Inquiry of the n-body problem

Regan Clare

2023

Contents

Chapter 1

Glossary

- **Chaotic Systems** - Chaotic systems arise when systems are predictable for a period of time and then appear to become random. A minute tweak in the initial values can completely change the outcome of the system. This is also known as the butterfly effect.
- **Sum/Summation** - summations use the Greek notation capital Σ and is the addition of a sequence of any kind of number. The upper bound refers to how far the sequence will go, and the lower bound refers to where the variable will start. E.g. $\sum_{n=1}^{100} n = 1 + 2 + 3 + ... + 99 + 100 = 5050$. This is an example of an arithmetic series. Series and sums are used literally everywhere in reality, physics, finance, computer science, and mathematics. They're so common because they are such a powerful tool for turning real-world complex problems into simple, and manageable forms. They often lead to efficient algorithms, precise approximations, and more graceful solutions.
- **Integrals** - integrals refer to the area under a curve. Similar to summations, they are infinitesimal additions of infinitesimal rectangles that depict an area for a given function. $\int_a^b x$ is referred to as a definite integral due to its bounds, while $\int x$ is referred to as an indefinite integral. Both calculate the area under a curve, however, the definite integral calculates the area of said curve within the respective boundaries. Double and triple integrals can be used for calculating volumes, averages, surface areas, mass, and inertia in three-dimensions, etc over a two/three-dimensional region. $\mathbb{R}^2$ or $\mathbb{R}^3$.
- **ODEs, PDEs, and DEs** - Ordinary Differential equations are used to describe the behaviour of systems that change with only one independent variable, such as time. Partial Differential Equations are used to describe systems that change with multiple independent variables, such as space and time. Differential Equations in general are all about modeling change. PDEs can typically be denoted through the notation δ. An example of a DE is $a = \frac{\Delta v}{\Delta t}$.
- **Vectors and Matrices** - vectors can be thought of as arrows pointing in space, and what defines a given vector is its direction and magnitude. They have the notation $\vec{x}$. Matrices can be thought of as a way of representing multiple vectors in a given system by using a rectangular array $\begin{bmatrix} x & y & z \\ a & b & c \\ 1 & 2 & 3 \end{bmatrix}$. This 3x3 vector represents three different vectors that have direction and magnitude in $\mathbb{R}^3$.
- **Inertial reference frame** - a frame of reference that is not undergoing any acceleration, and objects inside the frame of reference will continue to be stationary/moving at a constant speed unless acted upon by an external force.
- **Unit-normal vector** - a basis vector/unit vector is a vector with a magnitude of 1
- **L2-norm of a number** - also known as Euclidean norm, is a measure of the magnitude of a vector in a multi-dimensional space. It is a way of find the distance between two points in space. E.g. vector x with n elements can be represented as $||\vec{x}||^2 = \sqrt{x_1^2 + x_2^2 + x_3^2 + ... + x_n^2}$. The L2-norm is always positive (modulus) and equal to 0 only when all elements of the vector are 0. For example, the l^2-norm of the vector $x = (x_1, x_2, x_3)$ is given by $||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$.
- **Newton's Law of Universal Gravitation** - is an equation which states that the gravitational force between two objects with mass, is equal to those two masses multiplied and divided by the square of the distance between their centres. This is all then multiplied by the gravitational constant G, which is = $6.67430 \cdot 10^-11 \frac{N \cdot m^2}{kg^2}$. The equation goes as follows: $F_g = G\frac{m_i m_j}{r^2}$

- **Newton's Second Law -** $F = ma$ is an equation that states the force an object has, is equal to its mass multiplied by its acceleration.
- **Square-cubed Law -** this law is a mathematical principle that describes the relationship between volume and surface area as a shape increases/decreases in size. It states that when an object undergoes a proportional increase in size, its surface area increases by the square of the multiplier, while its volume increase by the cube of the multiplier. For area, $A_2 = A_1(\frac{l_2}{l_1})^2$, and for volume, $V_2 = V_1(\frac{l_2}{l_1})^3$. Where V_1 and A_1 are the original values for volume and area respectively. l_1 and l_2 are the original and new length respectively. This should seem fairly intuitive when presented with the example of a 3x3 metre box; its area = 9 metres squared, while its volume is equal to 27 metres cubed. Increasing the box proportionally by a scalar value of 2, the new box is going to be 6x6 metres. We already know the area will be 36 metres square, and the volume 216 metres cubed, but we can use the provided equation as well. Take Volume for example, $V_2 = 27 \cdot (\frac{6}{3})^3 = 27 \cdot 8 = 216m^3$.
- **Lagrange Point -** in celestial mechanics, Lagrange points are points in space where the gravitational forces between small bodies in the gravitational field of two large bodies are in equilibrium. For example, the Earth/Sun system has 5 Lagrange points: L1, L2, L3, L4, and L5.

Chapter 2

What is the n-body problem?

2.1 Introduction

The n-body problem is a mathematical challenge in astrophysics to calculate the motion of multiple celestial objects (such as planets, stars, or asteroids) that are interacting with each other through gravitational forces. The problem is considered difficult because the mutual gravitational attraction between each pair of objects makes it impossible to predict the motion of each object individually. Instead, one must calculate the motion of all objects simultaneously, which can become extremely complex and computationally expensive as the number of objects increases. Physicists refer to this system as **chaotic** because one tiny change in the initial conditions and the entire outcome of the system can change. The derivation can become extremely technical and sophisticated, far beyond my current level of knowledge. However, I still have a grasp on the basic mathematical concepts, so I will be summarising and may refer to mathematical concepts such as **sums, integrals, ODEs/PDEs/DEs, vectors, matrices**. Basically, a fundamental grasp of the concepts of differential calculus, integral calculus, classical mechanics, and linear algebra will provide a sound understanding. If this is not the case, I will still be summarising every equation and piece of notation.

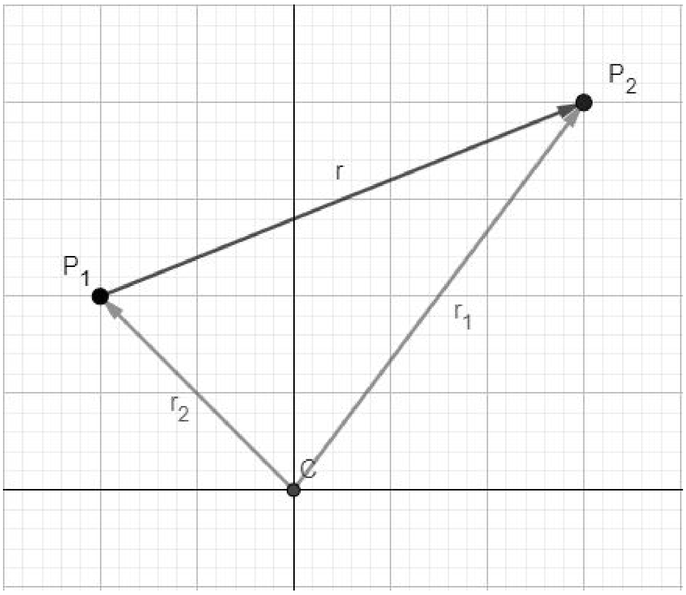
While we have equations that can completely predict the motions of two gravitating masses, our analytical tools fall short when faced with n $>$ 2 masses. It's impossible to know all the terms of a general formula that can exactly describe the motion n $\geq$ 3 gravitating objects. This is because there are too many unknown variables in n-body systems. Thanks to Isaac Newton, we have equations that describe gravitational force acting on masses. However, when we try to find a general solution to the unknown variables using these equations, we are faced with a constraint; for each unknown variable, there must be at least one equation that independently describes it. So we'll begin by covering the basic equations behind the problem and work up the derivation. Below is **Newton's law of universal gravitation**, where F = gravitational force, G = gravitational constant, m_1 = mass of object 1, m_2 = mass of object 2, and r = distance between the centre of both masses. The problem is formulated using Newton's law of gravitation.

$$F_g = G\frac{m_1 m_2}{r^2} \tag{2.1}$$

This sets the foundation for the n-body equations of motion. There are two interesting characteristics of this equation. The first being that the distance portion is quadratic; meaning if you halve the distance between two objects, the force between them quadruples. Second, there is a singularity when there is no distance between the two bodies. We have the magnitude, so now we need to incorporate a term for direction. For this, we need to know the relative positions of the two particles as a **unit-normal vector**. A unit vector is a vector with a magnitude of one.

$$F_g = \frac{Gm_1m_2}{r^2} \cdot \frac{r}{r} = \frac{Gm_1m_2}{||r||^3}\mathbf{r} \tag{2.2}$$

Note that 'r' is in bold as it represents a vector (we'll change it next, just wanted to clarify) and that the four vertical lines represent the **two norm of a number**. Which is, in summary, a way to measure the magnitude of a vector in multi-dimensional space. Before we continue, a quick note on vector addition and subtraction: if we have positions P1 and P2 of the two particles on a coordinate system, the method we use to get their relative positions is by adding the two vectors r_1 and r_2. This results in finding r, which is the relative position between the two.



$$r = r_1 - r_2 = r_{21} \tag{2.3}$$

We label the resultant vector as r_{21} because it goes from particle 2 to particle 1. If we want to describe how to go from particle 1 to particle 2, we do the same thing and vice versa.

$$r_{12} = r_2 - r_1 \tag{2.4}$$

With this new information, we can re-write Newton's Law of Gravitation acting on point 1 from point 2 as

$$F_{g21} = \frac{Gm_1m_2}{||r_1 - r_2||^3}(r_1 - r_2) \tag{2.5}$$

We now apply to **Newton's second law** ($F = ma$, force is equal to the change in momentum per change in time; mass multiplied by the acceleration of the object). Note: i and j are just variables and have replaced 1 and 2 respectively due to them being numbers, and that $j \neq i$. The equation below is the result of the two laws (Newton's second law and law of gravitation).

$$m_i a_i = \sum_{j=1}^{N} F_{Gji} = \sum_{j=1}^{N} \frac{G m_i m_j}{||r_i - r_j||^3}(r_i - r_j) \tag{2.6}$$

In this equation, m_i is the mass of the ith point mass, and a_i is its acceleration. This is equal to the sum of F_{Gji}, meaning it is the sum of the gravitational forces exerted on it by all the other point masses in the current system. This gravitational force between two point masses is given by Newton's law of universal gravitation (reminder: which states the force is proportional to the product of their masses and is inversely proportional to the square of the distance between them). When we put these laws together, we derive the equation of motion for the n-body problem, which is the equation above.

This equation states that the sum over j, which ranges from 1 to N particles, represents the gravitational force exerted on point mass i, by all other point masses in the system, while $(r_i - r_j)$ represents the direction of said force, and the denominator is the cube of the distance between point masses i and j, which is required to account for the **inverse-square** relationship of the gravitational force. Note; j is an index variable, meaning it is a variable that is used to identify the elements of an array, and in this context, the index variable j is used to identify or reference the jth particle in the system, and to calculate the net gravitational force on the ith particle.

To summarise, for n-bodies in a system that exerts a gravitational force, we are unable to predict the motion of these objects when n $\geq$ 3. This is because the motion of these bodies becomes horrifically difficult to model and predict due to their seemingly random and chaotic motion.

2.2 History of the n-body problem

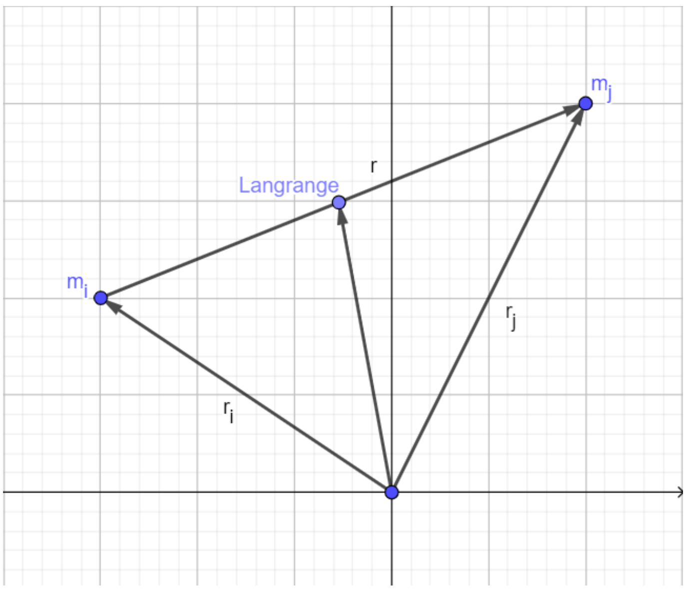
It all started with Isaac Newton in the early 17th century. He wanted to know if our solar system and the objects in it had a stable future, to do this, he would have to predict the motion of the planets. Before this however, Newton obtained three orbital positions of a planet's orbit from the astronomer John Flamsteed, and managed to produce an equation by straightforward analytical geometry, to predict any planets motion. But these equations, as he and other scientists soon found out, were not very good at predicting some orbits correctly, if at all. Newton, being the genius he was, quickly realised this was because of gravitational interactive forces among all the planets were affecting all their orbits. This is the essential concept of the n-body problem, in which it is not enough to specify initial positions and velocity, the gravitational forces have to be known.

2.3 What is its impact and importance?

Gravitational N-body simulations; which are numerical solutions of the equations of motions for N particles interacting gravitationally are widely used tools in astrophysics, that can be applied from a few celestial bodies in a solar system to large cosmological scales such as galaxies. As you can probably guess, n-body simulations and the n-body problem itself has had huge implications in fields such as celestial mechanics, astrophysics/astronomy, engineering and space exploration, other areas of physics, and even computer science. It has a large connection to Chaos Theory, as it is referred to as an unstable system due to its chaotic nature. In Astrophysics/Astronomy, the n-body problem helps us understand the motion and trajectories of celestial bodies, which in turn can be essential for making weather predictions and predictions of celestial events. In Engineering, the problem has a large impact on the designing aspect for spacecraft such as probes, satellites, and cargo vessels, as it allows them to accurately predict gravitational effects and how the objects will be affected by the gravitational force. Other chaotic systems in differing fields of physics also have a relation to n-body simulations as they play a role in understanding the behaviour of how fluids, particles, and celestial bodies, all move in an unstable system. For computer science, n-body simulations and the problem itself provide a useful benchmark test and evaluation of system capabilities and the development of a numerical algorithm and simulation techniques. I will expand on this in the next chapter; *How can computer science assist this problem?*. Note: this is the main inquiry question.

2.4 2-body problem

The two-body problem, as stated beforehand, is solvable, and is far easier to manage with respect to calculation than $n \geq 3$. This is because, obviously, we're assuming there are only two objects in a system inducing gravitational force. Now note, the derivation and intuition showed in the introductory section is for the two-body problem, as it accounts for two objects. Covering the two-body problem first, sets the foundation for covering the intuition and mathematical derivation of the three-body problem. Recognise that I will not be going over the entire mathematics for the two or three-body problem, but rather I am focusing on providing the reader with valuable insight and a fundamental understanding for what is actually happening.

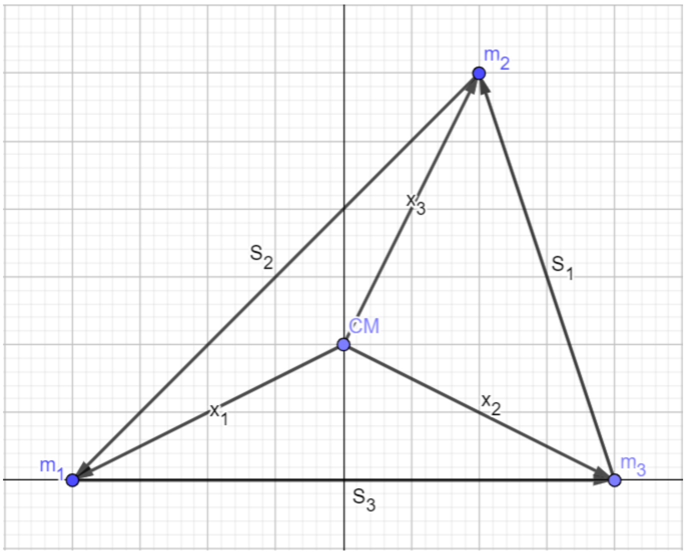


The **Lagrange point**, in summary, can be multiple points between two large orbiting bodies that are under a gravitational equilibrium. Note that two body systems have five Lagrange points, take Earth and Sun for example, three Lagrange points were discovered by Leonhard Euler around 1750, and the other two a decade later by Joseph-Louis Largrange, hence the name Lagrange.

One should recognise the simplicity of a two-body system, and I need not go any further, as the mathematical derivations were covered in the first section, and one should now have a good understanding of the intuition behind 2-body systems.

2.5 Restricted 3-body problem

If one mass in a group of three is so low relative to the others so that it exerts no significant gravitational force, then the system can be recognised to behave similarly (with very good approximation) to that of a two-body system. This is known as the Restricted 3-body problem, which proves to be extremely useful for simulating how celestial bodies act within this respective ratio of mass. Such as describing an asteroid in the Earth-Sun gravitational field, or a Planet in a Star/Black Hole gravitational field.



Here is where we see the transition into chaos. From 2-body to a restricted three body, we get a feel for how systems with greater than two objects can become very unpredictable, and increasingly harder to detect. There is enormous amounts of literature dedicated to this problem, analytically and numerically, so it would be not only naive, but impossible for me to summarise it all in a short paragraph. Therefore, this section and the 2-body section has been dedicated to remind the readers of the intuition behind the n-body problem, and attempts to make a connection between the point where systems begin to be chaotic.

Chapter 3

How can computer science assist this problem?

3.1 Use of simulations and analytical problems

Obviously, this problem isn't one that can just be thought of and solved; computers are needed to provide simulations, which do the calculations for us. Computers allow us to study physical phenomena that are otherwise impossible to observe directly, such as the n-body problem. Running simulations can provide insight which can contribute to a concise understanding of the problem itself, leading to new breakthroughs and discoveries. Although the n-body problem remains an area of active research and development, with the evolution of computer hardware and technological capabilities, the n-body problem can be solved for much larger systems and provides more accurate predictions for the motion of celestial bodies.

3.2 The benchmark

In computer science, the n-body problem is commonly used as a benchmark for evaluating performance of various algorithms and hardware (e.g. GPUs and CPUs). The reason it is used as a benchmark is because it's a challenging computational problem, that requires simulating the motion of objects, which will obviously have high demands from the computer hardware and will test the performance of it. The number of calculations required to simulate the motion of n objects grows exponentially when n becomes larger, meaning the simulation will require far greater computational resources, hence testing the true capabilities of the hardware of the computer. The time it takes to simulate an n-body system can be compared with multiple systems, to differentiate the most efficient computers from the rest (the fastest to simulate).

Chapter 4

Citations/References

https://gereshes.com/2018/05/07/what-is-the-n-body-problem/
https://www.britannica.com/science/celestial-mechanics-physics/The-n-body-problem
https://en.wikipedia.org/wiki/N-body_problem
https://en.wikipedia.org/wiki/Three-body_problem
https://en.wikipedia.org/wiki/Two-body_problem
http://www.physics.drexel.edu/~steve/Courses/Physics-431/two-body.pdf
https://www.math.arizona.edu/~dwang/485home-10/N-body.pdf
https://www.phys.lsu.edu/faculty/gonzalez/Teaching/Phys7221/ThreeBodyProblem.pdf
https://www.britannica.com/science/celestial-mechanics-physics/The-three-body-problem