tx\| \leq c\|x\| < \varepsilon \Leftarrow \|x\| \leq \frac{\varepsilon}{c}$. This implies $T(K_{\frac{\varepsilon}{c}}(x)) \subseteq K_{\frac{\varepsilon}{c}}(Tx)$.

$\square$

Definition 3.2. Let E, F be normed spaces over $\mathbb{K}$. Then $\mathscr{L}(E, F) = T : E \to F \; |T \, is \, linear \, and \, bounded.$ If $E = F$ we write $\mathscr{L}(E)$. We define the operator norm on $\mathscr{L}(E, f)$ by $\|T\| = sup_{\|x\|_E \leq 1} \|Tx\|_F$.

Lemma 3.3. *Let E, F, G be normed spaces over $\mathbb{K}$:*

(i) $(\mathscr{L}(E, F), \|\cdot\|)$ is a normed linear space. Also for $T \in \mathscr{L}(E, F)$ we have $\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$.

(ii) If $T \in \mathscr{L}(E, F)$ and $S \in \mathscr{L}(F, G)$ then $T \circ S \in \mathscr{L}(E, G)$ and $\|S \circ T\| \leq \|S\|\|T\|$.

Proof. Excercise. $\square$

Definition 3.4. Let E, F be normed spaces over $\mathbb{K}$. Then the space of continuous linear functions $\mathscr{L}(E, \mathbb{K})$ is the dual space E^* of E.

Theorem 3.5. *Let E be a normed space over $\mathbb{K}$ and let F be a Banach space over $\mathbb{K}$. Then $\mathscr{L}(E, F)$ is a Banach space. In particular E^* is a Banach space.*

Proof. Let $(T_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathscr{L}(E, F)$. Then since $\|T_n x - T_m x\| \leq \|T_n - T_m\| \cdot \|x\|$, also $(T_n x)_{n \in \mathbb{N}}$ is a Cauchy sequence in F for each $x \in E$. Since F is complete we can find $Tx = \lim_{n \to \infty} T_n x$. This defines $T : E \to F$.

1) Let $x, y \in E$ $\lambda, \mu \in \mathbb{K}$ then $T(\lambda x + \mu y) = \lim_{n \to \infty} T_n(\lambda x + \mu y) = \lambda \cdot \lim_{n \to \infty} T_n x + \mu \cdot \lim_{n \to \infty} T_n y = \lambda \cdot Tx + \mu \cdot Ty$

2) Let $\varepsilon \in \mathbb{R}_{>0}$ $N_\varepsilon \in \mathbb{N}$ such that $\|T_n - T_m\| < \varepsilon \, \forall m, n \geq N_\varepsilon$. This implies $\|T_n x - T_m x\| \leq \|T_n - T_m\|\|x\| < \varepsilon\|x\| \, \forall m, n \geq N_\varepsilon$ and $\forall x \in E$. Letting $n \to \infty$ we obtain $\|Tx - T_m x\| \leq \varepsilon \cdot \|x\| \forall m \geq N_\varepsilon$ hence $\|Tx - T_{N_\varepsilon} x\| \leq \varepsilon \cdot \|x\|$. This implies $\|Tx\| \leq \varepsilon \cdot \|x\| + \|T_{N_\varepsilon} x\| \leq (\varepsilon + \|T_{N_\varepsilon}\|) \cdot \|x\| \; \forall x \in E$. Hence $T \in \mathscr{L}(E, F)$

3) $\|Tx - T_{N_\varepsilon x}\| \leq \varepsilon \cdot \|x\| \forall m \geq N_\varepsilon$ $\|T - T_{N_\varepsilon}\| \leq \varepsilon$. Hence $\lim_{m \to \infty} T_m = T$.

□

Theorem 3.6. *Let E be a normed space over $\mathbb{K}$ and let F be a Banach space over $\mathbb{K}$, $L \subseteq E$ a linear subspace, F be a Banach space, and $T : L \to F$ a continuous linear operator. Then theere exists $S : \bar{L} \to F$ with $S_{|L} = T$, S is continuous, linear and $\|S\| = \|T\|$, $S \in \mathscr{L}(\bar{L}, F)$.*

Proof.

1) Let $x \in \bar{L}$ and $(x_n)_{n\in\mathbb{N}} \subseteq L$, $x = \lim_{m\to\infty} x_n$ we observe: $\|Tx_n - Tx_m\| \leq \|T\|\|x_n - x_m\|$. This implies that $(Tx_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since F is a Banach space, hence $(Tx_n)_{n\in\mathbb{N}}$ converges. Now chosse $(y_n)_{n\in\mathbb{N}} \subseteq L$ with $x = \lim_{n\to\infty} y_n$ then $\|Tx_n - Ty_n\| \leq \|T\|\|x_n - y_n\|$, which implies $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Ty_n$. This means tahat the map S defined by $Sx = \lim_{n\to\infty} Ty_n$ is well defined.

2) Let $x, y \in \bar{L}$ be arbitrary, choose $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq L$ with $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $\lambda, \mu \in \mathbb{K}$. Therefore $S(\lambda x + \mu y) = \lim_{n\to\infty} T(\lambda x_n + \mu y_n) = \lambda \lim_{n\to\infty} Tx_n + \mu \lim_{n\to\infty} Ty_n = \lambda Sx + \mu Sy$, which implies that S is linear.

3) Let $x \in \bar{L}$, $\lim_{n\to\infty} x_n = x$, $x_n \in L$, $\|x\| \leq 1$. Then $\|Sx\| = norm \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} \|Tx_n\| = \|Tx\| \leq \|T\|$. Therefore $\|S\| \leq \|T\|$. Since $L \subseteq \bar{L}$ $S_{|L} = T$ we have $\|T\| \leq \|S\|$ hence $\|S\| = \|T\|$.

4) From 3) it follows that $\|Sx\| \leq \|S\|\|x\| = \|T\|\|x\| < \infty$. This implies that S is contiuous.

5) Now choose $R \in \mathscr{L}(\bar(L), F)$ with $R_{|L} = T$. Then all $x \in \bar{L}$ and sequences $(x_n)_{n\in\mathbb{N}}$ with $\lambda \lim_{n\to\infty} x_n = x$ satisfy $Rx = \lim_{n\to\infty} Rx_n = \lim_{n\to\infty} Tx_n = S_x$, which shows the uniqueness of S.

□

Lemma 3.7. *Let E be a normed space over $\mathbb{K}$, $L \in E$ a subspace, F a Banach space and $T : L \to F$ a continous linear operator. Then there exists a unique $S \in \mathscr{L}(\overline{L}, F)$ with $S_{|_L} = T$. There holds*

$$\|S\| = \|T\|$$

Proof. Let $x \in \bar{L}$ and let $(x_n) \subseteq L$ be a sequence which converges to x. We observe

$$\|Tx_n - Tx_m\| \leq \|T\|\|x_n - x_m\|$$

This implies that (Tx_n) is a Cauchy sequence in F. F is complete - hence (Tx_n) converges. For any other sequence (y_n) which also converges to x we have

$$\|Tx_n - Ty_n\| \leq \|T\|\|x_n - y_n\|$$

Thus $\lim Tx_n = \lim Ty_n$. We can now define

$$Sx = \lim Tx_n$$

and have no concernes about well-defining issues.

It follows immediately $S_{|_L} = T$, because for $x \in L$ we can just choose the constant sequence $x_n := x, \forall n$ as "defining sequence". The linearity of S is also easily proven:

$$S(x + y) = \lim T(x_n + y_n) = \lim Tx_n + Ty_n = Sx + Sy$$

and analogously $S(\lambda x) = \lambda Sx$.

Moreover, $\|S\| \geq \|S_{|_L}\| = \|T\|$. To prove the reverse inequality, let $x \neq 0 \in \bar{L}$ and (x_n) be a sequence in L converging to x. For large n, $x_n \neq 0$ and therefore we can consider

$$\frac{\|Sx\|}{\|x\|} = \lim \frac{\| Tx_n\|}{\|x_n\|} \leq \lim \|T\| = \|T\|$$

This proves $\|S\| = \|T\|$, thus S is bounded.

It only remains to prove the uniqueness of S. Consider another continous linear operator R with $R_{|_L} = T$, and $x \in \bar{L}$. For any sequence converging to x there follows

$$Rx = \lim Rx_n = \lim Tx_n = Sx$$

We used the continuity of R. □

We draw a corollary. It follows from the uniqueness part of the lemma.

Corollary 3.8. *If two bounded linear operators $S, T \in \mathscr{L}(E, F)$, where F is a Banach space, coincides in a dense subspace of E, then they coincide in E*

Lemma 3.9. *Let E, F be normed spaces over $\mathbb{K}$ and $T : E \to F$ linear. Then the following are equivalent*

(i) There exists a linear, cond:continousinous, inverse operator

$$T^{-1} : T(E) \to E$$

(ii) There exists $c > 0$ so that $c\|x\| \leq \|Tx\|$

Proof. (i) $\Rightarrow$ (ii) Assume T^{-1} exists. The cond:continousinuity of it gives us the existence of a $\gamma > 0$ such that:

$$\|T^{-1}y\| \leq \gamma\|y\|$$

for an arbitrary x we put $y = Tx$ to obtain

$$\|x\| \leq \gamma\|Tx\|$$

Putting $c := \frac{1}{\gamma}$ we have proven (ii).

(ii) $\Rightarrow$ (i) We observe that i secures the injectivity of T (if $x \in \ker(T)$, then $\|x\| = 0$). Thus $T^{-1} : \operatorname{ran} E \to E$ exists.

Now letting $y = Tx$ in (i) assures the existence of a $c > 0$ with

$$c\|T^{-1}y\| \leq \|y\| \quad \forall y \in T(E)$$

Thus the inverse operator is cond:continousinous. □

Definition 3.10 (Graph of T)**.** Let E, F be normed spaces, $L \subseteq E$ a subspace and $T : L \to F$ a linear operator.

(i)
$$G_T = \{(x, Tx), x \in L\} \subseteq L \times F$$
is called the *graph of T*.

(ii) If G_T is closed, T is *closed*.

Lemma 3.11. *Let E, F, T and L be as above. Then the following are equivalent*

(i) T is closed

(ii) If $(x_n) \subseteq (L)$ converges to $x \in E$ and (Tx_n) to $y \in F$, then $x \in L, y = Tx$

Proof. Since
$$\|(x_n, Tx_n) - (x, y)\| = \max(\|x_n - x\|, \|Tx_n - y\|)$$
we have that if $x_n \to x, Tx_n \to y$, then

$$\lim_{n\to\infty} (x_n, Tx_n) = (x, y)$$

Because of G_T being closed $(x, y) \in G_T$. Thus $x \in L, y = Tx$.

Now consider a convergent sequence $(x_n, y_n) \to (x, y)$ in G_T. Because of the convergence of $y_n = Tx_n$ and (ii), there follows $x \in L, y = Tx$, thus $(x, y) \in G_T$. □

Remark 3.12. If L is closed, and T is cond:continousinous, then T is closed. In particular, all $T \in \mathscr{L}(E, F)$ are closed.

Proof. If $(x_n, Tx_n) \to (x, y)$, then (T closed) $x \in L$. Continuity of T now implies $Tx_n \to Tx$, thus $(x, y) \in G_T$. □

Theorem 3.13. *Let $1 \leq p < \infty$. Define q so that $\frac{1}{p} + \frac{1}{q} = 1$, i.e.*

$$q = \begin{cases} \frac{p}{p-1} & : 1 < p < \infty \\ \infty & : p = 1 \end{cases}$$

Moreover define for $y \in l_q$.

$$f_y : l_p \to \mathbb{K}, x = (x_n) \mapsto \sum_{n=1}^{\infty} x_n y_n$$

Then $f_y \in l_p^*$ and $y \mapsto f_y$ is an isomorphism. In particular $l_q \sim l_p^*$.

Proof. First of all, f_y is welldefined (i.e. the series converges) because of the H"older-inequality:
$$\sum_{n=1}^{\infty} |x_n y_n| \leq \|x\|_p \|y\|_q$$
That f_y is linear is evident. Furthermore, by the above, we have

$$\|f_y(x)\| \leq \|x\|_p \|y\|_q$$

Thus f_y bounded. We conclude that $f_y \in l_p^*$.

We now claim that $\|f_y\| = \|y\|_q$. We already proved $\|f_y\| \leq \|y\|_q$. To prove the inverse inequality, we consider the cases $p = 1, p > 1$ separately.

$\underline{p = 1:}$ Let $\epsilon > 0$. There exists an $n \in \mathbb{N}$ such that

$$|y_n| \geq \|y\|_q - \epsilon$$

Set $x = e_n \in l_1$. We obtain $f_y(x) = y_n$. Because of $\|x\| = 1$ there follows $\|f_y\| \geq \|y\|_\infty - \epsilon$. Because of ϵ arbitrary, we conclude $\|f_y\| \geq \|y\|_q$

$\underline{p > 1}$ Define x by

$$x_n = \begin{cases} 0 & y_n = 0 \\ \frac{|y_n|^q}{y_n} & \text{else} \end{cases}$$

Then $\sum_{n=1}^{\infty} |x_n|^p = \sum_{n=1}^{\infty} |y_n|^{p(q-1)} = \sum_{n=1}^{\infty} |y_n|^q < \infty$, thus $x \in l_p$.

We now compute $f_y(x)$

$$f_y(x) = \sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} |y_n|^q = \|y\|_q^q$$

Thus $\frac{|f_y(x)|}{\|x\|} = \|y\|^{\frac{q(p-1)}{p}} = \|y\|_q$, and we conclude $\|f_y\| \geq \|y\|_q$.

This proves the injectivity of $y \mapsto f_y$. Now the surjectivity. Let $x \in l_p^*$ and put

$$y_n := f(e_n)$$

To prove $y \in l_q$, we again treat the two cases $p = 1$, $p > 1$ separately.

$\underline{p = 1}$ We have for all n

$$|y_n| = |f(e_n)| \leq \|f\| \|e_n\| = \|f\|$$

Thus $\|y\|_\infty \leq \|f\|$, $y \in l_\infty$.

$\underline{p > 1}$ For all $m \in \mathbb{N}$ there holds

$$\sum_{n=1}^{m} |y_n|^q = \sum_{\substack{n=1 \\ y_n \neq 0}}^{m} \frac{|y_n|^q}{y_n} f(e_n) = f\Big(\sum_{\substack{n=1 \\ y_n \neq 0}}^{m} \frac{|y_n|^q}{y_n} e_n \Big) \leq \|f\| \| \sum_{\substack{n=1 \\ y_n \neq 0}}^{m} \frac{|y_n|^q}{y_n} e_n \|_p$$

We have

$$\| \sum_{\substack{n=1 \\ y_n \neq 0}}^{m} \frac{|y_n|^q}{y_n} e_n \|_p = \Big(\sum_{\substack{n=1 \\ y_n \neq 0}} |y_n|^{p(q-1)} \Big)^{\frac{1}{p}}$$

And thus

$$\sum_{n=1}^{m} |y_n|^q \leq \|f\| \Big(\sum_{n=1}^{m} |y_n|^q \Big)^{\frac{1}{p}}$$

which implies

$$\Big(\sum_{n=1}^{m} |y_n|^q \Big)^{\frac{1}{q}} \leq \|f\|$$

Letting $m \to \infty$, we get $\|y\|_q < \infty \Rightarrow y \in l_q$.

Finally, $f = f_y$ because of for $x = \sum_{n=1}^{m} x_n e_n$, there holds

$$f(x) = \sum_{n=1}^{m} x_n f(e_n) = \sum_{n=1}^{m} x_n y_n = f_y(x)$$

Thus f coincides with f_n on the dense linear subspace $\operatorname{span}(e_n)_{n \in \mathbb{N}}$, hence (Lemma (3.7), $\mathbb{K}$ is a Banach space) they are equal. $\square$

Lemma 3.14. *Let E, F be normed spaces, and let $T \in \mathscr{L}(E, F)$- Then the operator $T^* : F^* \to E^*$ defined by*

$$T^*(\phi)(x) = \phi \circ T(x)$$

satisfies $T^* \in \mathscr{L}(F^*, E^*)$ and $\|T^*\| = \|T\|$.

Proof. T^* is obviously linear. Further

$$\|(T^*\phi)x\| = \|\phi T x\|\|\| \leq \|phi\|\|T\|\|x\|$$

This proves that T^* is bounded and $\|T^*\| \leq \|T\|$. The inverse inequality is an exercize. $\square$

Let E, F be normed spaces and $T \in \mathscr{L}(E, F)$. The $T^* : F^* \to E^*, T^*\phi = \phi \circ T$ is called the *dual operator* of T

4 Hahn-Banach Theorem and Corollaries

Definition 4.1. a) Let E be a linear space. Then the algebraic dual space of E which is the space of linear maps $E \to \mathbb{K}$, is denoted E'.

b) Let E be an $\mathbb{R}$-vector space. Then $p : E \to \mathbb{R}$ is a sublinear functional on E if all $x, y \in E$ and $\lambda \in \mathbb{R}$ satisfy $p(x+y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$,

Lemma 4.2. *... to be continued*

WARNING The first lemma of the day uses the Lemma of Zorn. If you're not comfortable with it, then essentially everything in this lecture will be very hard to swallow!

When now everyone has been warned it is time for

Lemma 4.3. *Let E be an $\mathbb{R}$-vector space, $F \subseteq E$ a linear subspace and p a sublinear functional on E. Further let $f \in F'$ with*

$$f(x) \leq p(x) \quad \forall x \in F$$

Then there exists $l \in E'$ with $l|_F = f$ and $l(x) \leq p(x) \forall x \in E$.

Proof. Set

$$\begin{aligned} \mathcal{L} = \{(L, l) \quad : \quad & L \text{ linear subspace of } E \text{ with } L \supset F \\ & \text{and } l \in L' \text{ with } l|_F = f \text{ and } l(x) \leq p(x) \forall x \in L\} \end{aligned}$$

To prove is the existence of a pair of the form $(E, l) \in \mathscr{L}$. To do this, we define an ordering on $\mathscr{L}$.

$$(L_1, l_1) \leq (L_2, l_2) \Leftrightarrow L_1 \subset L_2 \wedge l_2|_{L_1} = l_1$$

We know that $\mathcal{L} \neq \varnothing$, then we have $(F, f) \in \mathscr{L}$.

We want to use the Lemma of Zorn. Let $\mathscr{K}$ be a chain in $\mathscr{L}$. To prove is that $\mathcal{K}$ has an upper bound in $\mathscr{L}$. We claim that if we set

$$\tilde{L} = \{L : \exists l' \text{ with } (L, l') \in \mathscr{K}\}$$

then $\tilde{L}$ is a linear subspace since $\mathcal{K}$ is linearly ordered. define $\tilde{l} : \tilde{L} \to \mathbb{R}$ through

$$\begin{aligned} \tilde{l}(x) := l(x) \qquad & \text{if } x \in L \text{ and} \\ & l \in L' \text{ with } (L, l) \in \mathscr{K} \end{aligned}$$

We know by the definition of $\tilde{L}$ that such an l always exists. In fact, this definition is well-defined, since if $(L_2, l_2) \in \mathcal{K}$ is another pair with $x \in L_2$, then one of the pairs is bigger with respect to the ordering ($\mathscr{K}$ is a chain), WLOG $(L_1, l_1) \leq (L_2, l_2)$. Then, because of $x \in L_1$ we have $l_2(x) = l_1(x)$.

$\tilde{l} \in \tilde{L}'$. To prove is only the linearity. Let $x_i \in L_i, \alpha_i \in \mathbb{R}, (L_i, l_i) \in \mathscr{K}, i = 1, 2$. Then WLOG $(L_1, l_1) \leq (L_2, l_2)$. Therefore $x_1 \in L_2$ and thus $\alpha_1 x_1 + \alpha_2 x_2 \in L_2$. There follows

$$\begin{aligned} \tilde{l}(\alpha_1 x_1 + \alpha_2 x_2) &= l_2(\alpha_2 x_1 + \alpha_2 x_2) = \alpha_1 l_2(x_1) + \alpha_2 l_2(x_2) \\ &= \alpha_1 l_1(x_1) + \alpha_2 l_2(x_2) = \alpha_1 \tilde{l}(x_1) + \alpha_2 \tilde{l}(x_2) \end{aligned}$$

Finally $\tilde{l}(x) = l(x) \leq p(x)$. The Lemma of Zorn now provides the existence of a maximal element of $\mathcal{L}$ $(L, l) \in \mathcal{L}$. To prove is $L = E$.

Suppose the opposite, then there exists $x_0 \in E \backslash L$. By Lemma 4.2, there exists $g \in (L + \mathbb{R}x_0)'$ with $g|_L = l$ and $g(x) \leq p(x) \quad \forall x \in (L + \mathbb{R}x_0)$. Then we although have $(L, l) < (L + \mathbb{R}x_0, g)$. Contradiction!

Thus $L = E$, this proves the Lemma. □

Theorem 4.4. *Let E be a vector space over $\mathbb{K}$, F a linear subspace and $f \in F'$.*

Let $p : E \to \mathbb{R}$ be a seminorm on E, i.e. for all $x, y \in E, \lambda \in \mathbb{K}$

$$\begin{aligned} p(x+y) &\leq p(x) + p(y) \\ p(\lambda x) &= |\lambda| p(x) \end{aligned}$$

Suppose that $|f(x)| \leq p(x)$ for all $x \in F$. Then there exists an $l \in E'$ with $l|_F = f$ and $|l(x)| \leq p(x) \quad \forall x \in E$.

Proof. First consider $\mathbb{K} = \mathbb{R}$. Then $f(x) \leq p(x)$ for all $x \in F$ and $p(\alpha x) = \alpha p(x)$ for all $x \in E, \alpha \geq 0$. By Lemma 4.3, there exists some $l \in E'$ with $l|_F = f$ and $l(x) \leq p(x) \quad \forall x \in E$. Since also

$$-l(x) = l(-x) \leq p(-x) = p(x)$$

we have $|l(x)| \leq p(x)$.

The case $\mathbb{K} = \mathbb{C}$ will be discussed in the exercises. □

Theorem 4.5. *(Hahn-Banach Theorem) Let E be a normed space, F a linear subspace of E. Then for each $f \in F^*$ there exists some $l \in E^*$ with*

$$l|_F = f \wedge \|l\| = \|f\|$$

Proof. Let p be defined by

$$p(x) = \|f\| \|x\|$$

Then p is a seminorm - the properties are inherited from the norm properties of $\|\cdot\|$. Furthermore

$$|f(x)| \leq p(x) \quad \forall x \in F$$

By Theorem 4.4, there exists $l \in E'$ with $l|_F = f$ and $|l(x)| \leq p(x) = \|f\| \|x\|$. This proves in particular that $l \in E^*$ and $\|l\| \leq \|f\|$. Because of $l|_F = f$, the reverse inequality holds. Thus $\|l\| = \|f\|$. □

Corollary 4.6. *Let E be a normed space and F a linear subspace of E and $x \in E$ such that*

$$\delta := \inf_{y \in F} \|x - y\| > 0$$

Then there exists an $l \in E^*$ with

$$l|_F = 0, \|l\| = 1 \text{ and } l(x) = \delta$$

In particular, for any $x \neq 0$ there exists an $l \in E^*$ with $\|l\| = 1$ and $l(x) = \|x\|$.

Proof. Let $G = F + \mathbb{K}x$ and $g : G \to \mathbb{K}$ be defined through

$$g(y + \lambda x) = \lambda \delta \quad \forall y \in F, \lambda \in \mathbb{K}$$

g is well defined, then because of $x \notin F$ we have $G = F \oplus \mathbb{K}x$. Further g is linear, $g|_F = 0$ and $g(x) = \delta$.

We know claim $\|g\| = 1$. There holds

$$\begin{aligned} |g(y + \lambda x)| = |\lambda| \delta &= |\lambda| \inf_{z \in F} \|z - x\| \\ &= \inf_{z \in F} \|\lambda z - \lambda x\| = \inf_{z \in F} \|z + \lambda x\| \leq \|y + \lambda x\| \end{aligned}$$

thus $\|g\| \leq 1$.

Secondly, there exists for every $\epsilon > 0$ a $z_\epsilon \in F$ with $\delta \leq \|z_\epsilon + x\| \leq \delta + \epsilon$. There follows

$$\begin{aligned} g(x + z_\epsilon) &= \delta \geq \|x + z_\epsilon\| - \epsilon \Leftarrow \\ g(\|x + z_\epsilon\|^{-1}(x + z_\epsilon)) &= \frac{\delta}{\|x + z_\epsilon\|} \geq 1 - \frac{\epsilon}{\|x + z_\epsilon} \geq 1 - \frac{\epsilon}{\delta} \end{aligned}$$

Now apply Theorem 4.5 to lift g up to E^*.

For the in-particular part, choose $F = \{0\}$. $\square$

We know define an important concept

Definition 4.7. Let E be a normed space, $M \subset E$ an arbitrary subset of E and $L \in E^*$ one of E^*. Then the annihilator of M in E^* is

$$M^\perp := \{l \in E^* : l(x) = 0 \forall x \in M\}$$

and the annihilator of L in E is

$$L_\perp = \{x \in E : l(x) = 0 \forall l \in L\}$$

Remark 4.8. The annihilators are closed linear subspaces of E^* and E, respectively. This follows from the continuity of $l \mapsto l(x), x \mapsto l(x)$.

Lemma 4.9. *Let E be a normed space and $\varnothing \neq M \subseteq E$. Then $(M^\perp)_\perp$ is the closed linear hull of M, i.e. the smallest closed linear subspace of E which contains M.*

Proof. If $x \in M$, then $l(x) = 0$ for all $l \in M^\perp$, thus $x \in (M^\perp)_\perp \Leftarrow M \subseteq (M^\perp)_\perp$.

Now let F be the closed linear hull of M. By the remark, $F \subseteq (M^\perp)_\perp$. Now assume there exists $x \in (M^\perp)_\perp \backslash F$. Corollary 4.6 secures the existence of an $l \in (M^\perp)_\perp^*$ with $l|_F = 0$ and $l(x) = 0$.

Theorem 4.5 now implies the existence of an $f \in E^*$ with $f|_{(M^\perp)_\perp} = l$. f is in $M^\perp$ because of $f|_F = l|_F = 0$ and $M \subseteq F$. But $f(x) \neq 0$ - Contradiction! $\square$

Theorem 4.10. *Let E be a normed space over $\mathbb{K}$, and $F \subseteq E$ a linear subspace.*

(i) The linear operator

$$\Phi : E^*/F^\perp \to F^*, \quad \Phi(f + F^\perp) = f|_F,\ f \in E^*,$$

is an isometric isomorphism.

(ii) If F is closed, then the linear operator

$$\Phi : (E/F)^* \to F^\perp, \quad (\Phi f)(x) = f(x + F),\ x \in E,\ f \in (E/F)^*,$$

is an isometric isomorphism.

Proof. (i). Consider the map $T : E^* \to F^*$, $Tf := f|_F$, $f \in E^*$. We have $\ker T = F^\perp$. Hence, Φ is well-defined, linear, and injective. By Theorem 4.5, for each $\ell \in F^*$ there exists some $f \in E^*$ with $f|_F = \ell$. Hence, Φ is surjective. Finally, let $f \in E^*$, and choose $g \in E^*$ such that Reference!

$$\|g\| = \|f|_F\| \quad \text{and} \quad g|_F = f|_F.$$

Then, we obtain

$$\|f + F^\perp\| \le \|f + (g - f)\| = \|g\| = \|f|_F\| = \|\Phi(f + F^\perp)\|.$$

On the other hand, for all $g \in F^\perp$,

$$\|\Phi(f + F^\perp)\| = \|f|_F\| = \|(f + g)|_F\| \le \|f + g\|.$$

Hence, $\|\Phi(f + F^\perp)\| \le \|f + F^\perp\|$.

(ii). First, $\Phi f : E \to \mathbb{K}$, $x \mapsto f(x + F)$, is linear. Since

$$|(\Phi f)(x)| = |f(x + F)| \le \|f\| \cdot \|x + F\| \le \|f\| \cdot \|x\|,$$

we have $\Phi f \in E^*$. If $x \in F$, then $(\Phi f)(x) = f(F) = 0$, hence $\Phi f \in F^\perp$. Therefore, indeed $\Phi : (E/F)^* \to F^\perp$. It is obvious that Φ is linear and injective. To prove surjectivity, let $g \in F^\perp$, and let $f : E/F \to \mathbb{K}$ be defined by

$$f(x + F) = g(x), \quad x \in E.$$

The functional f is is well-defined since $F \subseteq \ker g$. Moreover, f is linear, and for all $x \in E$, $y \in F$ we have

$$|f(x + F)| = |g(x)| = |g(x + y)| \le \|g\| \cdot \|x + y\|,$$

which implies $f \in (E/F)^*$. In addition, $\Phi f = g$, and surjectivity is proved.

It remains to show that Φ is isometric. For this, note that $|(\Phi f)(x)| \le \|f\| \cdot \|x\|$, $x \in E$, implies $\|\Phi f\| \le \|f\|$ for all $f \in (E/F)^*$. On the other hand, for each $\varepsilon > 0$ there exists $x \in E$ with

$$\|x + F\| = 1 \quad \text{and} \quad |f(x + F)| \ge \|f\| - \varepsilon.$$

Since $1 = \|x + F\| = \inf_{y \in F} \|x + y\|$, there exists $y \in F$ with $\|x + y\| \le 1 + \varepsilon$. This implies $\|\frac{x+y}{1+\varepsilon}\| \le 1$ and hence

$$\left| (\Phi f) \left(\frac{x + y}{1 + \varepsilon} \right) \right| = \frac{|f(x + F)|}{1 + \varepsilon} \ge \frac{\|f\| - \varepsilon}{1 + \varepsilon}.$$

Thus $\|\Phi f\| \ge \frac{\|f\|-\varepsilon}{1+\varepsilon}$, which yields $\|\Phi f\| \ge \|f\|$. This proves that Φ is isometric. □

Lemma 4.11. *Let E be a normed space over $\mathbb{K}$. For $x \in E$ let $\hat{x} : E^* \to \mathbb{K}$ be defined by $\hat{x}(\ell) = \ell(x)$, $\ell \in E^*$. Then $\Lambda_E : E \to (E^*)^*$, $\Lambda_E x = \hat{x}$, is an isometric linear operator.*

Proof. Λ_E is linear, and for $x \in E$ we have

$$\sup\{|\hat{x}(\ell)| : \ell \in E^*, \|\ell\| = 1\} = \sup\{|\ell(x)| : \ell \in E^*, \|\ell\| = 1\} = \|x\|,$$

where the last equality follows from Corollary 4.6. This proves that indeed $\hat{x} \in (E^*)^*$ and that $\|\Lambda_E x\| = \|\hat{x}\| = \|x\|$ for all $x \in E$. □ Reference!

Definition 4.12. Let E be a normed space over $\mathbb{K}$, and let Λ_E be defined as above. Then Λ_E is called *canonical map* or *embedding* of E in $E^{**} := (E^*)^*$. E is *reflexive*, if Λ_E is surjective. E^{**} is the *bi-dual* of E, and $\overline{\Lambda_E(E)}$ is the *completion* of E.

Remark 4.13. Only Banach spaces can be reflexive. The class of reflexive spaces is a highly important class of Banach spaces. Intriguingly, there exist non-reflexive Banach spaces, which are isometrically isomorphic to their bi-dual. Moreover, notice that finite-dimensional spaces are always reflexive because of $\dim E^{**} = \dim E$.

Theorem 4.14. *Let E be a normed space over $\mathbb{K}$.*

(i) If E is reflexive and $F \subseteq E$ a closed linear subspace, then F is also reflexive.

(ii) If E is a Banach space, then E is reflexive if and only if E^* is reflexive.

Proof. (i). We have to show that for each $\varphi \in F^{**}$ there exists some $y \in F$ with $\varphi(f) = f(y)$ for all $f \in F^*$. For this, let $\varphi \in F^{**}$ and let $\psi : E^* \to \mathbb{K}$ be defined by $\psi(\ell) := \varphi(\ell|_F)$, $\ell \in E^*$. Since

$$|\psi(\ell)| \leq \|\varphi\| \cdot \|\ell|_F\| \leq \|\varphi\| \cdot \|\ell\|,$$

we have $\psi \in E^{**}$. E being reflexive then implies that there exists $y \in E$ with $\psi(\ell) = \ell(y)$ for all $\ell \in E^*$. Next, towards a contradiction, assume that $y \notin F$. Then there exists some $\ell \in E^*$ with $\ell(y) \neq 0$ and $\ell|_F = 0$. Hence, $0 \neq \ell(y) = \psi(\ell) = \varphi(\ell|_F) = 0$. A contradiction. Finally, for $f \in F^*$ there exists some $\ell \in E^*$ with $\ell|_F = f$, hence

$$\varphi(f) = \varphi(\ell|_F) = \psi(\ell) = \ell(y) = f(y).$$

This shows that F is reflexive.

(ii). Let E be reflexive. We need to show that for each $u \in E^{***}$ there exists some $f \in E^*$ with $u(\varphi) = \varphi(f)$ for all $\varphi \in E^{**}$. For this, let $u \in E^{***}$, and set $f(x) := u(\hat{x})$, $x \in E$. Then $f \in E^*$. Next, let $\varphi \in E^{**}$. Since there hence exists some $x \in E$ with $\hat{x} = \varphi$, we obtain

$$u(\varphi) = u(\hat{x}) = f(x) = \hat{x}(f) = \varphi(f).$$

Therefore, E^* is reflexive.

For the converse, let E^* be reflexive. Then, by the above, E^{**} is reflexive. By (i), also $\Lambda_E(E) = \overline{\Lambda_E(E)}$ is reflexive. The claim now follows from the fact that E and $\Lambda_E(E)$ are (isometrically) isomorphic (see Exercise Sheet 6, Exercise 1(ii)). $\square$

Theorem 4.15. *Let E be a Banach space and $F \subseteq E$ a closed linear subspace. Then the following are equivalent:*

(i) E is reflexive.

(ii) F and E/F are reflexive.

Proof. (i)⇒(ii). By (i) and Theorem 4.14(i), also F is reflexive. By Theorem 4.14(ii), E^* is reflexive, hence $F^\perp$ is reflexive. By Theorem 4.10(ii), $F^\perp$ is isometrically isomorphic to $(E/F)^*$ which is therefore also reflexive. By Theorem 4.14(ii), this finally implies that E/F is reflexive.

(ii)⇒(i). Let $\varphi \in E^{**}$. We will again use the isometric isomorphism

$$\Phi : (E/F)^* \to F^\perp \subseteq E^*, \quad (\Phi u)(x) = u(x + F),\ u \in (E/F)^*,\ x \in E,$$

from Theorem 4.10. Then we can define $\psi \in (E/F)^{**}$ by

$$\psi(u) := \varphi(\Phi u), \quad u \in (E/F)^*.$$

Since E/F is reflexive, there exists some $x \in E$ with $\widehat{x + F} = \psi$, i.e.

$$\varphi(\Phi u) = \psi(u) = (\widehat{x + F})(u) = u(x + F) = (\Phi u)(x) = \hat{x}(\Phi u), \quad u \in (E/F)^*.$$

Hence, $(\varphi - \hat{x})|_{F^\perp} = 0$.

To utilize the reflexivity of F, we next define a suitable $\rho \in F^{**}$. For each $f \in F^*$, choose some $g \in E^*$ with $g|_F = f$ and $\|g\| = \|f\|$. Then define

$$\rho(f) := (\varphi - \hat{x})(g).$$

This is a proper definition since for two extensions $g, h \in E^*$ of f we have $(g - h)|_F = 0$ and thus $g - h \in F^\perp$. A similar argument shows that ρ is linear. Moreover,

$$|\rho(f)| \leq \|\varphi - \hat{x}\| \cdot \|g\| = \|\varphi - \hat{x}\| \cdot \|f\|.$$

Thus, $\rho \in F^{**}$. As F is reflexive, there exists some $y \in F$ with $\rho(f) = f(y)$ for all $f \in F^*$. Now, we conclude that for all $g \in E^*$ we have

$$\hat{y}(g) = g(y) = (g|_F)(y) = \rho(g|_F) = (\varphi - \hat{x})(h)$$

with some $h \in E^*$ satisfying $h|_F = g|_F$ and $\|h\| = \|g|_F\|$. Hence, $h - g \in F^\perp$ and thus $\hat{y}(g) = (\varphi - \hat{x})(g)$ for all $g \in E^*$. Equivalently, $\varphi = \hat{x} + \hat{y} = \widehat{x + y} \in \Lambda_E(E)$, which shows that E is reflexive. $\square$

Theorem 4.16. *Let E and F be normed spaces, $E \neq \{0\}$. If $\mathcal{L}(E, F)$ is complete, then so is F.*

Proof. First, choose $x_0 \in E$ with $\|x_0\| = 1$. Then there exists some $f \in E^*$ with $f(x_0) = \|x_0\| = 1 = \|f\|$. Next, let $(y_n)_{n\in\mathbb{N}} \subseteq F$ be a Cauchy sequence, and define $T_n : E \to F$ by

$$T_n x := f(x) y_n, \quad x \in E.$$

Since $\|T_n x\| \leq \|f\|\|x\|\|y_n\| = \|y_n\|\|x\|$ for $x \in E$, we have $T_n \in \mathcal{L}(E, F)$. Further, $\|T_n x - T_m x\| = |f(x)|\|y_n - y_m\| \leq \|y_n - y_m\|\|x\|$ implies

$$\|T_n - T_m\| \leq \|y_n - y_m\|.$$

Hence $(T_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}(E, F)$ and thus converges to some $T \in \mathcal{L}(E, F)$. This implies $y_n = T_n x_0 \to T x_0$ as $n \to \infty$, i.e. $(y_n)_{n\in\mathbb{N}}$ converges in F. $\square$

5 The open mapping, closed graph and Banach-Steinkraus theorem

Lemma 5.1. *Let E be a normedm F a Banach space and $T \in \mathcal{L}(E,F)$ surjective. Then*

$$K_r(0_F) \subset \overline{T(K_r(0_E))} \qquad \text{for some } r > 0$$

Theorem 5.2 (Uniform Boundedness Priniple)**.** *Let E be a Banach space, F a normed space and let $\mathcal{T} \in \mathcal{L}(E,F)$. Let $\mathcal{T}$ be pointwise bounded, i.e., for each $x \in E$ there exists $M_x < \infty$ sucht that $\|Tx\| \leq M_x \quad \forall T \in \mathcal{T}$. Then $\mathcal{T}$ is bounded, i.e., there exists some $M < \infty$ sucht that $\|T\| < M \quad \forall T \in \mathcal{T}$.*

Proof. For $n \in \mathbb{N}$ let

$$E_n = \{x \in E : \|Tx\| \leq m \quad \forall T \in \mathcal{T}\}.$$

By hypothesis, $E = \bigcup_{n=1}^{\infty} E_n$. Let $x = \lim_{j\to\infty} x_j$ with $x_j \in E_n$ for fixed n. Since $\|Tx_j\| \leq n$ for all j we have

$$\|Tx\| = \lim_{j\to\infty} \|Tx_j\| \leq m.$$

Thus $x \in E_n$, and hence E_n is closed. By remark 1.11 on Baire's Theorem, there exists some $n_0 \in \mathbb{N}$ with

$$\mathring{E}_{n_0} \neq \varnothing.$$

Hence, $K_r(x) \subseteq E_{n_0}$ for some $x \in E_n$, $r > 0$. Let $y \in E$ with $\|y\| \leq r$. Then $y + x \in K_r(x) = x + K_r(0)$. This implies

$$\begin{aligned} \|Ty\| &= \|T(y+x) - Tx\| \\ &\leq \|T(y+x)\| + \|Tx\| \leq 2n_0 \quad \forall T \in \mathcal{T}. \end{aligned} \tag{5.1}$$

Now let $y \in E$, $y \neq 0$ arbitrary. Then

$$\begin{array}{rrcl} & \frac{r}{\|y\|}\|Ty\| & = & \left\|T\left(\frac{ry}{\|y\|}\right)\right\| \overset{5.1}{\leq} 2n_0 \\ \Rightarrow & \|Ty\| & \leq & \frac{2n_0}{r}\|y\| \\ \Rightarrow & \|T\| & \leq & \frac{2n_0}{r}. \end{array}$$

□

Example 5.3. In general, Theorem 5.2 does not hold, if E is *not* a Banach space. Example:

$$\begin{aligned} E &= \{x = (x_n)_{n\in\mathbb{N}} \in \ell_1 : x_n = 0 \text{ for almost all } n \in \mathbb{N}\} \\ F &= \mathbb{K} \\ f_n(x) &= nx_n \quad \forall x \in E, n \in \mathbb{N} \\ \mathcal{T} &= \{f_n : n \in \mathbb{N}\}. \end{aligned}$$

We see that $\mathcal{T}$ is pointwise bounded, since $x_n = 0$ from some $n \geq N$ on.

BUT $\|f_n\| = n$, $n \in \mathbb{N}$.

Corollary 5.4. *Let E be a Banach Space, F a normed space and $T_n \in \mathcal{L}(E,F)$. Suppose for every $x \in E$, $(T_n x)_{n\in\mathbb{N}}$ is convergent in E. Then define $T \colon E \to F$ by*

$$Tx := \lim_{n\to\infty} T_n x.$$

Then $T \in \mathcal{L}(E,F)$, $(\|T_n\|)_{n\in\mathbb{N}}$ is bounded, and

$$\|T\| \leq \liminf_{n\to\infty} \|T_n\|.$$

Proof. By definition, T obviously is linear. $(\|T_n x\|)_{n\in\mathbb{N}}$ is bounded. By Theorem 5.2, $\|T_n\| \leq M$ for all $n \in \mathbb{N}$. Hence, for all $x \in E$

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq M\|x\|.$$

This shows that $T \in \mathcal{L}(E, F)$. Let $(\|T_{n_k}\|)_{k\in\mathbb{N}}$ be a convergent subsequence of $(\|T_n\|)_{n\in\mathbb{N}}$. Then

$$\|Tx\| = \lim_{k\to\infty} \|T_{n_k}\| \leq \|x\| \lim_{k\to\infty} \|T_{n_k}.$$

Thus

$$\|T\| \leq \lim_{k\to\infty} \|T_{n_k}\|,$$

and hence

$$\|T\| \leq \liminf_{n\to\infty} \|T_n\|.$$

$\square$

Lemma 5.5. *Let E be a normed space and F be a Banach space. Then let E_0 be a dense linear subspace of E, and $T_0 \in \mathcal{L}(E_0, F)$. There exists a unique $T \in \mathcal{L}(E, F)$ with*

$$T|_{E_0} = T_0 \quad \textit{and} \quad \|T\| = \|T_0\|.$$

Proof. Follows with Lemma 3.6. $\square$ Reference!

Theorem 5.6 (Banach-Steinhaus Theorem)**.**

(i) Let E be a Banach space and F a normed space. Further, let $T_n \in \mathcal{L}(E, F)$, $n \in \mathbb{N}$. If $(T_n)_{n\in\mathbb{N}}$ is pointwise convergent to some $T\colon E \to F$ which is linear, then

$$\sup_{n\in\mathbb{N}} \|T_n\| < \infty.$$

(ii) Let E be a normed space and F a Banach space. Further, let $T \in \mathcal{L}(E, F)$, $n \in \mathbb{N}$. If

(a) $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$

(b) there exists a dense linear subspace E_0 of E such that $(T_n x)_{n\in\mathbb{N}}$ is convergent in F for each $x \in E_0$,

then there existst some $T \in \mathcal{L}(E, F)$ with

$$Tx = \lim_{n\to\infty} T_n x \quad \textit{for all } x \in E.$$

Proof.

(i) This is Cor. 5.4

(ii) For each $y \in E_0$ set

$$T_0 y := \lim_{n\to\infty} T_n y. \quad \text{(exists by (b))}$$

T_0 is linear and

$$\|T_0 y\| = \lim_{n\to\infty} \|T_n y\| \leq \underbrace{\sup_{n\in\mathbb{N}} \|T_n\| \|y\|}_{<\infty \text{ by (a)}}.$$

Hence $T_0 \in \mathcal{L}(E,F)$. By Lemma 5.5, there exists some $T \in \mathcal{L}(E,F)$ with $T|_{E_0} = T_0$. Let $x \in E$, and $\varepsilon > 0$. Then let $y \in E_0$ with

$$\|x-y\| \leq \varepsilon \text{ and } N \in \mathbb{N} \text{ with } \|T_n y - T_0 y\| \leq \varepsilon \ \forall n \geq N. \quad \text{(b)}$$

Then for all $n \geq N$,

$$\begin{aligned}
\|T_n x - Tx\| &\leq \underbrace{\|T_n x - T_n y\|}_{\leq \|T_n\|\cdot\|x-y\|} + \underbrace{\|T_n y - T_0 y\|}_{\leq \varepsilon} + \underbrace{\|T_0 y - Tx\|}_{\leq \|T\|\cdot\|y-x\|} \\
&\leq \|T_n\|\|x-y\| + \varepsilon + \|T\|\|y-x\| \\
&\leq \varepsilon(\|T_n\| + 1 + \|T\|) \\
&\leq \varepsilon\Bigg(\underbrace{\sup_{n\in\mathbb{N}} \|T_n\|}_{<\infty \text{ by (a)}} + 1 + \|T\|\Bigg).
\end{aligned}$$

This implies $T_n x \to Tx$ as $n \to \infty$.

□

Theorem 5.7. *Let E be a normed space and $M \subseteq E$. Then the following conditions are equivalent:*

(i) M ist bounded.

(ii) $f(M) \subseteq \mathbb{K}$ is bounded for all $f \in E^*$.

Remark 5.8 (Geometric interpretation of 5.7)**.** Suppose that for every closed hyperplane H in E (kernel of f) some c exists with M between $H+c$ and $H-c$. Then M is already contained in a ball. ((ii)⇒(i))

Proof.

(i)⇒(ii) This follows from

$$\|x\| < c \Rightarrow \|f(x)\| \leq c\|f\| \quad \forall x \in M.$$

(ii)⇒(i) Consider the set

$$\hat{M} = \{\hat{x} \colon f \mapsto f(x) : x \in M\} \subseteq \mathcal{L}(E^*, \mathbb{K}) = E^{**}.$$

Since $\hat{M}(f) = f(M)$, by (ii) M is pointwise bounded. By Theorem 5.2, $\hat{M}$ is bounded. Since $\hat{}$ is isometric, also M is bounded.

□

Corollary 5.9. *Let E, F be normed spaces and $T \colon E \to F$ be linear. Then*

$$T \in \mathcal{L}(E,F) \Leftrightarrow f \circ T \in E^* \textit{ for all } f \in F^*.$$

Proof.

$$\begin{aligned}
rcl T \text{ is bounded} &\Leftrightarrow & T(K_1(0)) \text{ is bounded in } F \\
&\overset{5.7}{\Leftrightarrow} & f(T(K_1(0))) \text{ is bounded for all } f \in F^* \\
&\Leftrightarrow & f \circ T \in E^* \text{ for all } f \in F^*.
\end{aligned}$$

□

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