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INSAT
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Worksheet solution: Two continuous random variables.

Exercise 1: Let $f(x,y) = cxy$ for $0 < x < y < 1$ be a joint probability distribution function.

1. Determine the value of c.

$$\begin{aligned}
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)dxdy &= 1 \\
&= c\int_{\{0<x<y<1\}} xydxdy \\
&= c\int_0^1 dxx\left\{\int_x^1 ydy\right\} \\
&= c\int_0^1 dxx\left[\frac{y^2}{2}\right]_x^1 \\
&= \frac{c}{2}\int_0^1 dxx(1-x^2) \\
&= \frac{c}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1 = \frac{c}{2}(\frac{1}{2}-\frac{1}{4}) \\
&= \frac{c}{8},
\end{aligned}$$

   and so $c = 8$.

2. Compute $\mathbb{P}\{X < \frac{1}{2}, Y < 1\}$:

$$\begin{aligned}
\mathbb{P}\{X < \frac{1}{2}, Y < 1\} &= \int_{\{x<1/2, y<1\}} f_{XY}(x,y)dxdy \\
&= 8\int_{\{x<1/2,y<1\}\cap\{0<x<y<1\}} xydxdy \\
&= 8\int_{\{0<x<y<1 \ and \ x<1/2\}} xydxdy \\
&= 8\int_0^{1/2} dxx\left\{\int_x^1 ydy\right\} \\
&= 8\int_0^{1/2} dxx\left[\frac{y^2}{2}\right]_x^1 \\
&= 4\int_0^{1/2} dxx(1-x^2) \\
&= 4\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^{1/2} = 4\left[\frac{1}{8}-\frac{1}{64}\right] = \frac{7}{16}
\end{aligned}$$

3. Determine the marginal probability distributions of X and Y:

$$\begin{aligned} f_X(x) &= \int_x^1 f_{XY}(x,y)dy \\ &= 8x \int_x^1 y dy \mathbf{1}_{]0,1[}(x) \\ &= 8x \left[\frac{y^2}{2}\right]_x^1 \mathbf{1}_{]0,1[}(x) \\ &= 4x(1-x^2)\mathbf{1}_{]0,1[}(x) \end{aligned}$$

where $\mathbf{1}_{]0,1[}(x)$ is the indicator function of $]0,1[$ defined as follows

$$\mathbf{1}_{]0,1[}(x) = \begin{cases} 0 & if\ x \notin]0,1[\\ 1 & if\ x \in]0,1[\end{cases}$$

By doing the same computation we get the marginal distribution of Y

$$f_Y(y) = 4y^3\mathbf{1}_{]0,1[}(y).$$

4. Compute $\mathbb{E}(X)$ and $\mathbb{E}(Y)$:

$$\begin{aligned} \mathbb{E}(X) &= \int_{\mathbb{R}} x f_X(x)dx \\ &= \int_0^1 x4x(1-x^2)dx \\ &= 4\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 4(\frac{1}{3} - \frac{1}{5}) = 0.133. \end{aligned}$$

$$\begin{aligned} \mathbb{E}(Y) &= \int_0^1 y4y^3 dy \\ &= 4\left[\frac{y^5}{5}\right]_0^1 = \frac{4}{5} = 0.8. \end{aligned}$$

5. Determine the function $\varphi(x) = \mathbb{E}(Y|X=x)$ for each $x \in]0,1[$ and deduce the conditional expectation random variable $\mathbb{E}(Y|X)$: First we need to determine the conditional distribution $f_{Y|X=x}$ for all $x \in]0,1[$

$$\begin{aligned} f_{Y|X=x}(y) &= \frac{f_{XY}(x,y)}{f_X(x)} \\ &= \frac{8xy\mathbf{1}_{\{0<x<y<1\}}}{4x(1-x^2)\mathbf{1}_{]0,1[}(x)} \\ &= \left[\mathbf{1}_{]0,1[}(x)\frac{2}{1-x^2}\right] y\mathbf{1}_{]x,1[}(y), \end{aligned}$$

$$
\begin{aligned}
\varphi(x) &= \mathbb{E}(Y|X=x) \\
&= \mathbf{1}_{]0,1[}(x)\frac{2}{1-x^2}\int_x^1 y \times y\,dy \\
&= \mathbf{1}_{]0,1[}(x)\frac{2}{1-x^2}\left[\frac{y^3}{3}\right]_x^1 \\
&= \frac{2}{3}\frac{1-x^3}{1-x^2}\mathbf{1}_{]0,1[}(x) \\
&= \frac{2}{3}\frac{1+x+x^2}{1+x}\mathbf{1}_{]0,1[}(x),
\end{aligned}
$$

and so

$$\mathbb{E}(Y|X) = \varphi(X) = \frac{2}{3}\frac{1+X+X^2}{1+X}.$$

6. What about the strongness and the nature of the relationship between X and Y? The relationship is definitely non-linear, to see the strongness we need to compute the determination coefficient, that is

$$R^2_{Y|X} = \frac{\mathbb{V}(\mathbb{E}(Y|X))}{\mathbb{V}(Y)}.$$

For this we have to compute

$$\mathbb{E}\left[\varphi(X)\right] = \mathbb{E}\left[\mathbb{E}(Y|X)\right] = \mathbb{E}(Y) = 0.8$$

and then

$$
\begin{aligned}
\mathbb{E}\left[\varphi(X)^2\right] &= \mathbb{E}\left[\frac{2}{3}\frac{1+X+X^2}{1+X}\right]^2 \\
&= \frac{4}{9}\mathbb{E}\left[\frac{1+2X+3X^2+2X^3+X^4}{1+X^2+2X}\right] \\
&= \frac{16}{9}\int_0^1\left[\frac{1+2x+3x^2+2x^3+x^4}{1+x^2+2x}\right]x(1-x^2)dx \\
&= \frac{16}{9}\int_0^1\left[\frac{1+2x+3x^2+2x^3+x^4}{1+x}\right]x(1-x)dx \\
&= 0.6465878,
\end{aligned}
$$

actually compute the last integral is too long and boring so I have used "R" to get it !

$$\mathbb{V}\left[\varphi(X)\right] = 0.006587802.$$

Now let us compute the variance of Y, we have

$$\mathbb{V}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = 4/6 - (4/5)^2 = 0.02666667.$$

Finally we get

$$R^2_{Y|X} = \frac{\mathbb{V}(\mathbb{E}(Y|X))}{\mathbb{V}(Y)} = \frac{0.006587802}{0.02666667} = 0.2470425.$$

This means that with X we can explain only 24.704% of Y, which is not strong enough.

Exercise 3: The conditional probability density of Y given $X = x$ is $f_{Y|X=x}(y) = xe^{-xy}$ for $0 < y$ and the marginal probability distribution of X is the uniform distribution $\mathcal{U}([0, 10])$.

1. Compute the probability $\mathbb{P}\{Y < 2|X = 2\}$. We just need to use the conditional density of $Y|X = 2$ given above:

$$\begin{aligned}
\mathbb{P}\{Y < 2|X = 2\} &= \int_{-\infty}^{2} f_{Y|X=2}(y)dy \\
&= \int_{-\infty}^{2} 2e^{-2y}\mathbf{1}_{]0,+\infty[}(y)dy \\
&= F_{\mathcal{E}(2)}(2) = 1 - e^{-4} = 98.16\%,
\end{aligned}$$

   where we have noted that $Y|X = x$ is following an exponential density $\mathcal{E}(x)$.

2. Determine the function $\varphi(x) = \mathbb{E}(Y|X = x)$ for all $x \in [0, 10]$ and deduce the conditional expectation random variable $\mathbb{E}(Y|X)$. Comment.

   Since $(Y|X = x) \sim \mathcal{E}(x)$ then $\mathbb{E}(Y|X = x) = \dfrac{1}{x}$ and then $\mathbb{E}(Y|X) = \dfrac{1}{X}$. The correlation between Y and X is definitely non-linear.

3. Determine the marginal distribution of Y, f_Y.

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{+\infty} \left[f_{Y|X=x}(y) \times f_X(x)\right] dx \\
&= \frac{1}{10}\int_{0}^{10} xe^{-xy}dx \\
&= \frac{-1}{10y}\int_{0}^{10} xd\left(e^{-xy}\right) dx \\
&= \frac{-1}{10y}\left\{\left[xe^{-xy}\right]_0^{10} - \int_0^{10} e^{-xy}dx\right\} \\
&= \frac{-1}{10y}\left(10e^{-10y} + \frac{1}{y}(e^{-10y} - 1)\right)
\end{aligned}$$

4. What about the strongness of the relationship between X and Y ?
   This question is left to students to solve it themselves.

Exercise 4: Consider the unit disc

$$D = \{(x, y) \,|\, x^2 + y^2 \leq 1\}.$$

Suppose that we choose a point (X, Y) uniformly at random in D. That is, the joint PDF of X and Y is given by

$$f_{XY}(x, y) = \begin{cases} c & (x, y) \in D \\ 0 & \text{otherwise} \end{cases}$$

i. Find the constant c.

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY(x,y)}dxdy &= 1\\ &= c\int_D dxdy\\ &= c\times \text{unit disc surface} = c\pi,\end{aligned}$$

and so $c = \dfrac{1}{\pi}$.

ii. Find the marginal PDFs f_X and f_Y.
Let $y \in [-1,1]$ and let $(x,y)\in D$ then $-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$, so

$$\begin{aligned} f_Y(y) &= \frac{1}{\pi}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\\ &= \frac{2}{\pi}\sqrt{1-y^2}.\end{aligned}$$

The joint density is symmetric and so $f_X(x) = \dfrac{2}{\pi}\sqrt{1-x^2}$ for $x \in [-1,1]$ and zero elsewhere.

iii. Find the conditional PDF of $(X|Y=y)$, for $-1 \le y \le 1$.
We note that while $Y=y$, X is varying uniformly in $[-\sqrt{1-y^2}, \sqrt{1-y^2}]$ i.e $(X|Y=y) \sim \mathcal{U}[-\sqrt{1-y^2}, \sqrt{1-y^2}]$ and so

$$f_{X|Y=y}(x) = \frac{1}{2\sqrt{1-y^2}}\mathbf{1}_{[-\sqrt{1-y^2},\sqrt{1-y^2}]}(y),$$

iv. Are X and Y independent?
X and Y are for sure dependent since $X^2+Y^2 \le 1$, but also because

$$f_{XY}(x,y) \neq f_X(x)\times f_Y(y).$$

We may add here the fact that

$$\mathbb{E}(X|Y=y) = \varphi_{X|Y}(y) = 0!$$

We should be careful here and note that even though the conditional expectation $\mathbb{E}(X|Y) = \mathbb{E}(X) = 0$, it doesn't imply that X is independent of Y, we just say that X is in average independent of Y.