Diffusion Exercises

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Note: Based on the instructions from the file ex1.pdf

1. (i)

Examining the graph generated from the file mean_sqr.txt, it is apparent that there is a linear relationship between the n and the mean squared values generated. In particular, we observe that:

$$\frac{\langle (x(t) - x(0))^2 \rangle}{n} = 1 = 2D\Delta t \Rightarrow D = \frac{1}{2\Delta t}$$

(ii)

Given $F_X(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$, we will show that, for $X_i = F_X(x)$ such that each X_i is independent:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} ((2 * X_i - 1) * \sqrt{3} * \sigma) \to N(0, \sigma)$$

Where $N(0, \sigma)$ is a normal distribution, which has the pdf:

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

Proof. We first observe that since each X_i is independent and identically distributed, the central limit theorem tells us that:

$$\lim_{n\to\infty} \sqrt{n} \left(\left(\frac{1}{n} \sum_{i=1}^{n} X_i \right) - \mu \right) \to N(0, \sigma^2)$$

For our given $F_X(x), \mu = 0.5$. We can double the range of $F_X(x)$ so that $x \in [0,2]$ by multiplying our finite sum of X_i values by two, noting that now we have $\mu = 1$, thus:

$$\lim_{n\to\infty} \sqrt{n} \left(\left(\frac{2}{n} \sum_{i=1}^{n} X_i \right) - 1 \right) \to N(0, \sigma^2)$$

We can condense this expression by observing that our two operations were equivalent to generating a new function:

$$F'_X(x) = \begin{cases} 1/2 & x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

Which we note is a PDF with $\mu = 0$. We will now generate X'_i values in the same manner that we generated our X_i values above, giving us that:

$$\lim_{n\to\infty} \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^{n} X_i \right) = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \to N(0, \sigma^2)$$

Now from our function definition we observe that each X_i has variance:

$$\mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathbb{E}[X^2] = \int_{-1}^{1} \frac{x^2}{2} = \sigma^2$$

This, however, only works for $\sigma = 1/\sqrt{3}$! To generalize, we realize that we can modify our variance by simply changing the bounds of $F_X'(x)$ by simply multiplying by a constant. Let us define some constant a such that:

$$F_X(x) = \begin{cases} \frac{1}{2a} & x \in [-a, a] \\ 0 & \text{otherwise} \end{cases}$$

This function has variance:

$$\mathbb{E}[X^2] = \int_{-a}^{a} \frac{x^2}{2a} = \frac{2a^3}{6a} = \frac{a^2}{3} = \sigma^2$$

Taking the square root of both sides yields that $a = \sqrt{3}\sigma$ for a given σ. Now we can put together all our described alterations to reveal that:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} ((2 * X_i - 1) * \sqrt{3} * \sigma) \to N(0, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

□

(iii)

We will let $x(0) = 0$ and observe that other cases follow by symmetry. Then:

$$p(x,t) = \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}$$

$$\frac{\partial p}{\partial t} = \frac{-e^{-x^2/4Dt}}{2t\sqrt{4\pi Dt}} + \frac{x^2 e^{-x^2/4Dt}}{4Dt^2\sqrt{4\pi Dt}}$$

$$\frac{\partial^2 p}{\partial x^2} = \frac{\partial p}{\partial x}\left(\frac{-xe^{-x^2/4Dt}}{2Dt\sqrt{4\pi Dt}}\right) = \frac{-e^{-x^2/4Dt}}{2Dt\sqrt{4\pi Dt}} + \frac{x^2 e^{-x^2/4Dt}}{4D^2t^2\sqrt{4\pi Dt}}$$

This gives us that:

$$D\frac{\partial^2 p}{\partial x^2} = \frac{-e^{-x^2/4Dt}}{2t\sqrt{4\pi Dt}} + \frac{x^2 e^{-x^2/4Dt}}{4Dt^2\sqrt{4\pi Dt}} = \frac{\partial p}{\partial t}$$

I'm not sure...