

# Linear Regression

Data Analysis for Networks - DataNets'19  
Anastasios Giovanidis

Sorbonne-LIP6



January 17, 2019

## Bibliography

- B.1** Gareth James, Daniela Witten, Trevor Hastie, Robert Tibshirani. "An introduction to statistical learning: with applications in R". Springer Texts in Statistics.  
📖 **Chapter 3**  
ISBN 978-1-4614-7137-0 (DOI 10.1007/978-1-4614-7138-7)
- B.2** H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at  
<https://www.probabilitycourse.com>, Kappa Research LLC, 2014.

# Intro

**Linear Regression:** A very simple approach for Supervised Learning:

- been around for a very long time...
- predicts a quantitative response.
- explains the relationship between two or more variables.

Many fancy statistical learning approaches can be seen as generalisation or extensions of linear regression.

This lecture: **Linear Regression, Least-Squares**

**Training data** available:

$$D_n = \{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n)\}$$

Data come in pairs  $(\mathbf{x}_i, y_i)$  of

- ▶  $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,K})$  **input data**, as a vector of size  $K \geq 1$ .
- ▶  $y_i$  **output data** of size 1.

**Goal:** Given set of known I/O pairs, "guess" a function  $f : \mathbb{R}^K \rightarrow \mathbb{R}$  that for any input  $\mathbf{x}^{(o)}$  predicts its output  $y^{(o)}$ .

☞ We hope to get a "good" prediction. What is "good"?

# Linear Regression

**Linear Regression:** Assumes that the output is an **affine** function of the input.

$$y = \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_K x_K + \beta_0 + \epsilon, \quad (1)$$

**Unknowns:**

- $(\beta_1, \dots, \beta_K) \in \mathbb{R}^K$  is a vector of *parameters/coefficients*.
- $\beta_0$  is the *bias parameter or intercept*.
- $\epsilon$  is a **random error term**; independent of  $\mathbf{x}$  with mean 0.

☞ Using known pairs, find the Regression line = “line of best fit”.

# Simple Linear Regression

Let us simplify for just 1-dimension,  $x \in \mathbb{R}$ .

$$y = \beta_1 x + \beta_0 + \epsilon \quad (2)$$

- ▶  $\beta_1$  is the **slope** (average increase in  $y$  for unit increase of  $x$ ).
- ▶  $\beta_0$  is the **intercept term** (expected value of  $y$  when  $x = 0$ ),
- ▶  $\epsilon$  is the **error term**, which summarises what we miss by using a simple linear model, e.g. non-linearities, other variables, or measurement errors.

## Simple Linear Regression, cont'd

**Simple Goal:** Use training data to produce estimates  $\hat{\beta}_0$  and  $\hat{\beta}_1$  for the model coefficients. Then we can make the prediction  $\hat{y}_o$  for  $y_o$ , on the basis of input  $x_o$ :

$$\hat{y} = \hat{\beta}_1 x + \hat{\beta}_0, \quad (3)$$

👉 Draw a line in the  $x - y$  plane that best "fits" our data points. This is called the **regression line** (without the error term).

# Examples

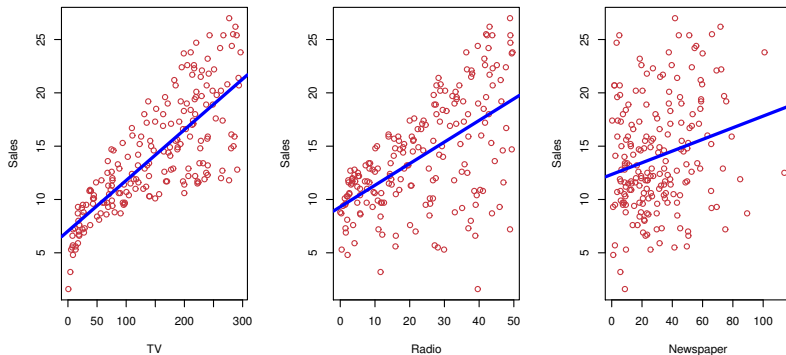


Figure: Advertisement Related Example. <sup>1</sup>

---

<sup>1</sup>Figure from [B.1] with permission from the authors: G. James, D. Witten, T. Hastie and R. Tibshirani



## How to draw the line?

We need to draw a line that "fits well" the **known data**. How?

- ▶ The available known data are the pairs  $(x_i, y_i)$ ,  $i = 1, \dots, n$ .
- ▶ The regression line gives  $\hat{y}_i = \hat{\beta}_1 x_i + \hat{\beta}_0$ , for input  $x_i$ .

**Definition:** the following quantities are called **residuals**

$$\epsilon_i := y_i - \hat{y}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i \quad . \quad (4)$$

# Residuals

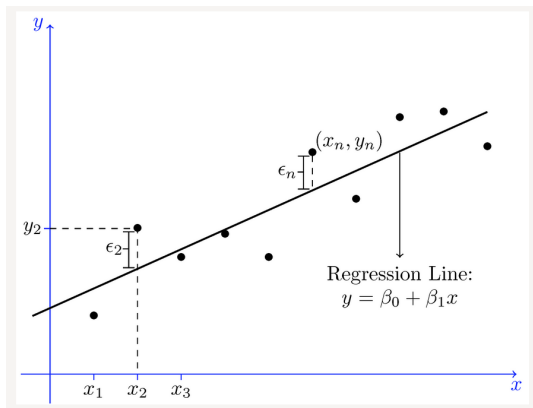


Figure: Regression Line and Residuals.<sup>2</sup>

---

<sup>2</sup>Source [B.2]

## Sum of Squares

The *Residual Sum of Squares* ( $RSS$ ) is a function of  $\hat{\beta}_0$ ,  $\hat{\beta}_1$ ,

$$RSS(\hat{\beta}_0, \hat{\beta}_1) = \sum_{i=1}^n \epsilon_i^2 := \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2.$$

It is always non-negative  $RSS \geq 0$ .

👉 Find and use the coefficients that **minimize** the  $RSS$ ! This should provide a good fit to the data.

## Least Squares

The optimal *least squares coefficient estimates* are found by:

$$\min \text{RSS}(\hat{\beta}_0, \hat{\beta}_1).$$

**Solution:**

$$\frac{\partial \text{RSS}}{\partial \hat{\beta}_0} = 2(-1) \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) = 0,$$

$$\frac{\partial \text{RSS}}{\partial \hat{\beta}_1} = 2(-1) \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) x_i = 0.$$

## Coefficient Estimates

Given  $n \geq 3$  observations  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ , we estimate  $\beta_0$  and  $\beta_1$  as

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}, \quad (5)$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}, \quad (6)$$

where  $\bar{y} := \frac{1}{n} \sum_{i=1}^n y_i$  and  $\bar{x} := \frac{1}{n} \sum_{i=1}^n x_i$  are the **sample means**.

## Alternative Method

Consider again the linear model with random errors.  
Error  $\epsilon$  has zero mean  $\mathbb{E}[\epsilon] = 0$  and is independent of  $x$ .

$$y = \beta_1 x + \beta_0 + \epsilon$$

👉 Applying expectation on both sides, we get

$$\begin{aligned}\mathbb{E}[y] &= \beta_1 \mathbb{E}[x] + \beta_0 + \mathbb{E}[\epsilon] \\ &= \beta_1 \mathbb{E}[x] + \beta_0.\end{aligned}$$

## Alternative Method cont'd

☞ We take **covariance** between  $x$  and  $y = \beta_0 + \beta_1 x + \epsilon$ .

$$\begin{aligned}\text{Cov}(x, y) &:= \mathbb{E}[(x - \mathbb{E}[x])(y - \mathbb{E}[y])] \\ &= \mathbb{E}[(x - \mathbb{E}[x])(\beta_0 + \beta_1 x + \epsilon - \beta_0 - \beta_1 \mathbb{E}[x] - \mathbb{E}[\epsilon])] \\ &= \mathbb{E}[(x - \mathbb{E}[x])(\epsilon + \beta_1(x - \mathbb{E}[x]))] \\ &= \beta_1 \mathbb{E}[(x - \mathbb{E}[x])^2] =: \beta_1 \cdot \text{Var}(x)\end{aligned}$$

## Alternative Method cont'd II

$$\beta_1 = \frac{\text{Cov}(x, y)}{\text{Var}(x)}, \quad (7)$$

$$\beta_0 = \mathbb{E}[y] - \beta_1 \mathbb{E}[x]. \quad (8)$$

☞ With the observed pairs  $(x_1, y_1), \dots, (x_n, y_n)$  we get the estimates  $\hat{\beta}_0, \hat{\beta}_1$ ,

$$\mathbb{E}[x] \approx \frac{1}{n} \sum_{i=1}^n x_i =: \bar{x}, \quad (9)$$

$$\mathbb{E}[y] \approx \frac{1}{n} \sum_{i=1}^n y_i =: \bar{y}, \quad (10)$$

$$\text{Var}(x) \approx \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 =: s_{xx} \quad (11)$$

$$\text{Cov}(x, y) \approx \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) := s_{xy} \quad (12)$$

**Remark:** For  $s_{xx}$  and  $s_{xy}$ , division by  $n$  or  $n-1$  does not affect the coefficients. But the same choice should be applied to both estimators.



## Numerical Example

Consider a data set of  $n = 4$  known  $(x, y)$  samples

$$D_4 = \{(1, 3), (2, 4), (3, 8), (4, 9)\}$$

Find:

- ▶ The coefficients  $\hat{\beta}_0, \hat{\beta}_1$  for the simple linear regression.
- ▶ The residuals  $\epsilon_i$  from the estimated values.

## Solution

$$\bar{x} = \frac{1 + 2 + 3 + 4}{4} = 2.5 \quad , \quad \bar{y} = \frac{3 + 4 + 8 + 9}{4} = 6.$$

$$s_{xx} = \frac{1}{4 - 1} [(1 - 2.5)^2 + (2 - 2.5)^2 + (3 - 2.5)^2 + (4 - 2.5)^2] = \frac{5}{3}$$

$$s_{xy} = \frac{1}{4 - 1} [(1 - 2.5)(3 - 6) + (2 - 2.5)(4 - 6) + \\ + (3 - 2.5)(8 - 6) + (4 - 2.5)(9 - 6)] = \frac{11}{3}$$

## Solution cont'd

$$\begin{aligned}\hat{\beta}_1 &= \frac{s_{xy}}{s_{xx}} = \frac{11}{5} = 2.2 \\ \hat{\beta}_0 &= 6 - (2.2)(2.5) = 0.5\end{aligned}$$

Regression-line:

$$\hat{y}_i = 0.5 + 2.2x_i,$$

Residuals:

$$\hat{y}_1 = 2.7, \hat{y}_1 = 4.9, \hat{y}_1 = 7.1 \hat{y}_1 = 9.3$$

$$\hat{e}_1 = 0.3, \hat{e}_2 = -0.9, \hat{e}_3 = 0.9 \hat{e}_4 = -0.3$$

🔍 **Verification:**  $\hat{e}_1 + \hat{e}_2 + \hat{e}_3 + \hat{e}_4 = 0$ . (Why?)

## Accuracy of the model

- ▶ *Mean Squared Error (MSE).*
- ▶ *Residual Standard Error (RSE).*
- ▶  $R^2$  statistic or *Coefficient of determination.*
- ▶ p-value.

## Mean Squared Error

One can use the **Mean Squared Error (MSE)**, defined as

$$MSE := \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2 \quad (13)$$

$$= \frac{1}{n} RSS \quad (14)$$

It is a measure of *lack of fit* for the model.

## Residual Standard Error

The **residual standard error** provides the average amount that the response  $y$  deviates from the true regression line  $\beta_0 + \beta_1 x$ .

$$RSE = \sqrt{\frac{1}{n-2} RSS} = \sqrt{\frac{1}{n-2} \sum_{i=1}^n (y_i - \hat{y}_i)^2}. \quad (15)$$

It is a measure of *lack of fit* for the model, having also the nice interpretation,

$$\text{Var}(\epsilon) \approx RSE^2. \quad (16)$$

## $R^2$ statistic

$R^2 \in [0, 1]$  is the proportion of variability in  $y$  that can be explained by knowing  $x$ . The closer to 1, the better the fit.

$$R^2 = \frac{\beta_1^2 \text{Var}(x)}{\text{Var}(y)} \stackrel{(7)}{=} \frac{[\text{Cov}(x, y)]^2}{\text{Var}(x) \text{Var}(y)} \approx \frac{s_{xy}^2}{s_{xx} s_{yy}} =: \rho^2. \quad (17)$$

☞ In this case equal to  $\rho^2$  the “sample correlation coefficient”.

## $R^2$ statistic cont'd

To better understand, let us take a look at the variance of  $y = \beta_0 + \beta_1 x + \epsilon$ ,

$$\text{Var}(y) \stackrel{\text{indep. } x, \epsilon}{=} \beta_1^2 \text{Var}(x) + \text{Var}(\epsilon).$$

1. Variance due to variation of  $x$ :  $\beta_1^2 \text{Var}(x)$
2. Variance of error:  $\text{Var}(\epsilon)$ . (Variance left in  $y$  after we know  $x$ )

☞ If  $\text{Var}(\epsilon)$  is small, then  $y$  will be close to  $\beta_0 + \beta_1 x$ , so that our linear regression model will successfully estimate  $y$ .



## $R^2$ example

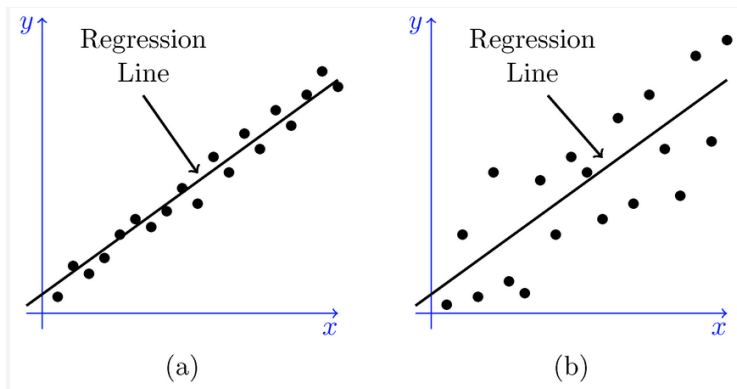


Figure: left: high value of  $R^2$ , right: low value of  $R^2$ .<sup>3</sup>

---

<sup>3</sup>Source [B.2]

## p-value: Are $x$ and $y$ related?

$$H_0 : \beta_1 = 0$$

versus

$$H_1 : \beta_1 \neq 0$$

If  $\beta_1 = 0$  then the model reduces to  $Y = \beta_0 + \epsilon$ , and  $x$  is not associated with  $y$ .

☞ To answer, we compute the **p-value**. This relates to the probability that  $\hat{\beta}_1$  is close to 0, related to the standard error.

If the **p-value** is very small (1% – 5%) we reject the null hypothesis, i.e. a relationship does exist between  $x$  and  $y$ .

## Numerical Example cont'd

- ▶  $RSS = \hat{\epsilon}_1^2 + \hat{\epsilon}_2^2 + \hat{\epsilon}_3^2 + \hat{\epsilon}_4^2 = 0.3^2 + 0.9^2 + 0.9^2 + 0.3^2 = 1.8$
- ▶  $MSE = \frac{RSS}{n} = \frac{1.8}{4} = 0.45.$
- ▶  $RSE = \sqrt{\frac{RSS}{n-2}} = \sqrt{\frac{1.8}{4-2}} \approx 0.949.$
- ▶  $R^2 = \frac{s_{xy}^2}{s_{xx}s_{yy}} = \frac{(11/3)^2}{(5/3) \cdot s_{yy}} = \frac{(11/3)^2}{(5/3) \cdot (26/3)} \approx 0.93.$
- ▶  $s_{yy} = \frac{1}{4-1} [(3-6)^2 + (4-6)^2 + (8-6)^2 + (9-6)^2] = \frac{26}{3}.$

☞  $R^2$ -statistic is 0.93, hence close to 1, and the fit is good!  
(Cannot tell just by RSE)

## Multiple Linear Regression Model

Very often, a response depends on several ( $K > 1$ ) features.

- ▶ **Naive approach:** Run one simple linear regression per feature. But! In this way the estimates ignore all other features left outside:  
Not always good, due to correlation among features.
- ▶ **Correct approach:** Extend the simple linear model to accommodate multiple predictors. Give to each predictor a **separate slope coefficient** in a single model,

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k + \epsilon. \quad (18)$$

## Coefficient Estimation

Again, the regression coefficients are unknown and must be estimated. We use the formula:

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \dots + \hat{\beta}_k x_k. \quad (19)$$

We choose the parameters that minimise again the RSS

$$\begin{aligned} RSS &= \sum_{i=1}^n (y_i - \hat{y}_i)^2 \\ &= \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_{i,1} - \hat{\beta}_2 x_{i,2} - \dots - \hat{\beta}_k x_{i,k})^2 \quad (20) \end{aligned}$$

☞ Complicated expressions, better use existing software packages.

## Least-squares plane

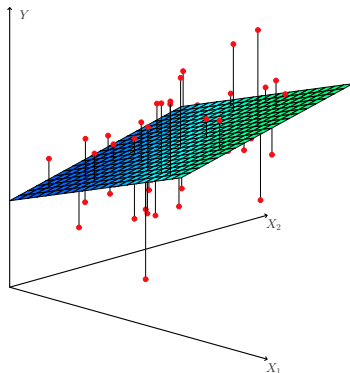


Figure: Regression plane for two features.<sup>4</sup>

---

<sup>4</sup>Source [B.1]

## Hypothesis test (again)

$$H_0 : \beta_1 = \beta_2 = \dots = \beta_k = 0$$

versus

$$H_a : \text{at least one } \beta_j \text{ non-zero.}$$

☞ To answer, we compute the *F-statistic*:

$$F = \frac{(TSS - RSS)/k}{RSS/(n - k - 1)}, \quad (21)$$

$RSS = \sum_{i=1}^n (y_i - \hat{y}_i)^2$ , and  $TSS = \sum_{i=1}^n (y_i - \bar{y})^2$  is the Total Sum of Squares.

If the *F-statistic* is very close to 1 then we expect no relationship between the response and the predictors. On the other hand, if  $H_a$  is true, we expect  $F > 1$ .

## Variable selection

How do we decide on the important variables that influence the  $y$ ?

- ☞ use the  $F$ -statistic and the individual  $p$ -value.
  - ▶ Use different combinations of features to derive the  $F$ -statistic. If for a specific combination among these the value of the statistic drops considerably, then this is an indicator that among the included features, some are unrelated to the response.
  - ▶ From the individual  $p$ -values, the one with the highest  $p$ -value is a candidate to be removed from the model.
  - ▶ The  $R^2$  value determines how much of the data variance is explained by the model. If it is low then the feature set used does not contain much info.
  - ▶ other selection methods (forward and backward selection...)



## Potential Problems of Linear Regression

- ▶ Non-linearity of the response-predictor relationships.
- ▶ Correlation of error terms.
- ▶ Non-constant variance of error terms ([heteroscedasticity](#))
- ▶ Outliers (e.g. incorrect data collection)
- ▶ High leverage points.

Use [Residual plots](#) to detect problems:  
 $(y_i - \hat{y}_i, x_i)$  or  $(y_i - \hat{y}_i, \hat{y}_i)$ .

## Non-linearity

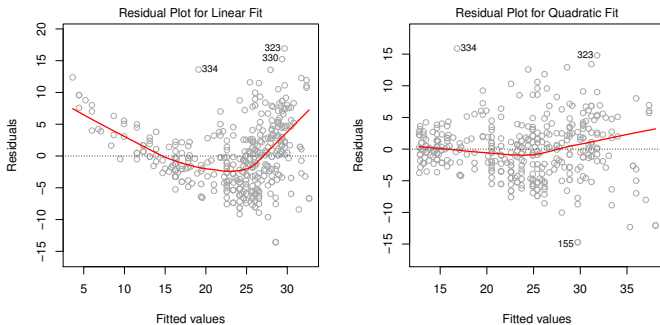


Figure: Residual plots for Linear and Polynomial fit.<sup>5</sup>

---

<sup>5</sup>Source [B.1]

## Error Correlation

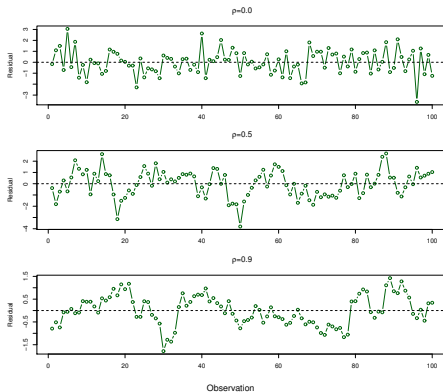


Figure: Residual plots for Time-series.<sup>6</sup>

---

<sup>6</sup>Source [B.1]

## Heteroscedasticity

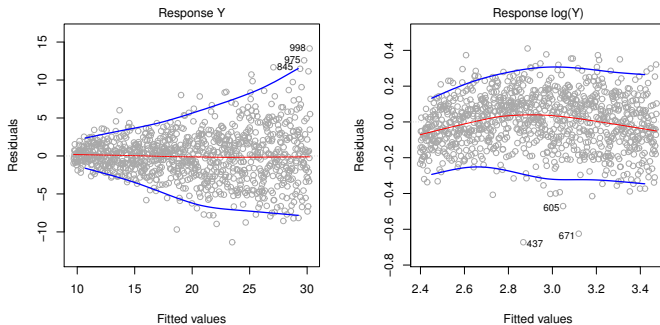


Figure: Residual plots for non-constant variance.<sup>7</sup>

<sup>7</sup>Source [B.1]

# Outliers

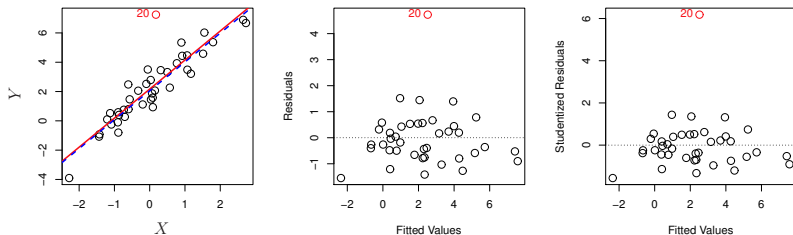


Figure: Effect of an outlier in least-squares fit.<sup>8</sup>

Outliers influences RSE, confidence intervals and p-values.

---

<sup>8</sup>Source [B.1]

## High Leverage Points

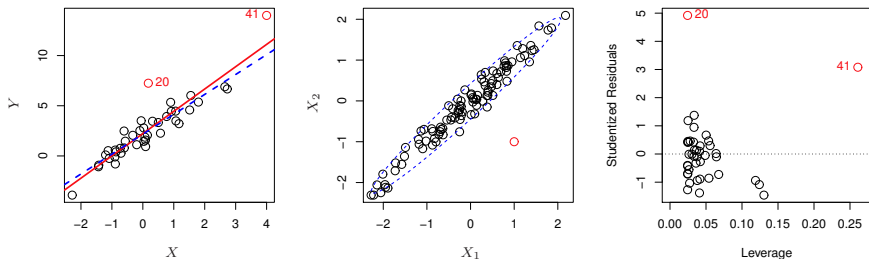


Figure: Effect of a high-leverage point in least-squares fit.<sup>9</sup>

<sup>9</sup>Source [B.1]

**END**