Simulation Methos for Barrier and Look-back Options

Course: Simulation Methods for Finance
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1 Generate Random Number

There are two methods we used to generate random numbers. The first one is to generate a uniform distribution and then transform it into Standard Normal Distribution. The second method we tried is the system build-in function ***Randon***, which can directly generate a variable follows Standard Normal Distribution.

To generate a uniform distribution we tried two ways: one can use the system build-in methods ***rand***. The function will return a number between 1 and $2^{15} - 1$ randomly. The range is around 30,000, way below 100,000, the sample size we plan to generate. In other words, when we use ***rand*** to simulate 100,000 entries, there will be numbers appear more than once, which will create bias. We prefer the second way to generate a uniform distriubtion: linear congruential generator.

$$n_i = (an_{i-1})\ mod\ m$$

for $i = 1, 2, ..., 10,000$, and we let $a = 7^5$, and $m = 2^{31} - 1$, which gives about 2 billion points. We run 100,000 times and will only use 0.005% of all points. Theoretically, no pattern should appear.

To transform the uniform distribution we got into Standard Normal Distribution, we have tried three methods. By Central Limit theory,

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}$$

where X_i is from the uniform distribution we generated previously. Z_n converges in distribution to SND, however, it requires n, the number of uniform distribution, to be sufficiently large. Therefore this method requires significant large simulations and speed is slow consequently.

We also tried Box-Muller methods

$$Z_1 = \sqrt{-2lnX_1} sin(2\pi X_2),\ \mathbb{Z}_2 = \sqrt{-2lnX_1} cos(2\pi X_2)$$

Since its simulation involves computation of *sine* and *cosine*, the speed is slow. We prefer the last method, Marsaglia Polar method.

$$let\ V_1 = 2U_1 - 1,\ V_1 = 2U_2 - 1.$$

where U_1 and U_2 are two independent uniform distribution. Let $W = V_1^2 + V_2^2$. If $W > 1$, return to the beginning. Otherwise,

$$N_1 = \sqrt{\frac{(-2logW)}{W}} V_1,\ N_2 = \sqrt{\frac{(-2logW)}{W}} V_2$$

As the computation does not involve sine and cosine, it is generally faster than Box-Muller.

The following is the simulation result of Standard Normal distribution and time used by different methods.

Method	Mean	Variance	Time
***rand* generated U + CLT**	cell5	cell6	..
***rand* generated U + Box-Muller**	cell8	cell9	..
***rand* generated U + Marsagilia Polar**	cell8	cell9	..
linear congruential generator + CLT	cell8	cell9	..
linear congruential generator + Box-Muller	cell8	cell9	..
linear congruential generator + Marsagilia	cell8	cell9	..
Randon	cell8	cell9	..

Error Analysis... For the tasks in this project, we use the combination of linear congruential generator and Marsagilia methods to simulate random variable.

2 Basic Task

The asset price in a risk neutral probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ follows Geometric Brownian Motion,

$$dS_t = rS_t dt + \sigma S_t dW_t,\ 0 \le t \le T$$

with initial price $S_0 = S$, where r is riskless interest rate, σ volatility, and W_t the standard Brownian motion. A European call option price at time t with maturity time T is given by

$$C_t = E[e^{-r(T-t)}(S_T - K)^+ | F_t]$$

For the basic task, we let $S_0 = 100$, $K = 100$, interest rate $r = 0.05$, volatility $\sigma = 0.4$, maturity time $T = 1$. We use Monte Carlo method to simulate the path of

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z}$$

and get sample distribution of $(S_T - K)^+$. By Black-Scholes formula,

$$C_{bs}(S_0, K) = N(d_1)S_0 - N(d_2)Ke^{-rT} \tag{1}$$

where

$$d_1(S_0, K) = \frac{1}{\sigma\sqrt{T}}[ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T],\ d_2(S_0, K) = d_1 - \sigma\sqrt{T} \tag{2}$$

The closed-form Greeks are calculated the following way:

$$\delta_{bs} = \frac{\partial C}{\partial S_0} = \Phi(d_1) \tag{3}$$

$$\gamma_{bs} = \frac{\partial^2 C}{\partial S_0^2} = \frac{\Phi'(d_1)}{S_0\sigma\sqrt{T}} \tag{4}$$

$$\nu_{bs} = \frac{\partial C}{\partial \sigma} = \Phi'(d_1)\sqrt{T} \tag{5}$$

To calculate the Greeks from simulation, we compare Likelihood ratio method and Pathwise method.As the Call option price is given by

$$C = e^{-rT}E[(S_T - K)^+] = e^{-rT}\int (S_T - K)^+ h_{S_0}(S_T)dS_T$$

where $h_{S_0}(S_T)$ is the probability density function of $(S_T - K)^+$. Then by Likelihood ratio methods, the partial dirivative of C with respect to S_0 is

$$\frac{\partial C}{\partial S_0} = \int (S_T - K)^+ \frac{d}{dS_0} h_{S_0}(S_T)dS_T = E[(S_T - K)^+ \frac{h'_{S_0}(S_T)}{h_{S_0}(S_T)}]$$

And the lognormal density function of S_T is given by

$$h(x) = \frac{1}{x\sigma\sqrt{T}}\phi(\xi(x)), \quad \xi(x) = \frac{ln(x/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Therefore,

$$Delta_{LR} = \frac{\partial C}{\partial S_0} = E[e^{-rT}(S_T - K)^+ \frac{Z}{S_0\sigma\sqrt{T}}]$$

where $Z\ N(0,1)$.

$$Gamma_{LRLR} = \frac{\partial^2 C}{\partial S_0^2} = E[e^{-rT}(S_T - K)^+ (\frac{Z^2 - 1}{S_0^2\sigma^2 T} - \frac{Z}{S_0^2\sigma\sqrt{T}})]$$

$$Vega_{LR} = \frac{\partial C}{\partial \sigma} = E[e^{-rT}(S_T - K)^+ \left(\frac{Z^2 - 1}{\sigma} - Z\sqrt{T}\right)]$$

By pathwise methods, the greeks are given as following[1], with $Z\ N(0,1)$:

$$Delta_{PW} = \frac{\partial C}{\partial S_0} = E[e^{-rT}\mathbb{1}_{S_T>K}\frac{S_T}{S_0}]$$

$$Gamma_{LRPW} = \frac{\partial^2 C}{\partial S_0^2} = E[e^{-rT}\mathbb{1}_{S_T>K}\frac{KZ}{S_0^2\sigma\sqrt{T}}]$$

$$Gamma_{PWLR} = \frac{\partial^2 C}{\partial S_0^2} = E[e^{-rT}\mathbb{1}_{S_T>K}\frac{S_T}{S_0^2}\left(\frac{Z}{\sigma\sqrt{T}} - 1\right)]$$

$$Vega_{PW} = \frac{\partial C}{\partial \sigma} = E[e^{-rT}\mathbb{1}_{S_T>K}S_T(\sqrt{T}Z - \sigma T)]$$

The following are the results calculated from the closed-form formulae and the simulations:

[1] For gamma, there are two different methods.

100 000 MC simulations	**Error Mean**	**Error Variance**	**Time (seconds)**
Option Price	na	na	5.0 e-6
Delta LR	4.1 e-3	9.7 e-6	1.4 e-3
Delta PW	1.8 e-3	1.8 e-6	9.9 e-4
Gamma PWLR	6.1 e-5	2.0 e-9	1.4 e-3
Gamma LRPW	3.5 e-5	7.4 e-10	1.4 e-3
Gamma LRLR	1.9 e-4	2.1 e-8	4.7 e-3
Vega LR	7.7 e-1	3.3 e-1	4.6 e-3
Vega PW	2.4 e-1	3.2 e-2	1.6 e-3

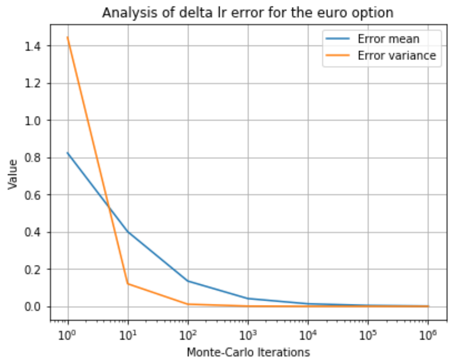
Table 1: Sample from our simlation dataset with 100 000 Monte-Carlo simlations. Note that the run time depends on the computer (here a 3.1 GHz Intel Core i5).

		Black -Scholes	**Monte Carlo Simulation**		
			1,000	**10,000**	**100,000**
Option Price		18.023	19.0162	17.6732	18.0014
Delta	**LR**	0.627409	0.662561	0.612362	0.627889
	PW		0.64485	0.623239	0.627453
Gamma	**PW LR**	0.0094605	0.0101213	0.00919597	0.00945029
	LR PW		0.00994418	0.00930474	0.00944593
	LR LR		0.00975465	0.00918059	0.0095502
Vega	**LR**	37.842	39.0186	36.7224	38.2008
	PW		40.4851	36.7839	37.8012

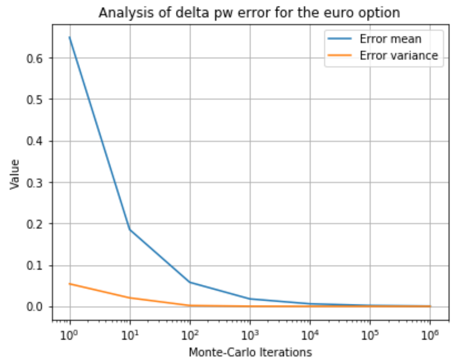
We can see from above table that simulation results improve and get closer to the theoretical value as simulation number increase. Hence to compare the results from different methods, we will only analyze the simulation result from the most simulation number - 100,000 in this case. This is even more clear on the graphs (Figure 2).

We now have a closer look at 100,000 simulations, which is the industry standard and which is baove the 1,000 simulations threshold we have noticed on the graphs. For delta, although the fastest methods is LR, it has a significantly lower accurany. Gamma LRLR is the most accurate but it is the slowest, so gamma PWLR seems to be a good compromise. Vega PW is both faster and more accurate than vega LR.

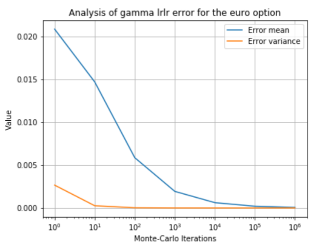
For the following section, PW methods will be infeasible to do for Barrier Option. Hense, we will use Likelihood ratio method when calculating the greeks.



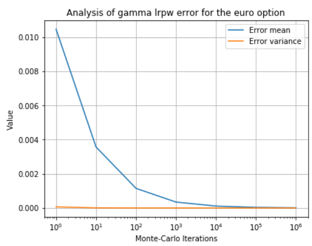
(a) Delta with LR method



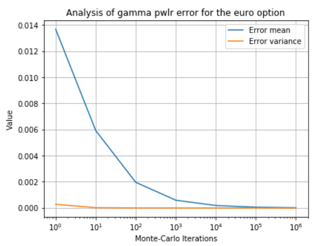
(b) Delta with PW method



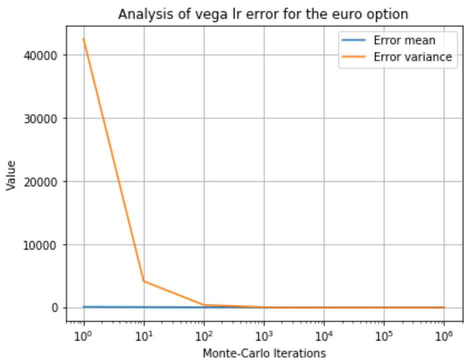
(c) Gamma with LRLR method



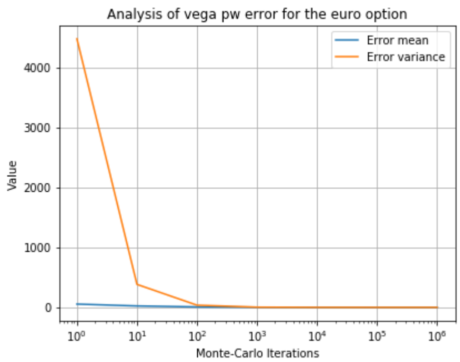
(d) Gamma with LRPW method



(e) Gamma with PWLR method



(f) Vega with LR method



(g) Vega with PW method

Figure 1: A graphical analysis of the error of the greeks depending on the number of simulations. We notice that there is a significant increase above 1000 simulations in all cases. This is very clear on our vega which tends to have a very high error variance below 1000 simulations. In order to generate these graphs, we have donne one thousand simlations for each value, at every power of 10.

3 Main Task - Barrier Option

For an up-and-out barrier call option $A_T = (S_T - K)^+ 1_{max 0 \le t \le T \ S_t \le B}$, where B is a barrier leve and 1_S is an indicator function. The closed form formula for the option price is

$$\begin{aligned} UOC(S_0, K, B) = 1_{B>K}\{&C_{bs}(S_0, K) - C_{bs}(S_0, B) - (B-K)e^{-rT}\Phi[d_1(S_0, B)] \\ &- \frac{B}{S_0}^{\frac{2v^2}{\sigma^2}} \left[C_{bs}(\frac{B^2}{S_0}, K) - C_{bs}(\frac{B^2}{S_0}, B) - (B_0 - K)e^{-rT}\Phi[d_1(S_0, B)]\right]\} \end{aligned} \tag{6}$$

Where C_{bs} and d_1 are as stated in the Black-Scholes formula (1) and (2), and $v = r - \frac{\sigma^2}{2}$.

The closed-form formulas for the greeks are the following:

$$\begin{aligned} DOC\delta = &\Phi\left(\frac{\log\frac{S_0}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - \left(\frac{B}{S_0}\right)^{r/\sigma^2 - 1} \\ &\times \left(-\frac{B}{S_0}^2 \Phi\left(\frac{log\frac{B^2}{S_0 K} + vT}{\sigma\sqrt{T}} + \sigma\sqrt{T}\right) - \frac{2vC_{BS}(B^2/S_0, K)}{(S_0\sigma^2)}\right) \end{aligned}$$

$$\begin{aligned} UOC\delta = &\delta_{BS}(S_0, K) - \delta_{BS}(S_0, B) - \frac{B-K}{\sigma S_0\sqrt{T}}e^{-rT}\Phi(d_2(S_0, B)) \\ &+ \left(\frac{2v}{\sigma^2 S_0}\left(\frac{B}{S_0}\right)^{2v/\sigma^2}\right) \\ &\times \left(C_{BS}\left(\frac{B^2}{S_0}, K\right) - C_{BS}\left(\frac{B^2}{S_0}, B\right) - (B-K)e^{-rT}\Phi(d_2(B, S_0))\right) \\ &- \left(\frac{B}{S_0}\right)^{2v/\sigma^2}(\left(\frac{-B}{S_0}\right)^2 \delta_{BS}\left(\frac{B^2}{S_0}, K\right) + \left(\frac{B}{S_0}\right)^2 \delta_{BS}\left(\frac{B^2}{S_0}, B\right) \\ &+ \left(\frac{B-K}{\sigma S_0\sqrt{T}}\right)e^{-rT}\Phi(d_2(B, S_0))); \end{aligned}$$

$$DOC\gamma = \frac{\phi(d_2)}{S_0\sigma\sqrt{T}} - \left(\frac{B}{S_0}\right)^{2v/\sigma^2}(\frac{4v^2 + 2v\sigma^2}{S_0^2\sigma^4}C_{BS}(B^2/S_0, K) + \gamma_{BS} - \frac{4v\delta_{BS}}{S_0\sigma^2})$$

We stimulate the path of S_T by taking the maturity time T into 10,000 steps, and we simulate 5,000 such paths. As standard normal distribution is symmetrically distributed, 5,000 paths can be treated as 10,000 paths by adding negative sign and creating the other 5,000.

To determine the greeks for Barrier Option, we first assessed the pathwise method. It is impossible to find the partial direvative of the indicator function $1_{max 0 \le t \le T \ S_t \le B}$

100 000 MC simulations	**Error Mean**	**Error Variance**	**Time**
Option Price			
Delta			
Gamma			
Vega			

Table 2: Sample from our simlation dataset with 100 000 Monte-Carlo simlations.

with respect to S_0. Hense pathwise method is eliminated by us. For the likelihood ratio method, we find the joint cdf of S_T and the $M_T = \max_{0 \le t \le T} S_t$ to be **

**closed form for greeks

To analyze further the accuracy of our simulation, we plot the simulated option prices and greaks versus their theoretical value as barrier increase.

**graph
**graph
**graph

4 Look-back Option

Let T denote option expiration time and $[0, T]$ lookback period. For $T_0 \le t \le T$ denote by

$$m_0^t = \min_{0 \le u \le t} S_u, \quad M_0^t = \max_{0 \le u \le t} S_u$$

minimum (maximum) value of realized asset. Lookback call option with fixed strik price K has payoff $(M_0^T - K)^+$. The call option price at time t is

$$c(S_0, K, t) = e^{-r(T-t)} E[(max(M_0^t, M_t^T) - K)^+ | \mathcal{F}_t]$$

The closed-form formula for fixed strike lookback call option at time 0 is

$$c(S_0, K, 0) = C_{bs}(S_0, K) + \frac{S_0 \sigma^2}{2r} \{\Phi[d_2(S_0, K)] - e^{-rT} \frac{S_0}{K}^{-\frac{2r}{\sigma^2}} \Phi[-d_1(K, S_0)]\}$$

**greeks closed form and simulation formulas

	Closed-form Formula	**Mean**	**Error**	**Variance**	**Time**
Option Price					
Delta					
Gamma					
Vega					

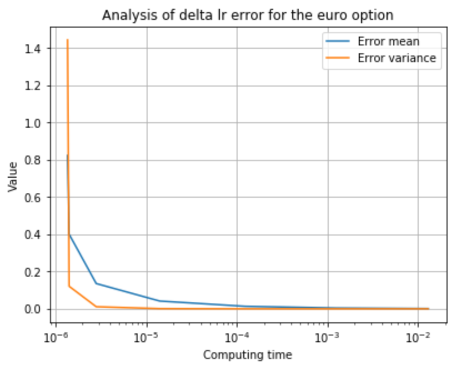
References

[1] A

[2] B

A Appendix: R code of part A

B Analysis of the error with time



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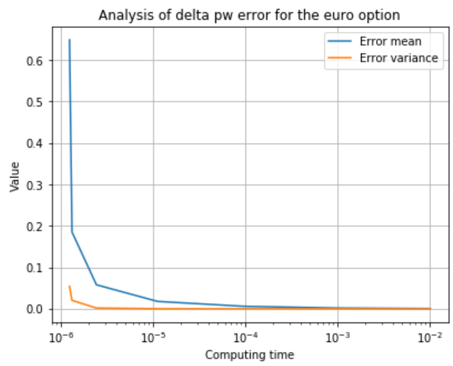
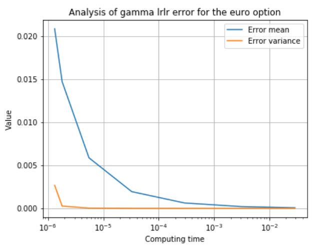


(b) Delta with PW method



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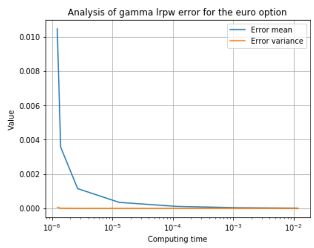
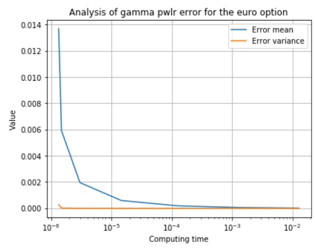
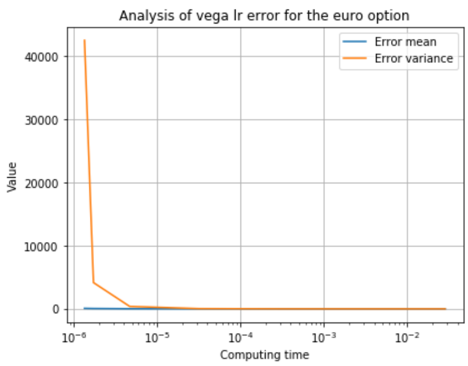


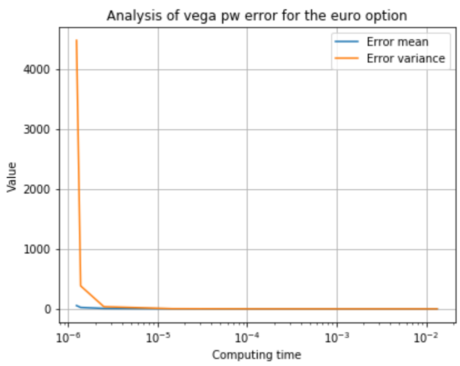
(d) Gamma with LRPW method



(e) Gamma with PWLR method



(f) Vega with LR method



(g) Vega with PW method

Figure 2: A graphical analysis of the error of the greeks depending on the time of computation. In order to generate these graphs, we have donne one thousand simlations for each value, at every power of 10.