

# Simulation Methods for Barrier and Look-back Options

Simulation Methods for Finance  
under the supervision of Professor Harry Zheng

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# Part I

## Basic Task

### 1 Generate Random Number

#### 1.1 Presentation

There are two methods we used to generate random numbers. The first one is to generate a uniform distribution and then transform it into Standard Normal Distribution. The second method we tried is the system build-in function `Random`, which can directly generate a variable follows Standard Normal Distribution.

#### 1.2 Generating the uniform distribution

To generate a uniform distribution we tried two ways: one can use the system build-in methods `rand`. The function will return a number between 1 and  $2^{15} - 1$  randomly. The range is around 30,000, way below 100,000, the sample size we plan to generate. In other words, when we use `rand` to simulate 100,000 entries, there will be numbers appear more than once, which will create bias. We prefer the second way to generate a uniform distribution: linear congruential generator.

$$n_i = (an_{i-1}) \bmod m$$

for  $i = 1, 2, \dots, 10,000$ , and we let  $a = 7^5$ , and  $m = 2^{31} - 1$ , which gives about 2 billion points. We run 100,000 times and will only use 0.005% of all points. Theoretically, no pattern should appear.

#### 1.3 Generating the normal distribution

To transform the uniform distribution we got into Standard Normal Distribution, we have tried three methods. By Central Limit theory,

$$Z_n = \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma}$$

where  $X_i$  is from the uniform distribution we generated previously.  $Z_n$  converges in distribution to SND, however, it requires  $n$ , the number of uniform distribution, to be sufficiently large. Therefore this method requires significant large simulations and speed is slow consequently.

We also tried Box-Muller methods [1]

$$Z_1 = \sqrt{-2\ln X_1} \sin(2\pi X_2), \quad Z_2 = \sqrt{-2\ln X_1} \cos(2\pi X_2)$$

Since its simulation involves computation of *sine* and *cosine*, the speed is slow. We prefer the last method, Marsaglia Polar method.

$$\text{let } V_1 = 2U_1 - 1, \quad V_2 = 2U_2 - 1.$$

where  $U_1$  and  $U_2$  are two independent uniform distribution. Let  $W = V_1^2 + V_2^2$ . If  $W > 1$ , return to the beginning. Otherwise,

$$N_1 = \sqrt{\frac{(-2\log W)}{W}} V_1, \quad N_2 = \sqrt{\frac{(-2\log W)}{W}} V_2$$

As the computation does not involve sine and cosine, it is generally faster than Box-Muller.

Method	Mean	Variance	Time
<i>rand</i> + CLT	1.77 e-5	0.999909	90.34
<i>rand</i> + Box-Muller	-6.80 e-5	1.000703	8.64
<i>rand</i> + Marsaglia Polar	1.70 e-5	1.000472	6.40
LCG + CLT	1.52 e-5	0.999973	236.3
LCG + Box-Muller	-1.27 e-4	0.999797	11.59
LCG + Marsaglia	7.36 e-5	0.999979	10.52
random	3.58 e-4	1.000210	1.498

Table 1: We note that the Marsaglia method is faster compared to the other methods. Note that the run time depends on the computer used for the simulation (here a 3.4 GHz Intel Core i7). We have generated 100,000,000 variables to obtain these results.

The following is the simulation result of Standard Normal distribution and time used by different methods from generating 100,000,000 random variables.

As shown in the table above, when using CLT generating standard normal distribution, the time needed is significantly longer than the rest methods. We use the combination of linear congruential generator and Marsaglia methods to simulate random variables. This was also the opportunity for us to have full control over the generation of random numbers, as it is a critical step towards efficient simulations [1]. It also features very good results in terms of statistical parameters and time.

## 2 European Option

### 2.1 Presentation

The asset price in a risk neutral probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$  follows Geometric Brownian Motion,

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad 0 \leq t \leq T$$

with initial price  $S_0 = S$ , where  $r$  is riskless interest rate,  $\sigma$  volatility, and  $W_t$  the standard Brownian motion. A European call option price at time  $t$  with maturity time  $T$  is given by

$$C_t = E[e^{-r(T-t)}(S_T - K)^+ | \mathcal{F}_t]$$

For the basic task, we let  $S_0 = 100$ ,  $K = 100$ , interest rate  $r = 0.05$ , volatility  $\sigma = 0.4$ , maturity time  $T = 1$ . We use Monte Carlo method to simulate the path of

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z}$$

and get sample distribution of  $(S_T - K)^+$ . By Black-Scholes formula,

$$C_{bs}(S_0, K) = N(d_1)S_0 - N(d_2)Ke^{-rT}$$

where

$$d_1(S_0, K) = \frac{1}{\sigma\sqrt{T}}[\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T], \quad d_2(S_0, K) = d_1 - \sigma\sqrt{T}$$

The closed-form Greeks are calculated the following way:

$$\delta_{bs} = \frac{\partial C}{\partial S_0} = \Phi(d_1), \quad \gamma_{bs} = \frac{\partial^2 C}{\partial S_0^2} = \frac{\Phi'(d_1)}{S_0\sigma\sqrt{T}}, \quad \nu_{bs} = \frac{\partial C}{\partial \sigma} = \Phi'(d_1)\sqrt{T}$$

## 2.2 Likelihood Ratio Greeks

To calculate the Greeks from simulation, we compare Likelihood ratio method and Pathwise method. As the Call option price is given by

$$C = e^{-rT} E[(S_T - K)^+] = e^{-rT} \int (S_T - K)^+ h_{S_0}(S_T) dS_T$$

where  $h_{S_0}(S_T)$  is the probability density function of  $(S_T - K)^+$ . Then by Likelihood ratio methods, the partial derivative of C with respect to  $S_0$  is

$$\frac{\partial C}{\partial S_0} = \int (S_T - K)^+ \frac{d}{dS_0} h_{S_0}(S_T) dS_T = E[(S_T - K)^+ \frac{h'_{S_0}(S_T)}{h_{S_0}(S_T)}]$$

And the lognormal density function of  $S_T$  is given by

$$h(x) = \frac{1}{x\sigma\sqrt{T}} \phi(\xi(x)), \quad \xi(x) = \frac{\ln(x/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Therefore [1],

$$Delta_{LR} = \frac{\partial C}{\partial S_0} = E[e^{-rT} (S_T - K)^+ \frac{Z}{S_0\sigma\sqrt{T}}]$$

where  $Z \sim N(0, 1)$ .

$$Gamma_{LRLR} = \frac{\partial^2 C}{\partial S_0^2} = E[e^{-rT} (S_T - K)^+ (\frac{Z^2 - 1}{S_0^2\sigma^2 T} - \frac{Z}{S_0^2\sigma\sqrt{T}})]$$

$$Vega_{LR} = \frac{\partial C}{\partial \sigma} = E[e^{-rT} (S_T - K)^+ \left( \frac{Z^2 - 1}{\sigma} - Z\sqrt{T} \right)]$$

## 2.3 Pathwise Derivative Greeks

By pathwise methods, the greeks are given as following<sup>1</sup> [1], with  $Z \sim N(0, 1)$ :

$$Delta_{PW} = \frac{\partial C}{\partial S_0} = E[e^{-rT} \mathbf{1}_{S_T > K} \frac{S_T}{S_0}]$$

$$Gamma_{LRPW} = \frac{\partial^2 C}{\partial S_0^2} = E[e^{-rT} \mathbf{1}_{S_T > K} \frac{KZ}{S_0^2\sigma\sqrt{T}}]$$

$$Gamma_{PWLRLR} = \frac{\partial^2 C}{\partial S_0^2} = E[e^{-rT} \mathbf{1}_{S_T > K} \frac{S_T}{S_0^2} \left( \frac{Z}{\sigma\sqrt{T}} - 1 \right)]$$

$$Vega_{PW} = \frac{\partial C}{\partial \sigma} = E[e^{-rT} \mathbf{1}_{S_T > K} S_T (\sqrt{T}Z - \sigma T)]$$

The following are the results calculated from the closed-form formulae and the simulations:

We can see from above table that simulation results improve and get closer to the theoretical value as simulation number increase. Hence to compare the results from different methods, we will only analyze the simulation result from the most simulation number - 100,000 in this case. This is even more clear on the graphs (Figure 1).

We now have a closer look at 100,000 simulations, which is the industry standard and which is above the 1,000 simulations threshold we have noticed on the graphs. For delta, although the fastest methods is LR, it has a significantly lower accuracy. Gamma LRLR is the most

<sup>1</sup>For gamma, there are two different methods.

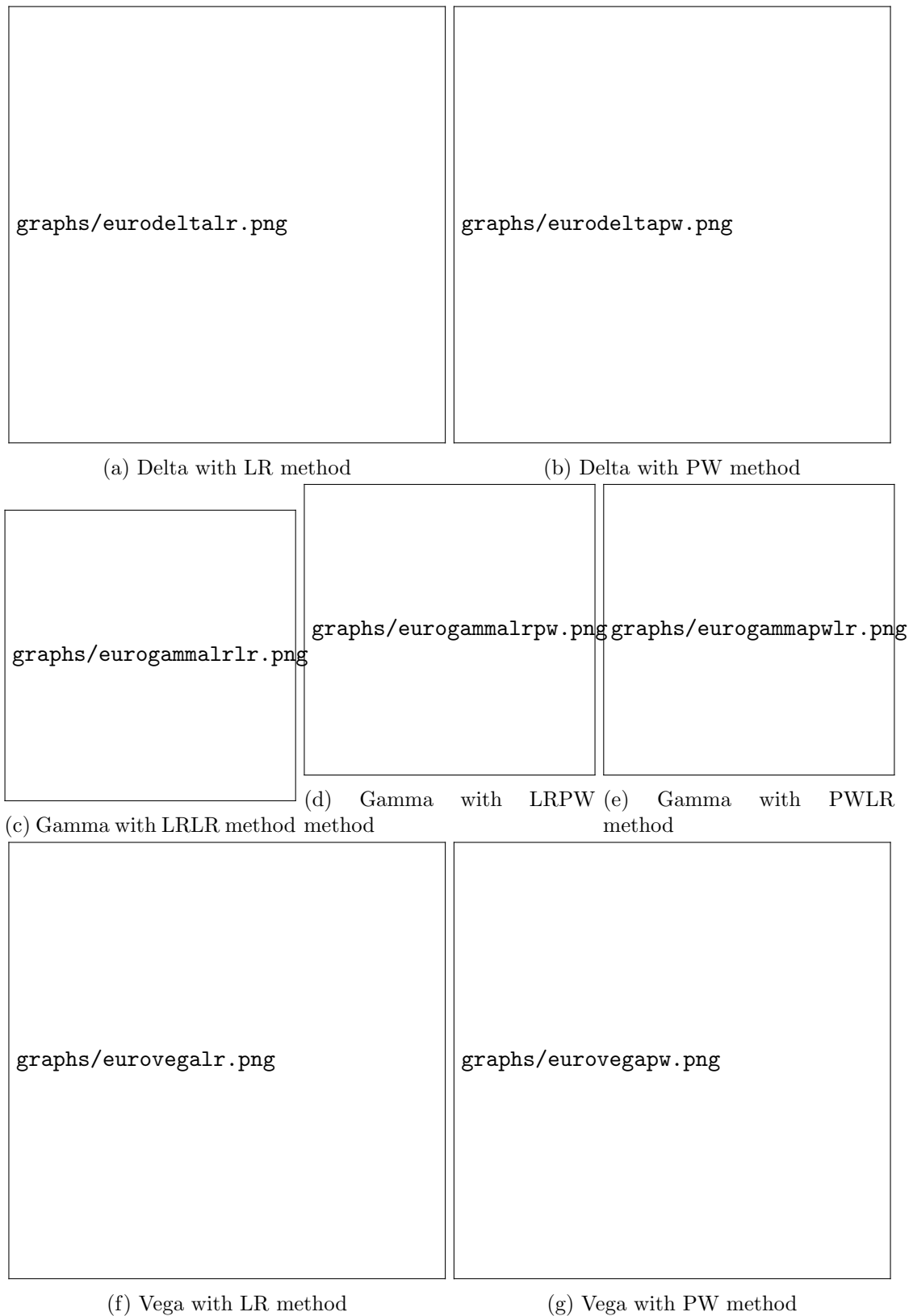


Figure 1: A graphical analysis of the error of the greeks depending on the number of simulations. We notice that there is a significant increase above 1000 simulations in all cases. This is very clear on our vega which tends to have a very high error variance below 1000 simulations. In order to generate these graphs, we have done one thousand simulations for each value, at every power of 10.

		Black -Scholes	Monte Carlo Simulation		
			1,000	10,000	100,000
<b>Option Price</b>		18.023	19.0162	17.6732	18.0014
<b>Delta</b>	<b>LR</b>	0.627409	0.662561	0.612362	0.627889
	<b>PW</b>		0.64485	0.623239	0.627453
<b>Gamma</b>	<b>PW LR</b>	0.0094605	0.0101213	0.00919597	0.00945029
	<b>LR PW</b>		0.00994418	0.00930474	0.00944593
	<b>LR LR</b>		0.00975465	0.00918059	0.0095502
<b>Vega</b>	<b>LR</b>	37.842	39.0186	36.7224	38.2008
	<b>PW</b>		40.4851	36.7839	37.8012

Table 2: Table with the result of our simulations for different Monte-Carlo iterations. We can clearly see the improvement between 1,000 and 10,000 simulations.

100 000 MC simulations	Error Mean	Error Variance	Time (seconds)
<b>Option Price</b>	na	na	5.0 e-6
<b>Delta LR</b>	4.1 e-3	9.7 e-6	1.4 e-3
<b>Delta PW</b>	1.8 e-3	1.8 e-6	9.9 e-4
<b>Gamma PWLR</b>	6.1 e-5	2.0 e-9	1.4 e-3
<b>Gamma LRPW</b>	3.5 e-5	7.4 e-10	1.4 e-3
<b>Gamma LRLR</b>	1.9 e-4	2.1 e-8	4.7 e-3
<b>Vega LR</b>	7.7 e-1	3.3 e-1	4.6 e-3
<b>Vega PW</b>	2.4 e-1	3.2 e-2	1.6 e-3

Table 3: Sample from our simulation dataset with 100 000 Monte-Carlo simulations. Note that the run time depends on the computer (here a 3.1 GHz Intel Core i5).

accurate but it is the slowest, so gamma PWLR seems to be a good compromise. Vega PW is both faster and more accurate than vega LR.

For the following section, PW methods will be infeasible to do for Barrier Option. Hence, we will use Likelihood ratio method when calculating the greeks.

## Part II

# Main Task

## 3 Barrier Option

### 3.1 Presentation

Let  $T$  denote option expiration time and  $[0, T]$  lookback period. For  $T_0 \leq t \leq T$  denote by

$$m_0^T = \min_{0 \leq t \leq T} S_t, \quad M_0^t = \max_{0 \leq t \leq T} S_t$$

For an up-and-out barrier call option  $A_T = (S_T - K)^+ 1_{\max_{0 \leq t \leq T} S_t \leq B}$ , where  $B$  is a barrier level and  $1_S$  is an indicator function.

We simulate the path of  $S_t$  by taking the maturity time  $T$  into 1,000 steps, and we simulate 5,000 such paths. As standard normal distribution is symmetrically distributed, 5,000 paths can be treated as 10,000 paths by adding negative sign and creating the other 5,000. We have an  $M_0^T$  for each path. However, this method is burdensome as we need to generate each step for each path. The simulation process is long.

### 3.2 Another Method using Rayleigh Distribution

Hence, we tried second method using Rayleigh Distribution to simulate the distribution of  $M_0^t$  directly. The maximum of a standard Brownian motion starting at the origin to be at  $b$  at time 1 over period  $[0, T]$  has the Rayleigh distribution [2, 3]

$$F(x) = 1 - e^{-2x(x-b)}, \quad x \geq b.$$

solving the equation  $F(x) = u, u \in (0, 1)$  has roots

$$x = \frac{b}{2} \pm \frac{\sqrt{b^2 - 2\log(1-u)}}{2}$$

Hence, at time  $T$  with  $S_T$ ,

$$M^T = \frac{S_T + \sqrt{S_T^2 - 2T\log U}}{2}$$

And we simulate  $S_T$  by Black-Scholes formula

$$S_T = S_0 e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z}$$

Therefore,

$$M^T = S_0 e^{\frac{1}{2}\log(\frac{S_T}{S_0}) + \sqrt{\log(\frac{S_T}{S_0})^2 - 2\sigma^2 T \log U}}$$

With this method, we only need to generate one  $S_T$  for each path and one random variable for  $U$  to get the  $M^T$ . It reduces the amount of simulation by 500 times (as before, 1,000  $S_t$  need to be generated for each path).

As we mentioned previously, it is impossible to find the partial derivative of the indicator function  $1_{\max_{0 \leq t \leq T} S_t \leq B}$  with respect to  $S_0$ . Hence pathwise method is eliminated by us. For the likelihood ratio method, we find the differentiation of joint cdf of  $S_T$  and the  $M_T$  to be

$$f_{uo}(x, m, T) = \frac{1}{\sqrt{T}} \left( \phi\left(\frac{x - \mu T}{\sqrt{T}}\right) - e^{2m\mu} \phi\left(\frac{x - 2m - \mu T}{\sqrt{T}}\right) \right)$$

where  $x = \frac{1}{\sigma} \ln \frac{x}{S_0}$  and  $m = \frac{1}{\sigma} \ln \frac{M_T}{S_0}$ . Then we can find the greeks according.

To analyze further the accuracy of our simulation, we plot the simulated option prices and greeks versus their theoretical value as barrier increase 2.





Figure 2: A graphical analysis of the error of the greeks depending on the computation time. We notice that the method with Rayleigh distribution is between 100 and 1,000 times faster. In order to generate these graphs, we have done one thousand simulations for each value, at every power of 10.

100 000 MC simulations	<b>Error Mean</b>	<b>Error Variance</b>	<b>Time (seconds)</b>
<b>Option Price</b>	na	na	..
<b>Delta LR</b>	1.036 e-2	3.029 e-5	9.284 e-2
<b>Gamma LRLR</b>	1.895 e-4	1.895 e-8	1.475 e-1
<b>Vega LR</b>	7.507 e-1	3.220 e-1	1.097 e-1

Table 4: Sample from our simulation dataset with the new fast method for barrier option simulation. It is clear that the results have significantly improved compared with the previous method. note that the run time depends on the computer used (here a 3.4 GHz Intel Core i7).

	<b>Closed-form Formula</b>	<b>Mean</b>	<b>Error</b>	<b>Variance</b>	<b>Time</b>
<b>Option Price</b>					
<b>Delta</b>					
<b>Gamma</b>					
<b>Vega</b>					

Table 5

## 4 Look-back Option

Lookback call option with fixed strik price  $K$  has payoff  $(M_0^T - K)^+$ . The call option price at time  $t$  is

$$c(S_0, K, t) = e^{-r(T-t)} E[(\max(M_0^t, M_t^T) - K)^+ | \mathcal{F}_t]$$

The closed-form formula for fixed strike lookback call option at time 0 [1] is

$$c(S_0, K, 0) = C_{bs}(S_0, K) + \frac{S_0 \sigma^2}{2r} \{ \Phi[d_2(S_0, K)] - e^{-rT} \frac{S_0^{-\frac{2r}{\sigma^2}}}{K} \Phi[-d_1(K, S_0)] \}$$

The pdf of the distribution of Maximum  $S_t$  for standard Brownian motion for the period of  $(0, T)$  is

$$f = \frac{2}{\sqrt{T}} \phi\left(\frac{m - aT}{\sqrt{T}}\right) - 2ae^{2am} \Phi\left(\frac{-m - aT}{\sqrt{T}}\right)$$

and the cdf is

$$F = \Phi\left(\frac{m - aT}{\sqrt{T}}\right) - e^{2am} \Phi\left(\frac{-m - aT}{\sqrt{T}}\right)$$

We used Newton-Raphson method to solve for the above cdf  $F = U$ . Hense, for each random number we generate from  $[0,1]$ , we receive one  $m$  in by solving for  $F$ . And to get  $M^T$  of stock price follows geometric Browninan motion with starting price  $S_0$ , we let  $M^T = S_0 e^{\sigma m}$ .

## References

- [1] Harry Zheng (Pr.), *Simulation Methods for Finance*, Imperial College London, London, 2018.
- [2] Diogo Monteiro da Costa Soares Justino, *Hedging of Barrier Options*, Instituto Universitário de Lisboa, Lisbon, 2010.
- [3] Paul Glasserman (Pr.), *Monte Carlo Methods in Financial Engineering*, Springer Sciencea, New York, 2013.

# Appendix

## A Closed-formed formula of Barrier Options

$$UOC(S_0, K, B) = 1_{B>K} \{ C_{bs}(S_0, K) - C_{bs}(S_0, B) - (B - K)e^{-rT} \Phi[d_1(S_0, B)] - \frac{B}{S_0} \frac{2v^2}{\sigma^2} \left[ C_{bs}\left(\frac{B^2}{S_0}, K\right) - C_{bs}\left(\frac{B^2}{S_0}, B\right) - (B_0 - K)e^{-rT} \Phi[d_1(S_0, B)] \right] \} \quad (1)$$

Where  $C_{bs}$  and  $d_1$  are as stated in the Black-Scholes formula (1) and (2), and  $v = r - \frac{\sigma^2}{2}$ .

closed form for DOC price is

$$C_{DO}(S_0, K, B) = C_{bs}(S_0, K) - \left(\frac{S_0}{B}\right)^{-2\frac{v}{\sigma^2}} C_{bs}\left(\frac{B^2}{S_0}, K\right)$$

where  $v = r - \frac{\sigma^2}{2}$ .

$$\begin{aligned} \delta_{DOC} = & \delta_{BS}(S_0, K) - \delta_{BS}(S_0, B) - \frac{B - K}{\sigma S_0 \sqrt{T}} e^{-rT} \Phi(d_2(S_0, B)) \\ & + \left( \frac{2v}{\sigma^2 S_0} \left( \frac{B}{S_0} \right)^{2v/\sigma^2} \right) \\ & \times \left( C_{BS}\left(\frac{B^2}{S_0}, K\right) - C_{BS}\left(\frac{B^2}{S_0}, B\right) - (B - K)e^{-rT} \Phi(d_2(B, S_0)) \right) \\ & - \left( \frac{B}{S_0} \right)^{2v/\sigma^2} \left( \left( \frac{-B}{S_0} \right)^2 \delta_{BS}\left(\frac{B^2}{S_0}, K\right) + \left( \frac{B}{S_0} \right)^2 \delta_{BS}\left(\frac{B^2}{S_0}, B\right) \right) \\ & + \left( \frac{B - K}{\sigma S_0 \sqrt{T}} \right) e^{-rT} \Phi(d_2(B, S_0)); \end{aligned}$$

$$\begin{aligned} \delta_{DOC} = & \Phi \left( \frac{\log \frac{S_0}{K} + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \right) - \left( \frac{B}{S_0} \right)^{r/\sigma^2 - 1} \\ & \times \left( -\frac{B^2}{S_0} \Phi \left( \frac{\log \frac{B^2}{S_0 K} + vT}{\sigma \sqrt{T}} + \sigma \sqrt{T} \right) - \frac{2v C_{BS}(B^2/S_0, K)}{(S_0 \sigma^2)} \right) \end{aligned}$$

$$\gamma_{DOC} = \frac{\phi(d_2)}{S_0 \sigma \sqrt{T}} - \left( \frac{B}{S_0} \right)^{2v/\sigma^2} \left( \frac{4v^2 + 2v\sigma^2}{S_0^2 \sigma^4} C_{BS}(B^2/S_0, K) + \gamma_{bs} - \frac{4v\delta_{bs}}{S_0 \sigma^2} \right)$$

$$\nu_{DOC} = S_0 \phi \left( \frac{\log(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma \sqrt{T}} \right) \sqrt{T} - \left( \frac{B}{S_0} \right)^{\frac{2v}{\sigma^2}} \left( \nu_{bs} - \frac{4r C_{bs}(\frac{B^2}{S_0}, K) \log(\frac{B}{S_0})}{\sigma^3} \right)$$

## **B App User Guide**

## **C Package Manual**