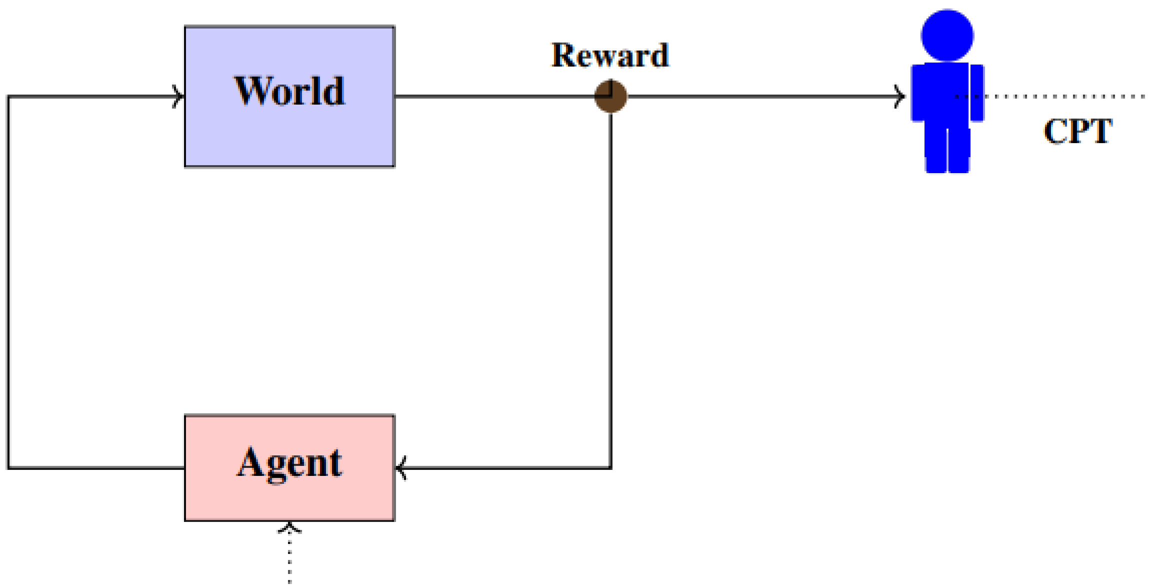
Cumulative Prospect Theory Meets Reinforcement Learning: Prediction and Control

- **Author**
  - Prashanth L.A. et al.
- **Publication**
  - ICML 2016
- **Related Work**
  - Reinforcement Learning With Function Approximation for Traffic Signal Control
    - Prashanth L. A. & Shalabh Bhatnagar
    - IEEE T-ITS 2011
  - Threshold Tuning Using Stochastic Optimization for Graded Signal Control
    - Prashanth L. A. & Shalabh Bhatnagar
    - IEEE T-VT 2012

Introduction

- **Contributions**
    - The first to define and investigate *human-centered RL problem*.
    - The first to combine CPT with RL.

- **Human-centered RL Problem**
    - Agent controls a system to produce returns that are maximally aligned with the preferences of one or possibly multiple humans.
    - Preferences of rational agents facing decisions with stochastic outcomes can be modeled using expected utilities.
    - Here the authors use CPT to model the stochastic outcomes.



Traffic Light Control (TLC) Problem

- **Target**
  - Train the traffic light to make a better traffic system
- **Environment**
  - Road network with signalled lanes that are spread across junctions and paths
  - Road users (cars)
- **Agent**
  - Traffic lights
- **States**
  - Queue length (pathwise)
  - Elapsed time (pathwise)
- **Actions**
  - feasible combinations of red and green
- **Reward**
  - CPT-value of differential delay X (because CPT needs gains and losses), i.e., $C(X) = \sum_{i=1}^{M} \mu_i C(X_i)$ where X_i is the differential delay (calculated by the elapsed time minus a baseline which is the elapsed time of a fixed-time signal control) of i-th path, μ_i is the proportion of road users on the i-th path and M is the num of paths.

CPT-value Estimation

- **Convergence assumptions (eithor or) for probability weighting function w**
  - Lipschitz continuous
  - $H\ddot{o}lder$ continuous

    Definition 1. ***(Hölder continuity)*** *A function $f \in C([a, b])$ is said to satisfy a Hölder condition of order $\alpha \in (0, 1]$ (or to be Hölder continuous of order α) if there exists $H > 0$, s.t.*

    $$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} \leq H.$$

    - Lipschitz continuity is the case that $\alpha = 1$.
  - Locally Lipschitz

- **Estimation scheme under each of the assumptions**
  - Under $H\ddot{o}lder$ continuity, the CPT-value can be estimatied by the discrete version.

    $$\sum_{i=-m}^{n} \pi_i v(x_i),$$

    where

    $$\pi_i = \begin{cases} w(p_i + \cdots + p_n) - w(p_{i+1} + \cdots + p_n) & \text{for} \quad 0 \leq i \leq n \\ w(p_{-m} + \cdots + p_i) - w(p_{-m} + \cdots + p_{i-1}) & \quad\quad\; -m \leq i < 0 \end{cases},$$

    - Algo:

1: Simulate n i.i.d. samples from the distribution of X.

2: Order the samples and label them as follows: $X_{[1]}, X_{[2]}, \ldots, X_{[n]}$. Note that $u^+(X_{[1]}), \ldots, u^+(X_{[n]})$ are also in ascending order.

3: Let

$$\overline{\mathbb{C}}_n^+ := \sum_{i=1}^{n} u^+(X_{[i]}) \left(w^+\left(\frac{n+1-i}{n}\right) - w^+\left(\frac{n-i}{n}\right)\right).$$

4: Apply u^- on the sequence $\{X_{[1]}, X_{[2]}, \ldots, X_{[n]}\}$; notice that $u^-(X_{[i]})$ is in descending order since u^- is a decreasing function.

5: Let

$$\overline{\mathbb{C}}_n^- := \sum_{i=1}^{n} u^-(X_{[i]}) \left(w^-\left(\frac{i}{n}\right) - w^-\left(\frac{i-1}{n}\right)\right).$$

6: Return $\overline{\mathbb{C}}_n = \overline{\mathbb{C}}_n^+ - \overline{\mathbb{C}}_n^-$.

- Under Locally Lipschitz
  - Omitted

Gradient-based Algo for CPT Optimization

- **Optimization Objective (Reward)**

$$\max_{\theta \in \Theta} \quad C(X^\theta)$$

  where the param vector θ with dimension d is contrained in a compact and convex real set Θ.

- **Gradient Estimation**
  - It is hard to get the gradient of $C(X^\theta)$, especially in high-dimension cases.
  - SPSA (Simultaneous Perturbation Stochastic Approximation)
    - A method to solve the problem above.
    - Use stochastic perturbation to get 2 sample values and approximate the gradient. The idea is similar to $\frac{\mathrm{d}f(x)}{\mathrm{d}x} \approx \frac{f(x+\varepsilon)-f(x-\varepsilon)}{2\varepsilon}$ where ε is a number close to 0.
    - At the n-th iteration, the gradient is estimated by

$$\hat{\nabla}_i C(X^\theta) = \frac{\bar{C}_n^{\theta_n+\delta_n\Delta_n} - \bar{C}_n^{\theta_n-\delta_n\Delta_n}}{2\delta_n\Delta_n^i}$$

      where $\bar{C}_n^{\theta_n}$ is the CPT-estimation calculated using the algo in *CPT-value Estimation* with m_n samples generated by param vector θ_n, δ_n tends to 0 as $n \to \infty$ (like ε above) and $\Delta_n = (\Delta_n^1, \ \cdots, \ \Delta_n^d)^\mathsf{T}$ where $\{\Delta_n^i\}_{i=1}^d \overset{\text{i.i.d.}}{\sim} Rademacher$ (Rademacher distribution is 1 with half prob and -1 with half prob) .

- **Update Gradient**
  - Update Rule (for the i-th component of θ):

$$\theta_{n+1}^i = \Gamma_i \left(\theta_n^i + \gamma_n \hat{\nabla}_i C\left(X^{\theta_n}\right)\right)$$

    where Γ_i is the clip operater to constrain θ in Θ and γ_n is the learning rate (also called the step size).
  - Convergence Condition
    - To guarantee that we can obtain the solution after several iterations, we need

$$\begin{cases} \gamma_n,\ \delta_n \to 0 \\ \frac{1}{m_n^{\frac{\alpha}{2}}\delta_n} \to 0 \\ \sum_n \gamma_n \to \infty \\ \sum_n \frac{\gamma_n^2}{\delta_n^2} < \infty \end{cases} \overset{\text{simple choice}}{\Longrightarrow} \begin{cases} \gamma_n = \frac{\gamma_0}{n} \\ m_n = m_0 n^{\nu} \\ \delta_n = \frac{\delta_0}{n^{\gamma}} \end{cases} \text{for} \begin{cases} \nu,\ \gamma > 0 \\ \gamma > \frac{\nu\alpha}{2} \end{cases}$$

where α is the $H\ddot{o}lder$ order (we choose $\alpha = 1$ for Lipschitz continuity).

- **Gradient Ascent Algo**

Input: initial parameter $\theta_0 \in \Theta$ where Θ is a compact and convex subset of $\mathbb{R}^d$, perturbation constants $\delta_n > 0$, sample sizes $\{m_n\}$, step-sizes $\{\gamma_n\}$, operator $\Gamma : \mathbb{R}^d \to \Theta$.

for $n = 0, 1, 2, \ldots$ **do**

Generate $\{\Delta_n^i, i = 1, \ldots, d\}$ using Rademacher distribution, independent of $\{\Delta_m, m = 0, 1, \ldots, n-1\}$.

CPT-value Estimation (Trajectory 1)

Simulate m_n samples using $(\theta_n + \delta_n \Delta_n)$.

Obtain CPT-value estimate $\overline{\mathbb{C}}_n^{\theta_n + \delta_n \Delta_n}$.

CPT-value Estimation (Trajectory 2)

Simulate m_n samples using $(\theta_n - \delta_n \Delta_n)$.

Obtain CPT-value estimate $\overline{\mathbb{C}}_n^{\theta_n - \delta_n \Delta_n}$.

Gradient Ascent

Update θ_n using the update rule above.

end for

Return θ_n.

> There still remains a problem: how the m_n samples are generated, i.e., what is the distribution (policy) we use? See the next chapter.

Function Approximation and Boltzmann Policy

- **Function Approximation**
  - Note that in Q-learning, the policy is dependent on the Q-table, which contains all $Q(s,\ a)$ for each state s and each action a. At some state s, the agent choose the action with biggest Q value. The dimension of the Q-table is $S \times A$ where S is the num of states and A is the num of feasible actions.
  - In problems with high-dimension of states and actions, Q-table becomes computationally expensive. Thus, we use function approximation to approximate the Q funtion:

$$Q(s,\ a) \approx \theta^{\mathsf{T}} \phi_{s,\ a}$$

    where $\phi_{s,\ a}$ is a d-dimensional vector describing the state-action feature, e.g., d can be $S + A$ if we construct features on each states and actions separately instead of considering their combinations.
  - In this paper, the authors use the following settings of state-action feature $\phi_{s,\ a}$:

State	Action	Feature
$q_i(n) < L_1$ and $t_i(n) < T_1$	RED GREEN	0 1
$q_i(n) < L_1$ and $t_i(n) \geq T_1$	RED GREEN	0.2 0.8
$L_1 \leq q_i(n) < L_2$ and $t_i(n) < T_1$	RED GREEN	0.4 0.6
$L_1 \leq q_i(n) < L_2$ and $t_i(n) \geq T_1$	RED GREEN	0.6 0.4
$q_i(n) \geq L_2$ and $t_i(n) < T_1$	RED GREEN	0.8 0.2
$q_i(n) \geq L_2$ and $t_i(n) \geq T_1$	RED GREEN	1 0

    where q represents the queue length, t represents the elapsed time and $L_1,\ L_2,\ T_1$ are thresholds. Now the dimension $d = 6A$ since they have shrinked the S states to be 6 states based on the thresholds.

- **Boltzmann Policy**

- Instead of choosing action with the biggest Q value, Boltzmann policy makes the Q values to be probs under some state.
- Formula:

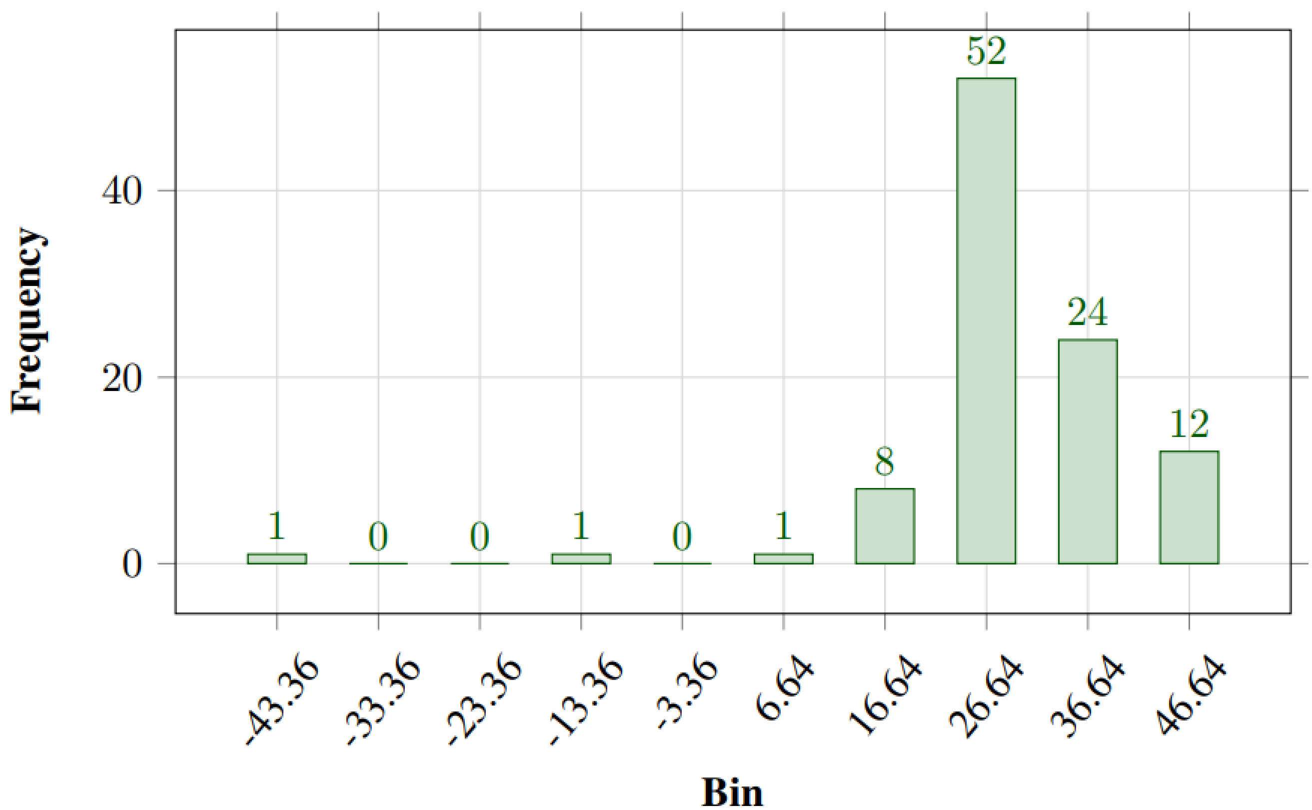
$$\pi_\theta(s,\ a) = \frac{e^{\theta^\top \phi_{s,\,a}}}{\sum_{a' \in A(s)} e^{\theta^\top \phi_{s,\,a'}}}$$

  where $A(s)$ is the feasible action set under a certain state s.
  - The exponential operator makes all values positive and thus can be transformed to be probs.

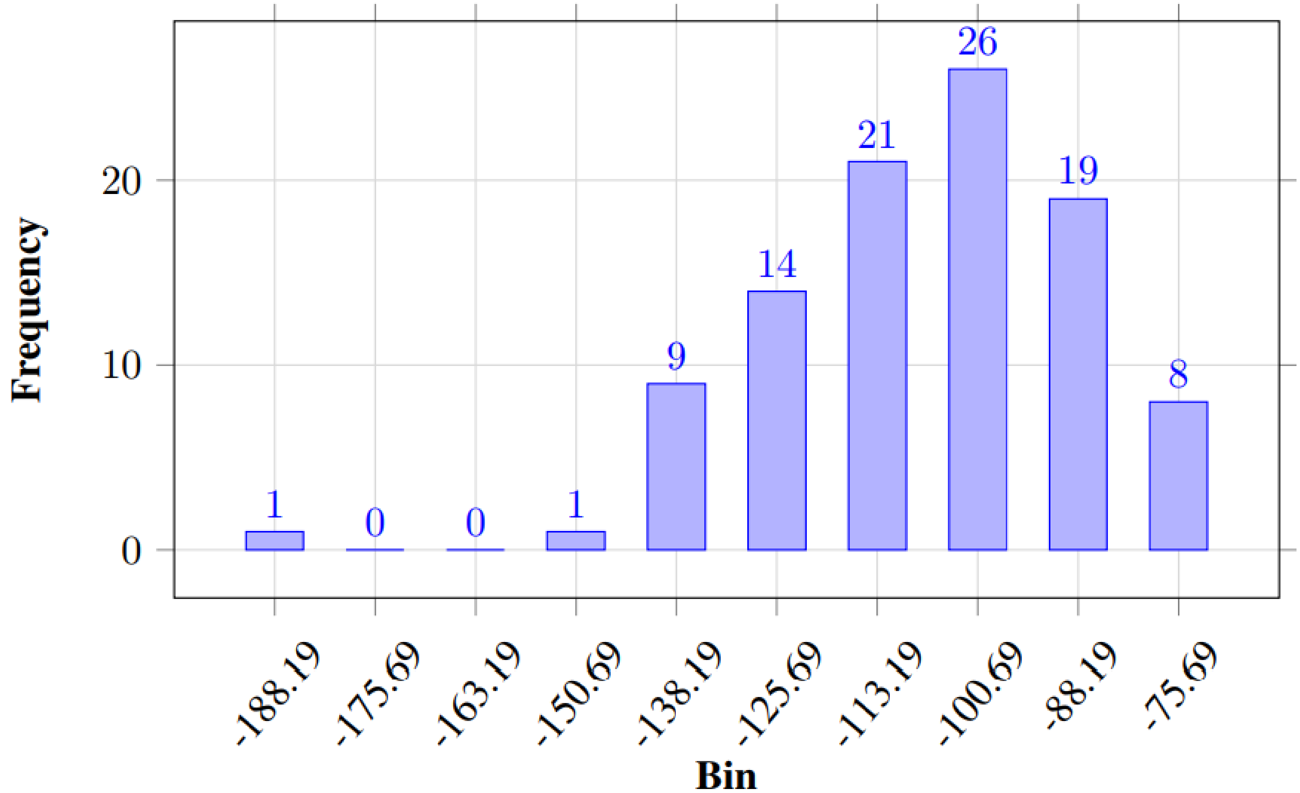
Experiments

- **Experiment Design**
  - CPT
    - Use CPT-value as reward.
  - EUT
    - Use value function u but do not use probability weighting function w.
  - AVG
    - Use neither value function nor probability weighting function, i.e., just use the simple mean.
  - Do 100 independent tests, calculate CPT-value of these 3 algos respectively.

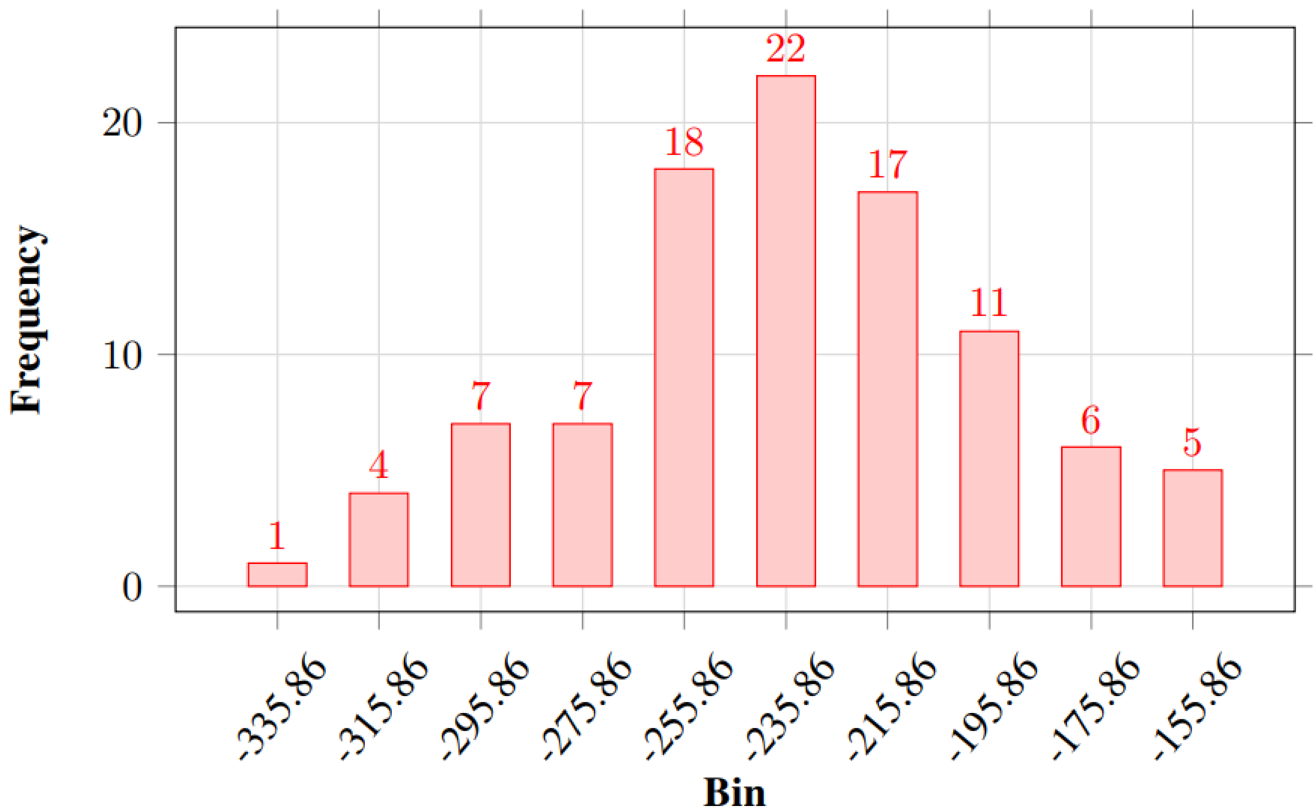
- **Results**
  - CPT



  - EUT



- AVG



- Results show that CPT > EUT > AVG. I think this is obvious and even need not be tested since the final measure is CPT-value and the CPT model is trained based on that.