A. Giovanidis 2019

Probability

Data Analysis for Networks - DataNets'19
Anastasios Giovanidis

Sorbonne-LIP6

June 7, 2019

Bibliography

A. Giovanidis 2019

B.1 Christopher M. Bishop, "Pattern Recognition and Machine Learning", Springer 2006.

☞ Chapter 1.2

B.2 H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

☞ Chapter 7.1

A. Giovanidis 2019

A. Giovanidis 2019

Uncertainty

The key concept we are dealing with is uncertainty.

Uncertainty arises through:

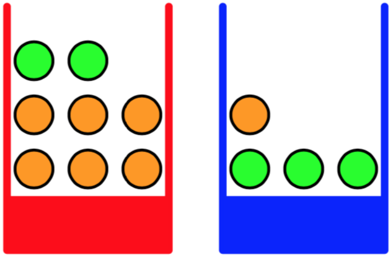
- noise on measurements
- finite size of data-sets

Probability theory provides a consistent framework for the quantification and manipulation of uncertainty.

A. Giovanidis 2019

Boxes, apples and oranges

Consider two boxes (red and blue), that contain apples and oranges.



☞ Pick one box at random, and then pick one fruit, note the result and put it back.

Event Probability

A. Giovanidis 2019

> **Probability** of an event is the fraction of times that the event occurs out of the total number of trials, in the limit that the number of trials goes to infinity.

☞ This is the frequentist definition, in contrast to the axiomatic one.

Let us suppose for the sake of example:

- We pick the red box 40% of times and
- the blue box 60% of times.

Event Probability

A. Giovanidis 2019

We write:

$$P(B = r) = \frac{4}{10}$$
$$P(B = b) = \frac{6}{10}$$

- Each probability must lie in the interval $[0, 1]$.
- If the events are mutually exclusive and if they include all possible outcomes (e.g. red, or blue box) then these probabilities must sum to 1.

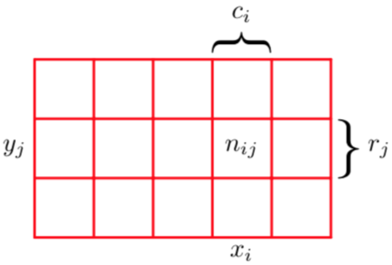
Questions

A. Giovanidis 2019

- "What is the overall probability that the selection procedure will pick an apple?"
- "Given we have chosen an orange, what is the probability that the box chosen is the blue one?

More general framework

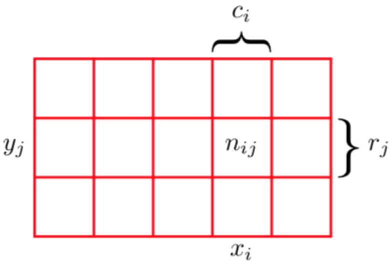
A. Giovanidis 2019



- Two random variables, X and Y.
- X takes any value x_i, $i = 1, \ldots, M$
- Y takes any value y_j, $j = 1, \ldots, L$.
- We sample both r.v.'s N times each.

A. Giovanidis 2019

Joint probability



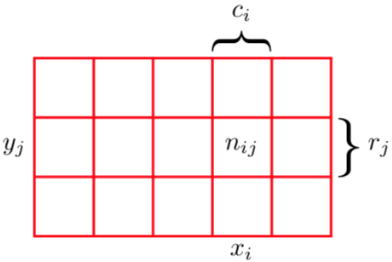
- The number of times we find $(X = x_i, Y = y_j)$ is n_{ij}.
- The joint probability reads

$$P(X = x_i, Y = y_j) = \frac{n_{ij}}{N}.$$

(we implicitly consider the limit $N \to \infty$)

Marginal probability

A. Giovanidis 2019



- The number of times we find $X = x_i$ (for any Y) is c_i.
- The number of times we find $Y = y_j$ (for any X) is r_j.
- The probability for $X = x_i$ reads (marginal probability)

$$P(X = x_i) = \frac{c_i}{N} \quad = \quad \frac{1}{N}\sum_{j=1}^{L} n_{ij} = \sum_{j=1}^{L} P(X = x_i, Y = y_j).$$

Conditional probability

A. Giovanidis 2019

By considering only those instances for which $X = x_i$:

- The fraction of instances for which $Y = y_j$ is

$$P(Y = y_j \mid X = x_i) \quad = \quad \frac{n_{ij}}{c_i}.$$

- We can derive the following relation

$$\begin{aligned} P\left(X = x_i, Y = y_j\right) \quad &= \quad \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i} \cdot \frac{c_i}{N} \\ &= \quad P(Y = y_j \mid X = x_i) \cdot P\left(X = x_i\right) \end{aligned}$$

☞ We denote by $P(X)$ the distribution over the random variable X.

The probability rules

A. Giovanidis 2019

sum rule $P(X) = \sum_Y P(X, Y)$

product rule $P(X, Y) = P(Y \mid X) P(X)$

A. Giovanidis 2019

The probability rules

sum rule $P(X) = \sum_Y P(X, Y)$

product rule $P(X, Y) = P(Y \mid X)P(X)$

☞ By symmetry we get

$$P(Y \mid X)P(X) \quad = \quad P(X \mid Y)P(Y)$$

which gives

Bayes' theorem $P(Y \mid X) = \frac{P(X \mid Y)P(Y)}{P(X)}$.

Back to apples and oranges

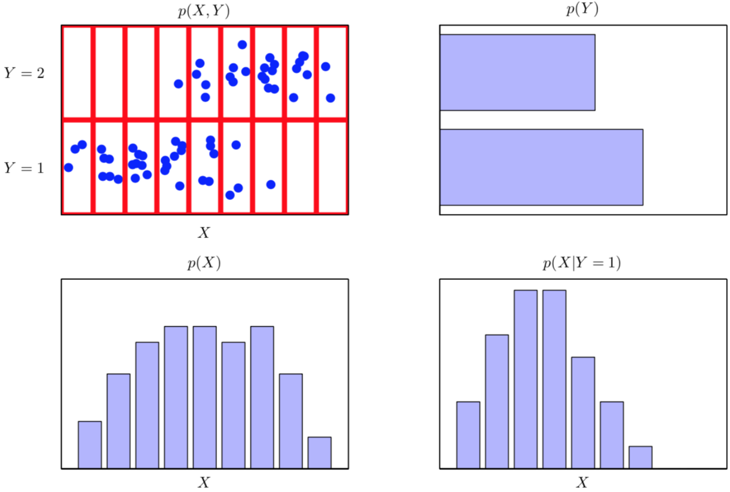
A. Giovanidis 2019

Given we picked an orange, what is the probability that it came from the red box?

$$
\begin{aligned}
P(B = \mathit{red} \mid F = \mathit{orange}) &= \frac{P(F = \mathit{orange} \mid B = \mathit{red}) \cdot P(B = \mathit{red})}{P(F = \mathit{orange})} \\
&= \frac{\frac{6}{8} \cdot \frac{4}{10}}{\frac{6}{8} \cdot \frac{4}{10} + \frac{1}{4} \cdot \frac{6}{10}} \\
&= \frac{24/80}{36/80} \\
&= \frac{2}{3}.
\end{aligned}
$$

Distribution over 2 variables

A. Giovanidis 2019



Elementary properties of probability

A. Giovanidis 2019

Consider events A, B as ensembles of realisations for the r.v. X, e.g. $A = \{x_1, x_4, x_{M-1}\}$, and $B = \{x_2\}$.

1. **Complementarity rule** $P(A^c) = 1 - P(A)$,
   where A^c is the complementary event of A, i.e. $A^c = \Omega/A$.
2. **Empty set** $P(\emptyset) = 0$.
3. **Inclusion property**: $B \subseteq A \Rightarrow P(B) \leq P(A)$ and $P(A - B) = P(A) - P(B)$.
4. **Additivity**: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
5. **Exclusivity**: $P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$, if the events A_K are mutually exclusive, i.e. $A_i \cap A_j = \emptyset$, for $i \neq j$.

The last property is due to the fact that for two exclusive events:

$$P(A_i \cap A_j) = P(\emptyset) = 0.$$

Conditional Probability and Independence

A. Giovanidis 2019

Two r.v.'s are independent when the probability of the one does not depend on the realization of the other

$$P(X|Y) \quad = \quad P(X),$$

so that conditioning does not add information. This definition has the following consequence. From the product rule

$$\begin{aligned} P(X,Y) &= P(X|Y) \cdot P(Y) \\ &\overset{indep.}{=} P(X) \cdot P(Y) \end{aligned}$$

☞ In our example, $P(X|Y) = P(X)$, when $\frac{n_{ij}}{r_j} = \frac{r_j}{N}$.

A. Giovanidis 2019

A. Giovanidis 2019

Mass and Density

- **Discrete** r.v. $\rightarrow$ probability mass function (p.m.f)

$$p_i := P(X = x_i), \quad \textit{for} \quad i = 1, \ldots, M$$

$$p_i \geq 0, \quad \textit{and} \quad \sum_{i=1}^{M} p_i = 1.$$

A. Giovanidis 2019

Mass and Density

- **Discrete** r.v. $\rightarrow$ probability mass function (p.m.f)

$$p_i := P(X = x_i), \quad for \quad i = 1, \ldots, M$$

$$p_i \geq 0, \quad and \quad \sum_{i=1}^{M} p_i = 1.$$

- **Continuous** r.v. $\rightarrow$ probability density function (p.d.f)

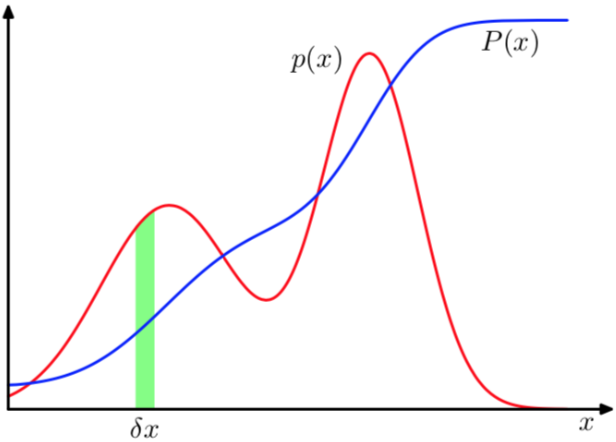
$$p(x)\delta x \quad for \quad (x, x + \delta x)$$

and when $\delta x \rightarrow 0$ then $p(x)$ is the probability density over x.

$$p(x) \geq 0, \quad and \quad \int_0^\infty p(x)dx = 1.$$

PDF

A. Giovanidis 2019



PDF and CDF

A. Giovanidis 2019

☞ Using the PDF we can derive the probability that the r.v. lies within a certain interval

$$P(X \in (a,b)) \;=\; \int_a^b p(x)dx$$

The probability that x lies in the interval $(-\infty, z)$ is given by the cumulative distribution function defined by

$$P(z) = \int_{-\infty}^{z} p(x)dx$$

which satisfies $dP(z)/dz|_{z=x} = p(x)$.

Sum and product rule

A. Giovanidis 2019

These rules apply also in the case of continuous r.v.'s for the PDF.

Sum Rule $$p(x) = \int p(x, y)dy$$

Product Rule $$p(x, y) = p(y|x)p(x).$$

Bayes' Theorem $$p(y|x) = \frac{p(x|y)p(y)}{p(x)}$$

Change of variables

A. Giovanidis 2019

Suppose we know the pdf $p_X(x)$ and we would like to find the pdf of a function of X, say $Y = f(X)$. So, we are looking for the pdf

$$p_X(x) \quad \rightarrow \quad p_Y(y)$$

Practically, since we know $p_X(x)$ we will need the inverse of f

$$X = f^{-1}(Y) \quad := \quad g(Y)$$

BUT WE CANNOT JUST WRITE $p_X\,(g(y))$!!

☞ Intuition: for small δx, δy the range $(x + \delta x)$ transforms to $(y + \delta y)$ and it holds $p_X(x)\delta x \approx p_Y(y)\delta y$, or

$$p_Y(y) \quad = \quad p_X(x)\frac{\delta x}{\delta y}.$$

Change of variables cont'd

A. Giovanidis 2019

$$\begin{aligned} p_Y(y) &= p_X(x)\left|\frac{dx}{dy}\right| \\ &= p_X(x)|g'(y)|. \end{aligned}$$

A. Giovanidis 2019

A. Giovanidis 2019

Expectation

The expectation is the weighted average value of some function of X, say $f(X)$ under a distribution p(X) and is defined as

Discrete $$\mathbb{E}[f] = \sum_x p(x) f(x)$$

Continuous $$\mathbb{E}[f] = \int p(x) f(x) dx.$$

Note that

$$\mathbb{E}_x[f(X,Y)] = g(Y)$$

$$\mathbb{E}_X[f(X,Y) \mid Y] = \sum_x p(x|Y) f(x)$$

for discrete variables, and using integrals for continuous.

Estimating Expectation from Samples

A. Giovanidis 2019

Suppose we are given N samples drawn from the $p(X)$, then the expectation can be approximated as a finite sum over these points:

$$\mathbb{E}[f] \quad \approx \quad \frac{1}{N}\sum_{n=1}^{N} f(x_n).$$

When $N \to \infty$ this converges to the true expectation, for both continuous and discrete variables.

Variance

A. Giovanidis 2019

The variance is a measure of variability of $f(X)$ around its mean $\mathbb{E}[f(X)]$.

Definition

$$\begin{aligned} Var(f(X)) &= \mathbb{E}[(f(X) - \mathbb{E}[f(X)])^2] \\ &= \mathbb{E}[f(X)^2] - \mathbb{E}[f(X)]^2. \end{aligned}$$

The variance of the variable X itself, is given as

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Covariance

A. Giovanidis 2019

The covariance expresses the extent to which X and Y vary together.

Definition

$$\begin{aligned} Cov(X,Y) &= \mathbb{E}_{X,Y}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] \\ &= \mathbb{E}_{X,Y}[XY]-\mathbb{E}[X]\mathbb{E}[Y]. \end{aligned}$$

Note here that, when X, Y are independent r.v.'s,
$P(X,Y) = P(X)P(Y)$, and hence

$$Cov(X,Y) \stackrel{indep.}{=} 0.$$

A. Giovanidis 2019

Bernoulli(p)

A. Giovanidis 2019

The result of an experiment is either positive or negatve

$$X = \begin{cases} 0, & \text{with probability } 1-p \\ 1, & \text{with probability } p \end{cases} .$$

$$\begin{aligned} \mathbb{E}[X] &= 1 \cdot p + 0 \cdot (1-p) = p \\ Var(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 1^2 \cdot p - p^2 = p(1-p). \end{aligned}$$

$\mathrm{Binomial(N, p)}$

A. Giovanidis 2019

We repeat the same experiment N times, where the result of each experiment is an independent $\mathrm{Bernoulli(p)}$ distribution. Here, X counts the number of times among the N efforts that the experiment was successful.

$$P(X = k) = \left(\begin{array}{c} N \\ k \end{array} \right) p^k (1-p)^{N-k}, \qquad k = 0, \ldots, N$$

where,

$$\left(\begin{array}{c} N \\ k \end{array} \right) = \frac{N!}{k!(N-k)!}.$$

counts the different combinations of k successes among the N efforts.
From the binomial formula

$$\sum_{k=0}^{N} P(X = k) = (p + (1-p))^N = 1.$$

$\mathrm{Binomial(N, p)}$ cont'd

A. Giovanidis 2019

$$\mathbb{E}(X) = \sum_{k=0}^{N} k \begin{pmatrix} N \\ k \end{pmatrix} p^k (1-p)^{N-k} = N \cdot p$$

$$Var(X) = Np(1-p).$$

Geometric(p)

A. Giovanidis 2019

Repeat independent $\mathrm{Bernoulli}(p)$ trials until observing the first success. Here, X is the number of efforts until success.

$$P(X=k)=(1-p)^{k-1}p, \qquad k=1,2,\ldots$$

The mean and variance are equal to

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=1}^{\infty} k(1-p)^{k-1}p=\frac{1}{p},\\ Var(X) &= \frac{1-p}{p^2}.\end{aligned}$$

Applications: ARQ retransmission, Wifi ALOHA, CSMA protocols, etc.

Poisson(λ)

A. Giovanidis 2019

Repeat independent $\mathrm{Bernoulli}(p)$ trials until observing the first success. Here, X is the number of efforts until success.

$$P(X = k) = \exp(-\lambda)\frac{\lambda^k}{k!} \qquad k = 0, 1, \ldots$$

The mean and variance are equal to

$$\begin{aligned} \mathbb{E}[X] &= \sum_{k=1}^{\infty} k \exp(-\lambda)\frac{\lambda^k}{k!} = \lambda, \\ Var(X) &= \lambda. \end{aligned}$$

Applications: models telephone calls, or packets within some time-interval.

☞ The Poisson distribution is the limit of a $\mathrm{Binomial}(N, p)$, as $N \to \infty$ and $p \to 0$, with $Np = \lambda$ fixed, for all pairs of values.

A. Giovanidis 2019

Uniform(a, b)

A. Giovanidis 2019

The Uniform r.v. X has a PDF which takes a constant value in a predefined interval, else 0. Hence the probability to find X in a small continuous subset of the interval, is proportional to the length of the subset,

$$p_X(x) = \begin{cases} \frac{1}{b-a}, & \text{for } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases} .$$

It is indeed a PDF because $1/(b-a) \geq 0$ and

$$\int_{-\infty}^{+\infty} \frac{1}{b-a} dx = \int_a^b \frac{1}{b-a} dx = 1.$$

and $\mathbb{E}[X] = \frac{a+b}{2}$, and $Var[X] = \frac{(b-a)^2}{12}$.

A. Giovanidis 2019

Exponential(λ)

A continuous random variable X is said to have an exponential distribution with parameter $\lambda > 0$, shown as $X \sim \mathrm{Exponential}(\lambda)$, if its PDF is given by

$$p_X(x) = \begin{cases} \lambda \exp(-\lambda x), & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

The mean and variance are equal to

$$\mathbb{E}[X] = \lambda^{-1} \qquad Var(X) = \lambda^{-2}.$$

Applications: Inter-arrival time between consecutive telephone calls, or packets. *Memoryless Property!
☞ The $\mathrm{Poisson}(\lambda)$ distribution in 1-D has the property that the distance between any two consecutive points is $\mathrm{Exp}(\lambda)$ distributed.

A. Giovanidis 2019

Standard Normal$(0,1)$

A continuous random variable Z is said to be a standard normal (standard Gaussian) r.v., shown as $Z \sim \mathcal{N}(0,1)$, if its PDF is given by

$$p_Z(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right).$$

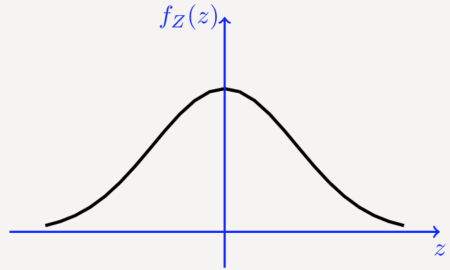


Fig.4.6 - PDF of the standard normal random variable.

A. Giovanidis 2019

Standard $\mathrm{Normal}(0,1)$ cont'd

A continuous random variable Z is said to be a standard normal (standard Gaussian) r.v., shown as $Z \sim \mathcal{N}(0,1)$, if its PDF is given by

$$p_Z(z) \quad = \quad \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right).$$

- The $1/\sqrt{2\pi}$ makes sures that the area under the PDF is equal to 1.
- It holds $\mathbb{E}[Z] = 0$ and $Var(Z) = 1$.

The CDF of the standard normal distribution is denoted by the Φ function:

$$\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(\frac{-u^2}{2}\right) du.$$

Standard Normal$(0, 1)$ ccont'd

A. Giovanidis 2019

☞ Because of the importance of the normal distribution, the values of the Φ function have been tabulated and many calculators and software packages have this function.

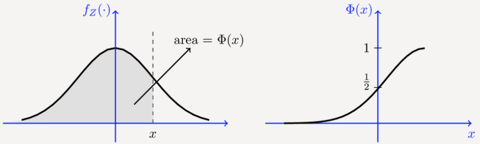


Fig.4.7 - The Φ function (CDF of standard normal).

Here are some properties of the Φ function that can be shown from its definition.

1. $\lim_{x\to\infty} \Phi(x) = 1, \quad \lim_{x\to-\infty} \Phi(x) = 0$;
2. $\Phi(0) = \frac{1}{2}$;
3. $\Phi(-x) = 1 - \Phi(x)$, for all $x \in \mathbb{R}$.

Normal(μ, σ^2)

A. Giovanidis 2019

We can obtain any normal random variable X by shifting and scaling a standard normal random variable Z.

$$X = \sigma Z + \mu,$$

where $\sigma > 0$. Then

$$\begin{aligned} \mathbb{E}[X] &= \sigma \mathbb{E}[Z] + \mu = \mu \\ Var(X) &= \sigma^2 Var(Z) = \sigma^2. \end{aligned}$$

Then X is a normal random variable with mean μ and variance σ^2.
We write $X \sim (\mu, \sigma^2)$.

$\mathrm{Normal}(\mu, \sigma^2)$ cont'd

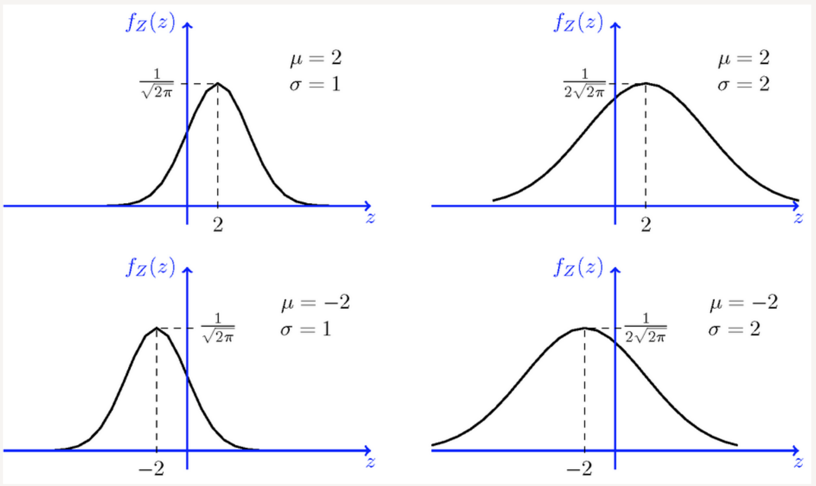
A. Giovanidis 2019

Conversely, if $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$
\begin{aligned}
p_X(x) &= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \\
F_X(x) = P(X \leq x) &= \Phi\left(\frac{x-\mu}{\sigma}\right), \\
P(a < X \leq b) &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).
\end{aligned}
$$

$\mathrm{Normal}(\mu, \sigma^2)$ Examples

A. Giovanidis 2019



A. Giovanidis 2019

Addition of 2 $\mathrm{Normal}(\mu_\mathrm{k}, \sigma_\mathrm{k}^2)$ r.v.'s

Suppose we would like to find the distribution of the sum

$$Y = X_1 + X_2,$$

where each $X_k \sim \mathcal{N}\left(\mu_k, \sigma_k^2\right)$, and they are independent. Then,

$$Y \sim \mathcal{N}\left(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2\right).$$

If they are not independent, but they are jointly normal then

$$Y \sim \mathcal{N}\left(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2\rho(X_1, X_2)\sigma_1\sigma_2\right).$$

where

$$\rho(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1)Var(X_2)}} = \frac{Cov(X_1, X_2)}{\sigma_1\sigma_2}.$$

A. Giovanidis 2019

A. Giovanidis 2019

1st important limit Theorem

- Law of Large Numbers (LLN),

LLN: the average of a large number of i.i.d. random variables converges to the expected value.

Sample mean

A. Giovanidis 2019

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables.

Sample Mean:

$$\overline{X} = \frac{X_1 + \ldots + X_n}{n}.$$

- The sample mean is a random variable.
- $\mathbb{E}[\overline{X}] = \mathbb{E}[X_1]$.
- $Var(\overline{X}) = \frac{Var(X_1)}{n}$.

Law of Large Numbers

A. Giovanidis 2019

☞ If you repeat an experiment independently a large number of times and average the result, what you obtain should be close to the expected value.

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with a finite expected value $E[X_i] = \mu < \infty$.

Weak Law of Large Numbers:
For any $\epsilon > 0$,

$$\lim_{n \to \infty} P\left(|\overline{X} - \mu| \geq \epsilon\right) = 0$$

❍ Convergence *in probability* (weak).

A Proof for WLLN

A. Giovanidis 2019

We assume for the samples $X_1, \ldots, X_n$ finite mean and variance:

$$
\begin{aligned}
P(|\overline{X} - \mu| \geq \epsilon) &\leq \frac{Var(\overline{X})}{\epsilon^2} \ (Chebyshev) \\
& \frac{Var(X_1)}{n\epsilon^2} \\
&\rightarrow 0
\end{aligned}
$$

2nd important limit Theorem

A. Giovanidis 2019

- Central Limit Theorem (CLT).

CLT: the sum of a large number of random variables has an approximately normal distribution (under some conditions).

Normalisation

A. Giovanidis 2019

Suppose $X_1, \ldots, X_n$ are i.i.d. random variables with expected values $\mathbb{E}[X_i] = \mu < \infty$ and variance $Var(X_i) = \sigma^2 < \infty$.

- Sample mean has $\mathbb{E}[\overline{X}] = \mu$ and $Var(\overline{X}) = \frac{\sigma^2}{n}$.
- Thus, the normalised random variable

$$Z_n = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + \ldots + X_n - n\mu}{\sigma\sqrt{n}}$$

has mean $\mathbb{E}[Z_n] = 0$ and variance $Var(Z_n) = 1$.

Remark: If r.v. Y has mean $M = 0$ and variance S^2, then with high probability all values fall within the interval $[0 - kS, 0 + kS]$. The new variable Y/S will have mean $M' = 0$ and variance $S' = 1$, and a range of values $[0 - k, 0 + k]$ with high probability. It is a shrinked version of Y.

CLT

A. Giovanidis 2019

The Central Limit Theorem (CLT)

Under the above assumptions (i.i.d. samples, finite mean, variance) the random variable

$$Z_n = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} = \frac{X_1 + \ldots + X_n - n\mu}{\sigma\sqrt{n}}$$

converges in distribution to the standard normal random variable as n goes to infinity, that is

$$\lim_{n\to\infty} P(Z_n \leq x) = \Phi(x), \quad \textit{for all} \quad x \in \mathbb{R},$$

where $\Phi(x)$ is the standard normal CDF.

CLT for all distributions!

A. Giovanidis 2019

The CLT does not care what the distribution of the X_i's is!
The X_i's can be discrete, continuous, or mixed r.v.'s.

☞ Example 1: X_i's are $\mathrm{Bernoulli}(p)$ (discrete with PMF).

- $\mathbb{E}[X_i] = p$, and $Var(X_i) = p(1-p)$.
- $Y_n = X_1 + X_2 + \ldots + X_n$ has $\mathrm{Binomial}(n, p)$ distribution.
- Normalisation

$$Z_n = \frac{Y_n - np}{\sqrt{np(1-p)}},$$

where $Y_n \sim \mathrm{Binomial}(n, p)$.

CLT for Bernoulli

A. Giovanidis 2019

Assumptions:

- $X_1, X_2 \ldots$ are iid Bernoulli(p).
- $Z_n = \dfrac{X_1 + X_2 + \ldots + X_n - np}{\sqrt{np(1-p)}}$.

We choose $p = \frac{1}{3}$.

$$Z_1 = \frac{X_1 - p}{\sqrt{p(1-p)}}$$

$$Z_2 = \frac{X_1 + X_2 - 2p}{\sqrt{2p(1-p)}}$$

$$Z_3 = \frac{X_1 + X_2 + X_3 - 3p}{\sqrt{3p(1-p)}}$$

$$Z_{30} = \frac{\sum_{i=1}^{30} X_i - 30p}{\sqrt{30p(1-p)}}$$

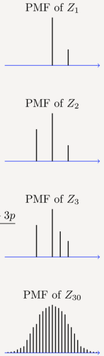


Fig.7.1 - Z_n is the normalized sum of n independent $Bernoulli(p)$ random variables. The shape of its PMF, $P_{Z_n}(z)$, resembles the normal curve as n increases.

Another example

A. Giovanidis 2019

☞ Example 2: X_i's are $\mathrm{Uniform}(0,1)$ (continous with PDF).

- $\mathbb{E}[X_i] = 1/2$, and $Var(X_i) = 1/12$.
- Normalisation

$$Z_n = \frac{X_1 + \ldots + X_n - n/2}{\sqrt{n/12}}.$$

CLT for Uniform

A. Giovanidis 2019

Assumptions:

- $X_1, X_2 \ldots$ are iid Uniform(0,1).
- $Z_n = \dfrac{X_1 + X_2 + \ldots + X_n - \frac{n}{2}}{\sqrt{\frac{n}{12}}}$.

$$Z_1 = \frac{X_1 - \frac{1}{2}}{\sqrt{\frac{1}{12}}}$$

$$Z_2 = \frac{X_1 + X_2 - 1}{\sqrt{\frac{2}{12}}}$$

$$Z_3 = \frac{X_1 + X_2 + X_3 - \frac{2}{3}}{\sqrt{\frac{3}{12}}}$$

$$Z_{30} = \frac{\sum_{i=1}^{30} X_i - \frac{30}{2}}{\sqrt{\frac{30}{12}}}$$

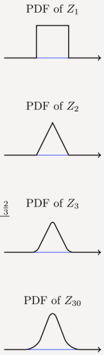


Fig. 7.2 - Z_n is the normalized sum of n independent $Uniform(0, 1)$ random variables. The shape of its PDF, $f_{Z_n}(z)$, gets closer to the normal curve as n increases.

CLT application 1

A. Giovanidis 2019

❑ Exercise 1: A data-centre serves requests of service that arrive over time. Suppose the service time X_i for request i has mean $\mathbb{E}[X_i] = 2$ [msec] and variance $Var[X_i] = 1$ [msec].

Let Y be the total time necessary to serve $n = 50$ requests.
What is the probability that this time ranges between $90 \leq Y \leq 110$ [msec]?

Solve application 1

A. Giovanidis 2019

Solution: We write the sum

$$Y_{50} = X_1 + \ldots + X_{50}$$

for the total time needed to serve 50 requests. This is a r.v..
We are looking for:

$$P\left(90 \leq Y_{50} \leq 110\right) = P\left(90 \leq X_1 + \ldots + X_{50} \leq 110\right)$$

The mean of Y is $50\mathbb{E}X_i = 100$ [msec] and the variance is $50Var[X_i] = 50$ [msec].

From the CLT, the normalised variable follows the Normal distribution

$$\frac{Y_{50} - 50\mathbb{E}X_i}{\sqrt{50Var(X_i)}} = \frac{Y_{50} - 100}{5\sqrt{2}} \sim \mathcal{N}\left(0,1\right).$$

A. Giovanidis 2019

Solve application 1 cont'd

Hence, we can use the Normal CDF

$$\begin{aligned} P\left(\frac{90-100}{5\sqrt{2}} \leq \frac{Y_{50}-100}{5\sqrt{2}} \leq \frac{110-100}{5\sqrt{2}}\right) &= \\ P\left(-\sqrt{2} \leq \frac{Y_{50}-100}{5\sqrt{2}} \leq \sqrt{2}\right) &= \\ \Phi\left(\sqrt{2}\right) - \Phi\left(-\sqrt{2}\right) &= 0.8427. \end{aligned}$$

We conclude that the service period is between 90 and 110 [msec] with probability 84.27%.

CLT application 2

A. Giovanidis 2019

❑ Exercise 2: A data packet consists of $n = 1000$ bits. Due to noise, bit errors occur independently, with probability $p = 0.1$.

What is the probability to have more that 120 errors within the same packet?

Solve application 2

A. Giovanidis 2019

Solution: We write the sum

$$Y_{1000} \quad = \quad X_1 + \ldots + X_{1000}$$

where X_1 takes 1 if an error occurs, otherwise 0. The Y_{1000} is a r.v..
We are looking for:

$$P\left(Y_{1000} > 120\right)$$

We known that $X_i \sim \mathrm{Bernoulli}(p = 0.1)$. The mean of Y is $1000p = 100$ [errors] and the variance is $1000p(1-p) = 90$ [errors].

From the CLT, the normalised variable follows the Normal distribution

$$\frac{Y_{1000} - 100}{\sqrt{90}} \sim \mathcal{N}\left(0, 1\right).$$

Solve application 2 cont'd

A. Giovanidis 2019

Hence, we can use the Normal CDF

$$\begin{aligned} P\left(\frac{Y_{1000}-100}{\sqrt{90}} > \frac{120-100}{\sqrt{90}}\right) &= \\ 1-\Phi\left(\frac{20}{\sqrt{90}}\right) &= 0.0175. \end{aligned}$$

We conclude that the probability to exceed 120 errors per packet is 1.75%.

A. Giovanidis 2019

END