博弈论

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Introduction

- **Guessing Game**
  - Everyone choose a number between $[0,\ 100]$.
  - The one whose number is close to $\frac{2}{3}$ of the average win.
  - If you assume all other people choose the average of the range, i.e., 50, then you may choose $50 \times \frac{2}{3} \approx 33$. If you assume all other people have thought about this, i.e., they will choose 33, then you may choose $33 \times \frac{2}{3} = 22$, and so on.
  - Another prospective: the answer cannot exceed $100 \times \frac{2}{3} \approx 67$, similarly we can repeat the process to get $67 \times \frac{2}{3} \approx 45$, $45 \times \frac{2}{3} = 30$, and so on.
  - Both ways of thinking will derive 0 ultimately, but choosing 0 must not be the optimal strategy.

- **Concepts in Game Theory**
  - Players: $I = \{1,\ \cdots,\ n\}$
  - Actions: $a = (a_1,\ \cdots,\ a_n) \in A = A_1 \times \cdots \times A_n$
  - Payoff function (or utility function): $u_i : A \to \mathbb{R}$ for player i
  - Example
    - Prisoner's Dilemma

		Person 2	
		C	D
Person 1	C	2, 2	0, 3
	D	3, 0	1, 1

      where action C means "Cooperate" while D means "Defect". The numbers in each cell are payoffs (the former one is the payoff of Person 1 and the latter is the payoff of Person 2).
    - There are 2 players in this example.
    - $A_1 = A_2 = \{C,\ D\}$, $A = A_1 \times A_2 = \{(C,\ C),\ (C,\ D),\ (D,\ C),\ (D,\ D)\}$
    - $u_1(a) = \begin{cases} 2, & a = (C,\ C) \\ 0, & a = (C,\ D) \\ 3, & a = (D,\ C) \\ 1, & a = (D,\ D) \end{cases}$, $u_2(a) = \begin{cases} 2, & a = (C,\ C) \\ 3, & a = (C,\ D) \\ 0, & a = (D,\ C) \\ 1, & a = (D,\ D) \end{cases}$

- **Best Response**
  - Denote $a_{-i} =< a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n >$ which is others' actions for player i. Now $a = (a_i, a_{-i})$.
  - Definition
    $a_i^* \in BR(a_{-i})$ iff $\forall a_i \in A_i,\ u_i((a_i^*, a_{-i})) \geqslant u_i((a_i, a_{-i}))$

- **Nash Equilibrium**
  - Definition
    a is a Nash equilibrium iff $\forall i,\ a_i \in BR(a_{-i})$
  - Discrete Example
    - Still consider the Prisoner's Dilemma described above.
    - If $a = (C, C)$, a_1 is not a best response and a_1^* should be D.
    - It is similar for $a = (C, D)$ and for $a = (D, C)$.
    - If $a = (D, D)$, a_1 and a_2 are both best response. Hence, (D, D) reaches Nash equilibrium.
  - Continuous Example
    - Price Competition
      - 2 restaurants with prices P_x and P_y.
      - Num of customers for each restaurant:

$$
\begin{aligned}
Q_x &= 44 - 2P_x + P_y \\
Q_y &= 44 - 2P_y + P_x
\end{aligned}
$$

      - Cost of serving each customer is 8.
      - Each restaurant's goal is to maximize its profit.
      - They set prices simultaneously.
    - Since the quantity functions are the same form, 2 restaurants are substitute.
    - The profit of restaurant x is

$$\Pi_x = (P_x - 8)(44 - 2P_x + P_y) = -2P_x^2 + (P_y + 60)P_x - 8(P_y + 44)$$

      By letting $\frac{\partial \Pi_x}{\partial P_x} = 0$ (first order condition), we get the best response function for x:

$$P_x^* = \frac{P_y + 60}{4} = \frac{P_y}{4} + 15$$

      Similarly for y, the best response function is $P_y^* = \frac{P_x}{4} + 15$.
    - According to the definition of Nash equilibrium, the prices should satisfy

$$
\begin{cases}
P_x = \frac{P_y}{4} + 15 \\
P_y = \frac{P_x}{4} + 15
\end{cases}
$$

which means $(P_x^*,\ P_y^*) = (20,\ 20)$.

- **Method of Scoring**
  - Control a_{-i} and underline the best response of i.
    - Example
      - Street Garden Game

        Emily, Nina and Talia have to choose simultaneously whether to contribute toward the creation of a flower garden for their small street.

        Talia

		Contribute	
		Nina	
		Contribute	Don‘t Contribute
Emily	Contribute	5,5,5	3,$\underline{6}$,3
	Don't Contribute	$\underline{6}$,3,3	$\underline{4}$,$\underline{4}$,1

		Don't Contribute	
		Nina	
		Contribute	Don't Contribute
Emily	Contribute	3,3,$\underline{6}$	1,$\underline{4}$,$\underline{4}$
	Don't Contribute	$\underline{4}$,1,$\underline{4}$	$\underline{2}$,$\underline{2}$,$\underline{2}$

        where numbers in each cell represent their utility respectively.
      - The Nash equilibrium is that they all choose to not contribute.

Dominated Strategies

- **Dominance**
  - Strict Dominance
    - Strategy $\overline{s}_i \in S_i$ is strictly dominated if there is some strategy $\hat{s}_i \in S_i$ s.t. $u_i((\hat{s}_i,\ s_{-i})) > u_i((\overline{s}_i,\ s_{-i}))$ for each $s_{-i} \in S_{-i}$.
  - Weak Dominance
    - Strategy $\overline{s}_i \in S_i$ is weakly dominated if there is some strategy $\hat{s}_i \in S_i$ s.t. $u_i((\hat{s}_i,\ s_{-i})) \geqslant u_i((\overline{s}_i,\ s_{-i}))$ for each $s_{-i} \in S_{-i}$ and $u_i((\hat{s}_i,\ s_{-i})) > u_i((\overline{s}_i,\ s_{-i}))$ for some s_{-i}.
  - Example
    - Second Price Auction
      - One indivisible unit of an object for sale.
      - n potential buyers with commonly known valuations $0 < v_1 < v_2 < \cdots < v_n$ for the object.
      - Buyers bid simultaneously and each submits bid $s_i \in [0,\ +\infty)$.
      - The bidder with the highest bid wins the auction and pays the second highest bid (if there are several winners, then randomly choose one).
      - Bidder i's payoff ($i = 1,\ 2,\ \cdots,\ n$) is given by

$$u_i = \begin{cases} 0, & s_i < \max\limits_{j \neq i} s_j \\ \frac{v_i - \max\limits_{j \neq i} s_j}{k}, & s_i = \max\limits_{j \neq i} s_j \\ v_i - \max\limits_{j \neq i} s_j, & s_i > \max\limits_{j \neq i} s_j \end{cases}$$

        where k is the num of winners.
      - Let $r = \max\limits_{j \neq i} s_j$, we can see the choice of r as the actions of buyers except i. The game between buyer i and other buyers can be written as the following payoff matrix:

	bidding $s_i'' < v_i$	bidding $s_i = v_i$	bidding $s_i' > v_i$
$r > s_i'$	0	0	0
$r = s_i'$	0	0	$-(s_i' - v_i)/k$
$r \in (v_i, s_i')$	0	0	$-(r - v_i)$
$r = v_i$	0	0	0
$r \in (s_i'', v_i)$	0	$v_i - r$	$v_i - r$
$r = s_i''$	$(v_i - r)/k$	$v_i - r$	$v_i - r$
$r < s_i''$	$v_i - r$	$v_i - r$	$v_i - r$

Note that bidding $s_i = v_i$ is a weakly dominant strategy even if buyer i does not know others' valuations.

- **Method of Eliminating**
  - We can get Nash equilibrium by eliminating strictly dominated strategies iteratively.
  - Example

	Left	Center	Right
Up	1, 0	1, 1	1, 0
Middle	0, 2	2, 3	2, 1
Down	-1, 2	1, 3	3, 2

    - We can easily find that for column player, "Left" and "Right" are both strictly dominated by "Center", which means we can eliminate them. Then, the matrix becomes 3×1 as follows:

	Left	Center	Right
Up	1, 0	1, 1	1, 0
Middle	0, 2	2, 3	2, 1
Down	-1, 2	1, 3	3, 2

  - Now we can see that for row player, "Up" and "Down" are both strictly dominated by "Middle". By eliminating these strategies we get the Nash equilibrium ("Middle", "Center").
- However, by eliminating weakly dominated strategies iteratively, we do not always get the Nash equilibrium (order matters).

Nash Equilibrium: Existence, Multiple Equilibria and Mixed Strategies

- **Existence**
  A Nash equilibrium exists in game $\Gamma_n = \{I,\ \{S_i\},\ \{u_i(\cdot)\}\}$ if $\forall i \in I$,
  - $\{S_i\}$ is a nonempty, convex and compact (closed and bounded) subset of some Euclidean space $\mathbb{R}^M$;
  - $u_i(s)$