NDA: Time Series Analysis (1/2)

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Bibliography

Content:

- Peter Brockwell and Richard Davis
  Introduction to Time Series and Forecasting
- William Thistleton and Tural Sadigov
  MOOC Coursera: *Practical Time Series Analysis*

Illustrative datasets:

- `https://data.world/datasets/time-series`
- `https://www.kaggle.com/tags/time-series`

Outline

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What is time series analysis

Definition

Set of observations $\{x_t\}$, recorded at time $t \in T_0$

Think of each x_t as a realization from a distribution

Specificities of the problem

A unique realization of the process

$\Rightarrow$ necessary to make assumptions

- observe time series, identify particularities
- choose a family of models X_t to represent data
- check the goodness of the model

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Assumptions for this course

Restrictions to a subfamily of problems

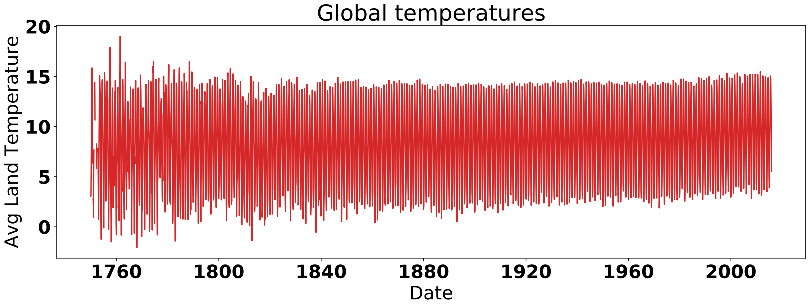
- discrete time series (discrete time set)
- fixed time steps (time resolution)
- univariate (one single variable over time)
  $\rightarrow$ processes have values in $\mathbb{R}$

And only a few approaches

- e.g. no Fourier analysis

A few examples

Average global land temperature (per month)



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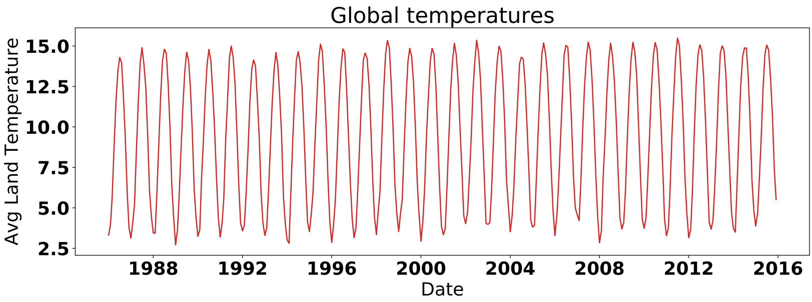
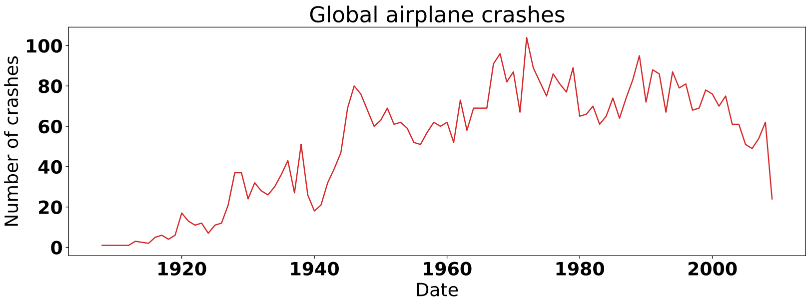


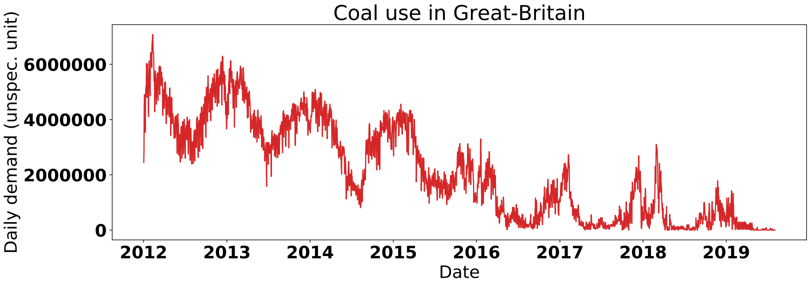
A few examples

Number of airplane crashes (per year)



A few examples

Daily demand of power obtained with coal in GB (per year)



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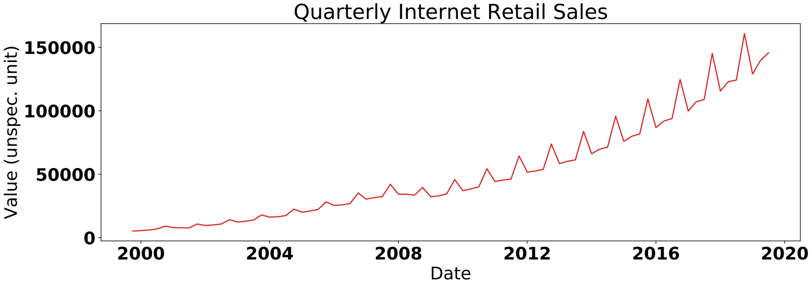


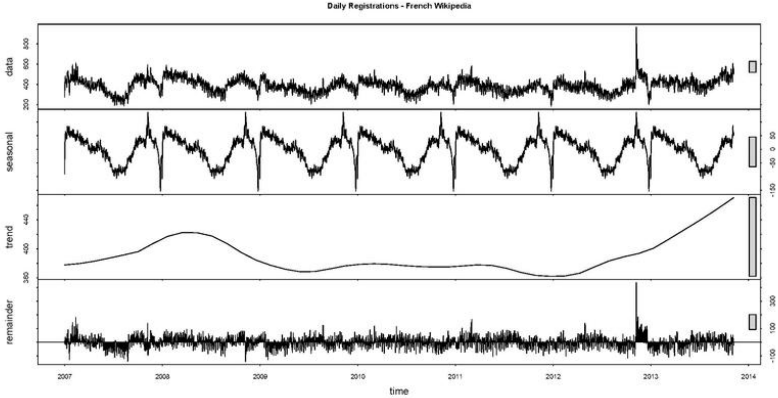
Goals of time series analysis

- Have a simplified description of the data
  $\rightarrow$ improve our understanding (*ex: climate data*)
- Test an assumption
  ex: is there a significant measurable global warming?
- Filter: separate signal from noise
  ex: known physical signal broadcast $\rightarrow$ *filter noise*
- Predict future values
  ex: predict the future demand for a product
- Simulate a process in a complex model
  ex: expectation for the GDP to predict economic activity

How to analyze a time series? (1)

Analyse from Greek *análusis* $\sim$ unravel $\Rightarrow$ decompose

Decompose the time series into parts, for example:

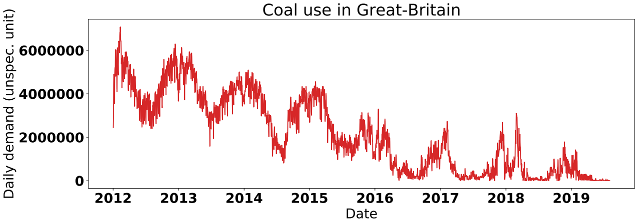


How to analyze a time series? (1)

First step

Plot the time series to:

- identify the existence of a trend *(tendance)*
- uncover seasonal variations *(variations saisonnières)*
- detect changes of behavior
- spot outliers *(valeurs aberrantes)*



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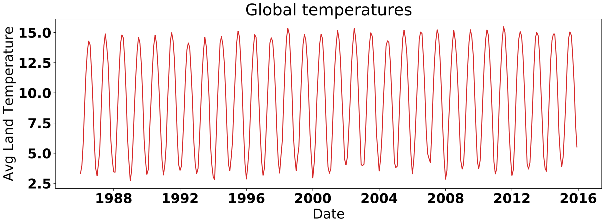

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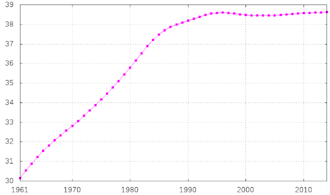
Poland population



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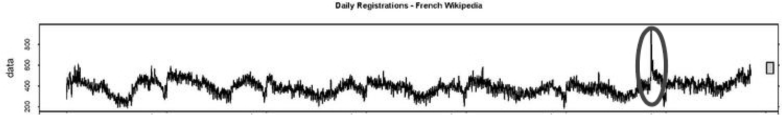

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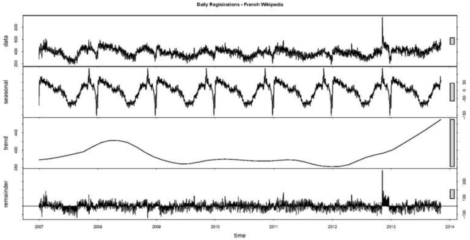
$\rightarrow$ subjective components in this analysis

The classical decomposition

Classical decomposition of the time series

$$X_t = s_t + m_t + r_t$$

- seasonality s_t
- trend m_t
- remainder r_t



Outline

Mean and covariance of a time series

Two fundamental definitions

Let $\{X_t\}$ a time series with $\mathbb{E}[X_t^2] < \infty$ (finite variance)

rk: here we consider X_t as a model

- **mean function** of X_t, defined for all t:

$$\mu_X(t) = \mathbb{E}[X_t]$$

- **covariance function** of X_t, defined for all r, s:

$$\gamma_X(r, s) = Cov(X_r, X_s) = \mathbb{E}[(X_r - \mu_X(r))(X_s - \mu_X(s))]$$

Stationarity

Informal definition

A process is **stationary** *(stationnaire)* if its statistical properties are similar when shifted in time

Remarks:

- stationarity is a property of a model (not of data)
- stationary processes are simpler to investigate
  $\Rightarrow$ usual to transform a TS to obtain a stationary process

Stationarity

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A process is **stationary** *(stationnaire)* if its statistical properties are similar when shifted in time

Formal definition

A process is said to be **weakly stationary** if

- the mean function $\mu_X(t)$ is independent of t
- $\gamma_X(t+h, t)$ is independent of t for any h

  h is called the *lag* *(décalage)*

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A process is **stationary** *(stationnaire)* if its statistical properties are similar when shifted in time

Formal definition

A process is said to be **strictly (or strongly) stationary** if

- $\forall n$ and $\forall h$

$$P(X_1 = x_1, \dots, X_n = x_n) = P(X_{1+h} = x_1, \dots, X_{n+h} = x_n)$$

Unless specified otherwise, we talk about weak stationarity in the following

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Autocorrelation function

Notice that for a stationary time series: $\gamma_X(t+h,t) = \gamma_X(h)$
$\Rightarrow$ the covariance function γ_X has one variable (the lag)

Definition

For a stationary time series:

- the **autocovariance function** at lag h is:

$$\gamma_X(h) = Cov(X_{t+h}, X_t)$$

- the **autocorrelation function** (ACF) at lag h is:

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)}$$

Equivalent on real data

Concepts well defined on models, but what about real data?
Let $\{x_1, \ldots, x_n\}$ be a series of observations

Sample mean

- the **sample mean** is

$$\overline{x} = \frac{1}{n} \sum_{t=1}^{n} x_t$$

Equivalent on real data

Concepts well defined on models, but what about real data?
Let $\{x_1, \ldots, x_n\}$ be a series of observations

Sample autocovariance function

- the **sample autocovariance function** is

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} (x_{t+|h|} - \overline{x}).(x_t - \overline{x}) \ , \ -n < h < n$$

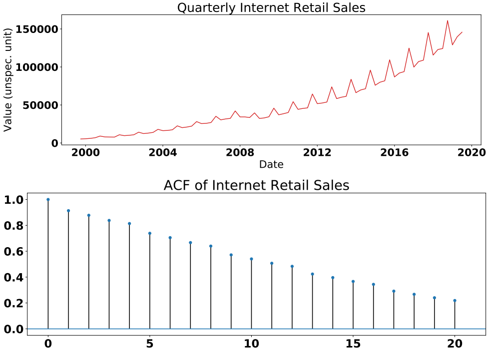
remark: notice the denominator (because of mathematical properties)

- the **sample autocorrelation function** is

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} \ , \ -n < h < n. \text{ Note that } \hat{\rho}(h) \in [-1; 1]$$

Equivalent on real data

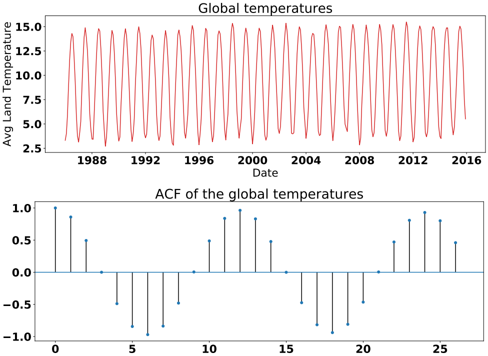
Data with strong trend:



slow decay of correlations with h

Equivalent on real data

Data with strong seasonality:



periodicity on the ACF (here monthly measures $\Rightarrow$ period = 12)

Outline

What is a time series model?

Definition

Time series model: specification of the joint distributions of a sequence of random variables X_t of which the observed data is supposed to be the realization

Remarks:

- suppose to know $\forall n$ the distribs $P(X_1 = x_1, \ldots, X_n = x_n)$ $\Rightarrow$ in most case too many parameters...
- in practice, we focus on first and second order moments:
  - expected values $\mathbb{E}[X_t]$
  - and expected products $\mathbb{E}[X_{t+h}X_t]$, $h = 1, 2, \ldots$

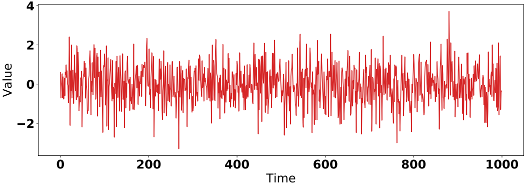
Independent Identically Distributed noise model

IID noise

- independant:
  $P(X_1 = x_1, \ldots, X_n = x_n) = P(X_1 = x_1) \cdot \ldots \cdot P(X_n = x_n)$
- identically distributed: $P(X_t = x) = P(X_{t'} = x)$

IID noise is obviously stationary

ex: repeated coin flipping with heads=1, tails=-1 should be IID noise



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White noise *(bruit blanc)*

Special case IID noise with

- 0 mean: $E[X_t] = 0$
- autocovariance function:
  $\gamma_X(h) = \sigma^2$ if $h = 0$ and $\gamma_X(h) = 0$ if $h \neq 0$

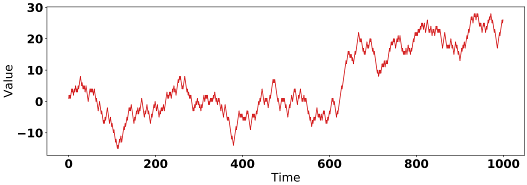
Random Walk model

How to build a random walk? *(marche aléatoire)*

Suppose $\{X_t\}$ is IID noise, then $\{S_t\}$ defined as:

$$S_t = X_1 + \ldots + X_t$$

is a random walk



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Remarks:

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Outline

How to analyze a time series? (2)

Second step:

- (if necessary) transform data
- remove the trend and seasonal components to get stationary residuals *(résidus)*

Residual time series obtained (remainder) is stationary, but not necessarily IID noise…

Back to the classical decomposition

Classical decomposition of the time series

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What is the difference between seasonality and trend?

$$s_{t+d} = s_t \text{ and } \sum_{j=1}^{d} s_j = 0$$

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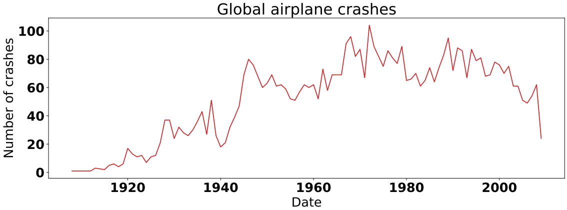
Isolate the trend component

Method 1: model and regression

$\rightarrow$ cf. course *Regression*

E.g.: 2^{nd} order polynomial model with least squares regression

Minimize $\sum_{t=1}^{n}(x_t - m_t)^2$, with $m_t = a_0 + a_1 t + a_2 t^2$



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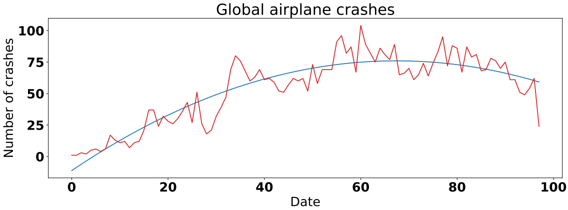
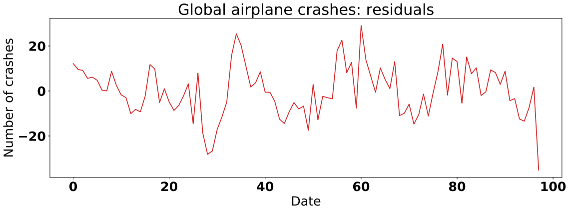


Isolate the trend component

Then we plot the residuals $\{x_t - m_t\}$



Questions to ask oneself:

- Does it look stationary? Perceptible trend?
- Does it look like noise? Is it smooth? Do we see stretch of values of the same sign?

Moving Average models: MA(1)

What is a moving average? *(moyenne mobile)*

To smooth a signal x_t, one possibility:

$$x'_t = \frac{1}{2q+1} \sum_{h=-q}^{h=+q} x_{t-h} \;,\; q < t < n-q$$

tool for signal processing *(low pass filter, filtre passe-bas)*

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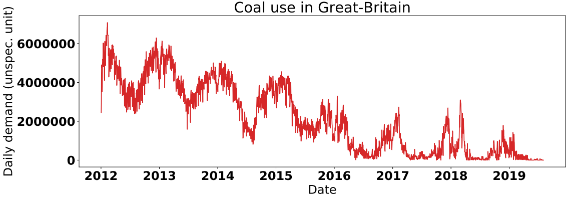

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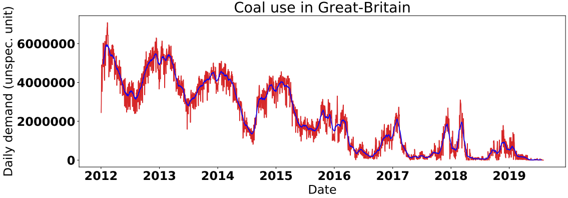

Isolate the trend component (2)

can also be seen as a method to isolate the trend

Method 2: moving average

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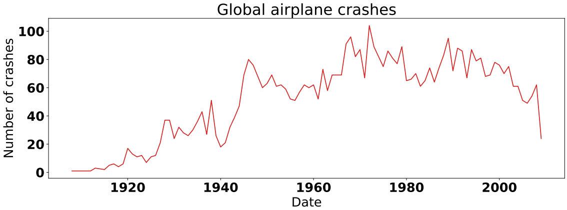

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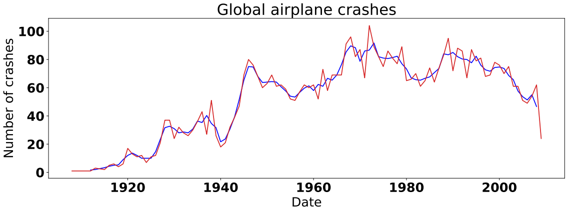

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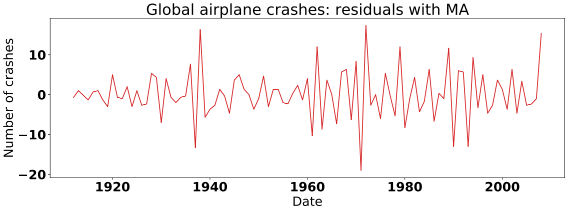

Isolate seasonal component

Regression

Which model? Harmonic regression

$$s_t = a_0 + \sum_{j=1}^{k} a_j cos\left(\frac{2\pi t}{T}\right) + b_j sin\left(\frac{2\pi t}{T}\right)$$

where T is the expected period of the process

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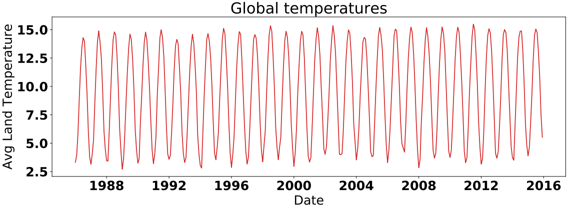

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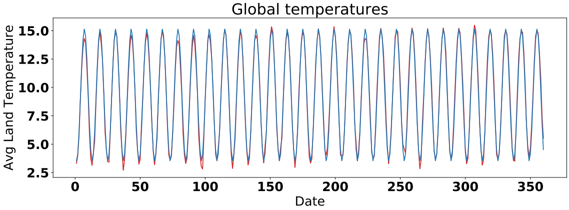

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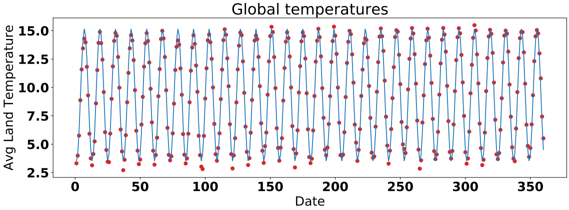

About pre-processing

Second step:

- (if necessary) transform data
- remove the trend and seasonal components to get stationary residuals

When is it necessary to transform data ?

Some cases

- if outliers → discard them if justified
  ex: external stimulus, mistake in data acquisition, ...
- if obvious different regimes
  → break data into homogeneous segments
- if noise or seasonality component increases with level
  → logarithmic transformation of the data

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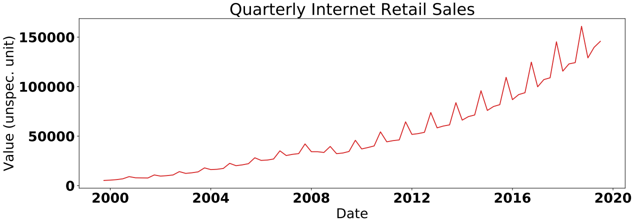
Logarithmic transformation

If fluctuations (seasonality, noise) grow with magnitude…



…after logarithmic transform

→ c.f. course *Regression (heteroscedasticity)*

Conduct similar analysis on the transformed time series and reverse the transformations in the end to make predictions etc.

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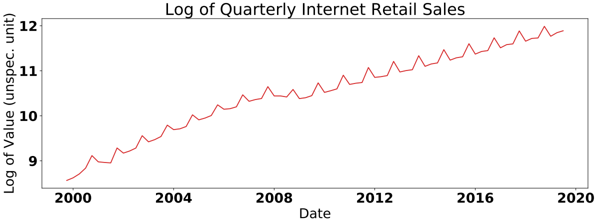

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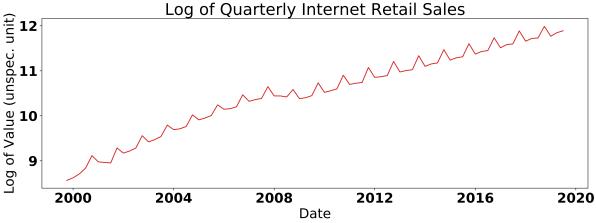

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Outline

How to analyze a time series? (3)

Third step:

- fit the residuals

For this purpose, we introduce new families of models

Auto-Regressive models: AR(1)

What is autoregression?

auto means self $\Rightarrow$ regression from itself

The most basic AR model: 1^{st} order regression or AR(1)

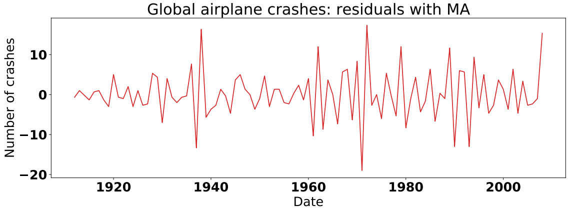
$\{X_t\}$ is a stationary series satisfying:

$$X_t = \phi X_{t-1} + W_t$$

where W_t is a white noise (0 means, σ^2 variance) and $|\phi| < 1$

we can check that $\mathbb{E}[X_t] = 0$ and $\gamma_X(h) = \phi^{|h|}\gamma_X(0) = \phi^{|h|}\frac{\sigma^2}{1-\phi^2}$

Auto-Regressive models: AR(1)



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- plot r_t as a function of r_{t-1}
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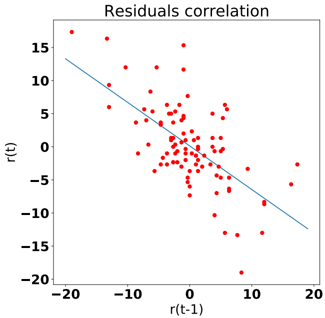

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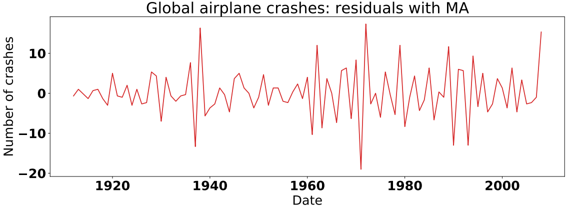
$$r_t - \phi r_{t-1}$$



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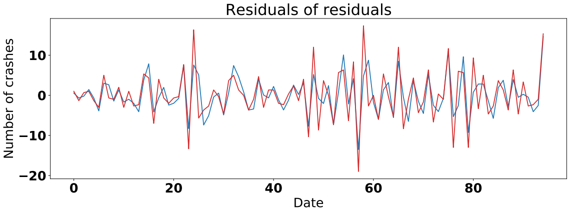

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Testing if the residuals time series is IID: method 1

General idea: suppose IID random variables
What should we oberve? Is it the case?

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Sample ACF criterion

Theorem (admitted):

- suppose x_t IID with mean 0 and variance 1 (white noise)
- if n large enough, $\hat{\rho}_x(h)$ is approx. distributed as $\mathcal{N}(0, \frac{1}{\sqrt{n}})$

In practice, consider the 95% confidence interval:
we measure how many values fall out of $\left[\frac{-1.96}{\sqrt{n}}, \frac{+1.96}{\sqrt{n}}\right]$

$\rightarrow$ c.f. course *Hypothesis testing*

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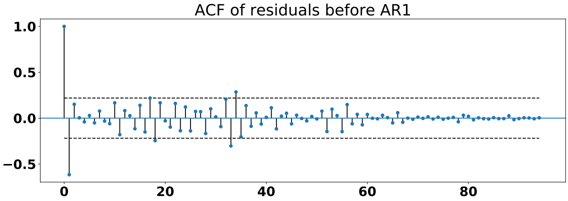

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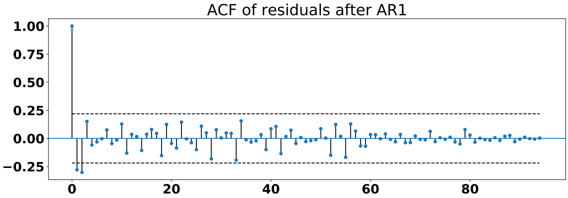

Testing if the residuals time series is IID: method 2

General idea: suppose IID random variables
What should we observe? Is it the case?

Turning point test

Turning point x_i (only defined for $1 < i < n$):
if $x_i \geq x_{i-1}$ and $x_i \geq x_{i+1}$ or x_i if $x_i \leq x_{i-1}$ and $x_i \leq x_{i+1}$

- Probability that a point is a turning point if IID? $\frac{2}{3}$
- $\Rightarrow \mu_T = \mathbb{E}[T_n] = \frac{2(n-2)}{3}$, with T_n number of turning points
- $Var(T_n) = \mathbb{E}[T_n^2] - \mathbb{E}[T_n]^2 \Rightarrow \sigma_T^2 = Var(T_n) = \frac{16n-29}{90}$

If x_i is IID, T_n is approximately $\mathcal{N}(\mu_T, \sigma_T^2)$

- test if $1.96 > \frac{T_n - \mu_T}{\sigma_T} > -1.96$ for the 95% CI

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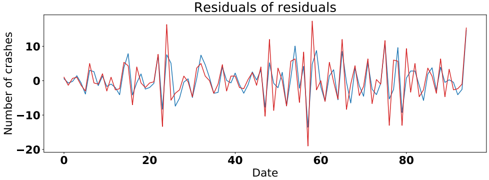
In our example (airplane crashes):



- T_n^{res} = 64 on the residuals
- T_n^{resAR1} = 59 on the residuals of the residuals (after AR1)

$\Rightarrow$ both pass this test

Studying time series in python

Among several options, `pandas` library

A few useful functions:

- Load data as dataframe:
  `read_csv` from `pandas` library
- Moving average:
  `rolling` from `pandas` library
- Fitting:
  `curve_fit` in `scipy.optimize` library
- Autocorrelation function:
  `plot_ACF` in `statsmodels` library