偏微分方程第3周作业

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1.2 令

$$v(x,t)=\begin{cases}-\infty, & \text{if } t=ax\\ 0, & \text{else}\end{cases}$$

那么可以得到

$$\begin{aligned}\langle u,\frac{\partial\varphi}{\partial t}\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(x,t)\frac{\partial\varphi}{\partial t}(x,t)dtdx\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{ax}\frac{\partial\varphi}{\partial t}(x,t)dtdx\\ &= \int_{-\infty}^{\infty}\varphi(x,ax)dx\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}-v(x,t)\varphi(x,t)dtdx\\ &= -\langle v,\varphi\rangle\end{aligned}$$

即 $u_t=v$.

当 $a=0$, $u_x(x,t)\equiv 0$, $u_x\neq u_t$.

当 $a>0$,

$$u(x,t)=\begin{cases}1, & \text{if } x\geq t/a\\ 0, & \text{else}\end{cases}$$

那么类似上述过程可以求得

$$u_t(x,t)=\begin{cases}\infty, & \text{if } x=t/a\\ 0, & \text{esle}\end{cases}$$

故 $u_x\neq u_t$.

当 $a<0$,

$$u(x,text)=\begin{cases}1, & \text{if } x\leq t/a\\ 0, & \text{else}\end{cases}$$

那么类似的, 可以求得

$$u_t(x,t)=\begin{cases}-\infty, & \text{if } x=t/a\\ 0, & \text{esle}\end{cases}$$

故 $u_x=u_t$.

综上, 当 $a<0$, $u_x=u_t$.

1.3 $\forall c\in\mathbb{R}$, 添加初值条件 $y(0)=c$, 方程转化为

$$\begin{aligned} y'-y&=0\\ y(0)&=c \end{aligned} \tag{1}$$

由定理12可知(1)解唯一. 容易验证, $y(x)=Ce^x$ 是(1)的解, 且 $y(x)\in\mathcal{D}(\mathbb{R})\subset\mathcal{D}'(\mathbb{R})$.

改变不同的初值条件, 可得所有满足 $y'-y=0$ 的 y 形如 $y(x)=Ce^x$.

1.4 (题中未说明定义域, 认为 $\mathbb{R}$) 若 u 是 $\mathcal{L}u(x)$ 的基本解, 则 $\mathcal{L}u(x)=\delta(x-x_0)$. 设 $h_{x_0}(x)=\int_{-\infty}^{\infty}\delta(x-x_0)dx$. 特别的, 一维时可以显式写出 $h_{x_0}(x)=\begin{cases}1, & \text{if } x\geq x_0\\ 0, & \text{else}\end{cases}$.

$$\mathcal{L}u(x)=\frac{d}{dx}(\frac{du}{dx}-u)(x)=\delta(x-x_0)$$

则

$$\frac{du}{dx}(x)-u(x)=h_{x_0}(x)+C_1$$

可以解得

$$\begin{aligned} u(x)&=C_2e^{-x}+e^{-x}\int_{-\infty}^{x}e^t(h_{x_0}(x)+C_1)dt\\ &=C_2e^{-x}+C_1+e^{-x}\int_{-\infty}^{x}e^th_{x_0}(x)dt\\ &=C_2e^{-x}+C_1+e^{-x}\int_{x_0}^{x}e^tdt\\ &=(C_2-e^{x_0})e^{-x}+C_1+1 \end{aligned}$$

故基本解为 $(C_2-e^{x_0})e^{-x}+C_1+1$ (当考虑更高维的情形时, 基本解应该是 $C_2e^{-x}+C_1+e^{-x}\int_{-\infty}^{x}e^th_{x_0}(x)dt$).

1.5 (题中未说明定义域, 认为 $\mathbb{R}^2$) 若 u 是 $\mathcal{L}u(x,y)$ 的基本解, 则 $\mathcal{L}u(x,y)=\delta(x-x_0,y-y_0)$.

$$\mathcal{L}u(x,y)=(\frac{d}{dx}-\frac{d}{dy})(\frac{d}{dx}-\frac{d}{dy})u(x,y)=\delta(x-x_0,y-y_0)$$

则

$$(\frac{d}{dx}-\frac{d}{dy})u(x,y)=\int_1^x\delta(t-x_0,-t+y_0+x+y)dpdt+f_1(x+y)=h(x,y)$$

其中 f_1 是任意函数, 记这个解为 $h(x,y)$. 继续解可得

$$\begin{aligned}
u(x,y) &= \int_1^x h(t,t-x+y)dt + f_2(x+y) \\
&= \int_1^x \left(\int_1^t \delta(p-x_0,-p+y_0+2t-x+y)dp + f_1(2t-x+y)\right)dt + f_2(x+y) \\
&= h(x,y) + g(x,y)
\end{aligned}$$

其中 f_2 是任意函数, $g(x,y)=\int_1^x f_1(2t-x+y)dt + f_2(x+y)$, $h(x,y)=\int_1^x\int_1^t \delta(p-x_0,-p+y_0+2t-x+y)dpdt$, 更具体的说

$$h(x,y) = \begin{cases} 1, & \text{if } (x,y)\in\Omega \\ 0, & \text{else} \end{cases}$$

其中 $\Omega=\{(x,y)\in\mathbb{R}^2 : x\geq 2x_0, x+y\geq 2x_0-y_0, x-y\geq -2x_0+y_0+2\}$.

为了满足 $y<0$ 时 $u(x,y)=0$,则要求 $g=-h+k$, 其中 $\forall y<0$, $k(x,y)=0$. 等式两边分别对 x,y 求偏导(以下导数都是在广义下)

$$\begin{aligned}
f_1(x+y)+f_1(-x+y+2)+f_2'(x+y) &= \frac{\partial}{\partial x}(-h+k)(x,y) \\
f_1(x+y)-f_1(-x+y+2)+f_2'(x+y) &= \frac{\partial}{\partial y}(-h+k)(x,y)
\end{aligned} \tag{2}$$

两式相减可得

$$\begin{aligned}
f_1(-x+y+2) &= \frac{1}{2}(\frac{\partial}{\partial x}-\frac{\partial}{\partial y})(-h+k)(x,y) \\
\Rightarrow f_1(x) &= \frac{1}{2}(\frac{\partial}{\partial x}-\frac{\partial}{\partial y})(-h+k)(t,t+x-2), \forall t\in\mathbb{R}
\end{aligned}$$

再考虑 f_2 . 将(2)中两式相加, 可得

$$\begin{aligned}
f_2'(x+y) &= \frac{1}{2}(\frac{\partial}{\partial x}+\frac{\partial}{\partial y})(-h+k)(x,y) - f_1(x+y) \\
\Rightarrow f_2'(x) &= \frac{1}{2}(\frac{\partial}{\partial x}+\frac{\partial}{\partial y})(-h+k)(t,x-t) - f_1(x), \forall t\in\mathbb{R}
\end{aligned}$$

t 的任意性要求, 对于几乎每个固定的 x , 存在常数 C_1,C_2 使下面等式对 $\forall t\in\mathbb{R}$ 成立

$$\begin{aligned}
(\frac{\partial}{\partial x}-\frac{\partial}{\partial y})(-h+k)(t,t+x-2) &\equiv C_1 \\
(\frac{\partial}{\partial x}+\frac{\partial}{\partial y})(-h+k)(t,x-t) &\equiv C_2
\end{aligned}$$

容易知道 $k\equiv 0$, 上式是可以满足的. 已知 $\forall y<0$, $k(x,y)=0$, 可得 $\forall x\in\mathbb{R}$

$$\begin{aligned}
&(\frac{\partial}{\partial x}-\frac{\partial}{\partial y})k(t,t+x-2) = (\frac{\partial}{\partial x}-\frac{\partial}{\partial y})k(-x,-2) = 0, \forall t\in\mathbb{R} \\
\Rightarrow &(\frac{\partial}{\partial x}-\frac{\partial}{\partial y})k(a,b) = (\frac{\partial}{\partial x}-\frac{\partial}{\partial y})k(a,a+(b-a+2)-2) = 0, \forall (a,b)\in\mathbb{R}^2 \\
\Rightarrow &\partial_x k \equiv \partial_y k
\end{aligned}$$

同理可得 $\partial_x k\equiv -\partial_y k$, 故 $\partial_x k\equiv\partial_y k\equiv 0 \Rightarrow k\equiv 0$

那么 $g=-h+k=-h$, 基本解为 $u=g+h\equiv 0$

1.6

$$
\begin{aligned}
\frac{\partial E}{\partial x_i} &= \frac{\partial E}{\partial r}\frac{\partial r}{\partial x_i} = \frac{\sin(\sqrt{c}r)\sqrt{c}r + \cos(\sqrt{c}r)}{4\pi r^3}x_i \\
\frac{\partial^2 E}{\partial x_i^2} &= \frac{\sqrt{c}r\sin(\sqrt{c}r) + \cos(\sqrt{c}r)}{4\pi r^3} + \frac{((cr^2-3)\cos(\sqrt{c}r) - 3\sqrt{c}r\sin(\sqrt{c}r))}{4\pi r^5}x_i^2 \\
\Delta E &= \sum_{i=1}^n \frac{\partial^2 E}{\partial x_i^2} = \frac{cr^2\cos(\sqrt{c}r) - 2\sqrt{c}r\sin(\sqrt{c}r) - 2\cos(\sqrt{c}r)}{4\pi r^3}
\end{aligned}
$$

若 E 是 $\Delta + c$ 的基本解, $\forall \varphi \in \mathcal{D}(\mathbb{R}^n)$

$$
\langle (\Delta + c)E, \varphi \rangle = \langle E, (\Delta + c)\varphi \rangle = \langle E, \Delta\varphi \rangle + c\langle E, \varphi \rangle
$$

$E(x, x_0)$ 是局部可和的, 则

$$
\begin{aligned}
\langle E, \Delta\varphi \rangle &= \int_{\mathbb{R}^n} E(x, x_0)\Delta\varphi(x)dx \\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus Q_\varepsilon^{x_0}} E(x, x_0)\Delta\varphi(x)dx
\end{aligned}
$$

其中 $Q_\varepsilon^{x_0}$ 表示中心在 x_0 , 半径为 ε 的球. 使用格林公式

$$
\begin{aligned}
&\int_{\mathbb{R}^n\setminus Q_\varepsilon^{x_0}} E(x, x_0)\Delta\varphi(x)dx \\
=&\int_{\mathbb{R}^n\setminus Q_\varepsilon^{x_0}} \Delta E(x, x_0)\varphi(x)dx + \int_{\partial Q_\varepsilon^{x_0}} \left(E\frac{\partial \varphi}{\partial \nu'} - \varphi\frac{\partial E}{\partial \nu'}\right)d\nu'
\end{aligned}
$$

已知在 $\mathbb{R}^n \setminus x_0$ 中, $(\Delta + c)E \equiv 0$, 则

$$
\begin{aligned}
&\int_{\mathbb{R}^n\setminus Q_\varepsilon^{x_0}} \Delta E(x, x_0)\varphi(x)dx \\
=&\int_{\mathbb{R}^n\setminus Q_\varepsilon^{x_0}} (\Delta + c)E(x, x_0)\varphi(x)dx - c\int_{\mathbb{R}^n\setminus Q_\varepsilon^{x_0}} E(x, x_0)\varphi(x)dx \\
=&-c\int_{\mathbb{R}^n\setminus Q_\varepsilon^{x_0}} E(x, x_0)\varphi(x)dx = -c\langle E, \varphi \rangle
\end{aligned}
$$

而另一个积分

$$
\int_{\partial Q_\varepsilon^{x_0}} \left(E\frac{\partial \varphi}{\partial \nu'} - \varphi\frac{\partial E}{\partial \nu'}\right) = \int_{\partial Q_\varepsilon^{x_0}} E\frac{\partial \varphi}{\partial \nu'}d\nu' - \int_{\partial Q_\varepsilon^{x_0}} \varphi\frac{\partial E}{\partial \nu'}d\nu'
$$

其中

$$
\begin{aligned}
\left|\int_{\partial Q_\varepsilon^{x_0}} E\frac{\partial \varphi}{\partial \nu'}d\nu'\right| &\le |C(\varepsilon)|\int_{\partial Q_\varepsilon^{x_0}} \left|\frac{\partial \varphi}{\partial \nu'}\right|d\nu' \\
&\le |C(\varepsilon)|\frac{1}{4\pi\varepsilon^3}\max_{x\in\partial Q_\varepsilon^{x_0}}\left|\frac{\partial \varphi}{\partial \nu'}(x)\right| \\
&\le K\frac{|C(\varepsilon)|}{\varepsilon^3}
\end{aligned}
$$

$C(\varepsilon)$ 是 $E(x,x_0)$ 在这个球面上的取值, K 是一个与 ε 无关的常数. 显然当 $\varepsilon \to 0$, $C(\varepsilon)/\varepsilon^3 \to 0$, 即 $\left|\int_{\partial Q_\varepsilon^{x_0}} E\frac{\partial\varphi}{\partial\nu'}d\nu'\right| \to 0$. 另外

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\partial Q_\varepsilon^{x_0}} \varphi\frac{\partial E}{\partial\nu'}d\nu' &= \lim_{\varepsilon\to 0}\frac{\sin(\sqrt{c}r)\sqrt{c}r+\cos(\sqrt{c}r)}{4\pi r^2}\int_{\partial Q_\varepsilon^{x_0}}\varphi d\nu' \\ &= \lim_{\varepsilon\to 0}\frac{\sin(\sqrt{c}r)\sqrt{c}}{4\pi r}\int_{\partial Q_\varepsilon^{x_0}}\varphi d\nu' \\ &= -\varphi(x_0)\end{aligned}$$

故

$$\begin{aligned}\langle(\Delta+c)E,\varphi\rangle &= \langle E,\Delta\varphi\rangle + c\langle E,\varphi\rangle \\ &= -c\langle E,\varphi\rangle + c\langle E,\varphi\rangle + \lim_{\varepsilon\to 0}\int_{\partial Q_\varepsilon^{x_0}} E\frac{\partial\varphi}{\partial\nu'}d\nu' - \lim_{\varepsilon\to 0}\int_{\partial Q_\varepsilon^{x_0}}\varphi\frac{\partial E}{\partial\nu'}d\nu' \\ &= \varphi(x_0) = \langle\delta(x-x_0),\varphi\rangle\end{aligned}$$

由此, 证明了 $E(x,x_0)$ 是 $\Delta+c$ 的基本解.

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(\varepsilon)d\varepsilon$$

1.10 a) 设 f 在 x_0 点处不连续, 则 $\mathcal{D}_x f(x_0) = \infty$ 或 $-\infty$.

若 f 的不连续点 x_0 孤立, 存在区间 $\varepsilon > 0$, s.t. $(x_0-\varepsilon, x_0+\varepsilon)$ 中只有唯一的不连续点, 则在这个区间上

$$\mathcal{D}_x f(x) = f'_L(x) + f'_R(x) + \delta(x-x_0)$$

其中 $f'_L(x) = \begin{cases} f'(x), & \text{if } x_0-\varepsilon < x < x_0 \\ 0, & \text{if } x_0 \le x < x_0+\varepsilon \end{cases}$, $f'_L(x) = \begin{cases} 0, & \text{if } x_0-\varepsilon < x \le x_0 \\ f'(x), & \text{if } x_0 < x < x_0+\varepsilon \end{cases}$

$\forall u \in C_0^\infty((0,1))$

$$\begin{aligned}\|f-u\| &= \left(\int_0^1 |\mathcal{D}_x(f-u)|^2 dx\right)^{\frac{1}{2}} \\ &\ge \left(\int_{x_0-\varepsilon}^{x_0+\varepsilon} |\mathcal{D}_x(f-u)|^2 dx\right)^{\frac{1}{2}} \\ &= \left(\int_{x_0-\varepsilon}^{x_0+\varepsilon} |f'_L(x)+f'_R(x)+\delta(x-x_0)-\mathcal{D}_x u(x)|^2 dx\right)^{\frac{1}{2}} \\ &\ge \left(\int_{x_0-\varepsilon}^{x_0+\varepsilon} \big(|f'_L(x)+f'_R(x)-\mathcal{D}_x u(x)| - |\delta(x-x_0)|\big)^2 dx\right)^{\frac{1}{2}} \\ &\ge \left(\int_{x_0-\varepsilon}^{x_0+\varepsilon} |\delta(x-x_0)|^2 dx\right)^{\frac{1}{2}} = 1\end{aligned}$$

若 f 的不连续点 x_0 不孤立 , 则存在区间 (a,b) , $\forall xin(a,b)$, $\mathcal{D}_x f(x) = \pm\infty$. 那么 $\forall u \in C_0^\infty((0,1))$

$$\|f-u\| \ge \left(\int_a^b |\mathcal{D}_x(f-u)|^2 dx\right)^{\frac{1}{2}} \ge \infty$$

故若 f 不连续, $f \notin \overset{\circ}{H}{}^1((0,1))$.

b) 是的. 若 f 连续, 则 $\max\limits_{0<x<1} |\mathcal{D}_x f(x)| < \infty$. 显然 $f^h \in C_0^\infty((0,1))$, 且由磨光子性质, 对 $\forall \varepsilon > 0$, $\exists h > 0$, 使 $\|\mathcal{D}_x f - (\mathcal{D}_x f)^h\|_{L_2} < \varepsilon$.

$$\begin{aligned}\|f - f^h\| &= \|\mathcal{D}_x f - \mathcal{D}_x f^h\|_{L_2} \\ &= \|\mathcal{D}_x f - (\mathcal{D}_x f)^h\|_{L_2} < \varepsilon\end{aligned}$$

故若 f 连续, $f \in \overset{\circ}{H}{}^1((0,1))$.

1.15 由1.10可以知, 需要 f 连续且 $f(1) = f(-1) = 0$, 容易得 $\alpha \geq 0, \beta = (\frac{1}{2} + k)\pi, k \in \mathbb{Z}$

1.16 类似1.15, 要求 f 连续且 $f(x) = 0$, $\forall x \in \partial B^n_{1/2}(0)$. 易知, 当 $\alpha \neq 0$, $f(x)$ 在0点不连续, 则 $\alpha = 0$. 当 $\forall x \in \partial B^n_{1/2}(0)$, $f(x) = \cos(\beta/2) = 0 \Rightarrow \beta = (1 + 2k)\pi$. 故 $\alpha = 0, \beta = (1 + 2k)\pi, k \in \mathbb{Z}$

2.4 原方程的特征形式为

$$\xi_1\xi_2 + (3x + y - z)\xi_1\xi_3 + (3x - y + z)\xi_2\xi_3 = 0$$

若为双曲形式, 则

$$3x + y - z \neq 0, \quad 3x - y + z \neq 0$$

故当 $x \neq \pm\frac{y-z}{3}$, 方程是双曲型的.

2.5 原方程的特征形式为

$$\xi_1^2 - y^2\xi_2^2 = 0$$

a) 当在点 $(1, 2)$, 方程是双曲型的, 特征为 $2x \pm ty = c$, $\forall c \in \mathbb{R}$.

b) 当在点 $(1, 0)$, 方程是抛物型的, 特征为 $y = c$, $\forall c \in \mathbb{R}$.

2.6 原方程的特征形式为

$$\xi_1\xi_2 - \xi_2^2 = 0$$

a) 方程是双曲型的, 特征为 $x = c$ 或 $x - y = c$, $\forall c \in \mathbb{R}$.

b) 设 $u(x, y) = e^{px+qy}$, 则原方程化为

$$(pq - q^2 - p + q)u(x, y) = 0$$

当 $p = q$ 或 $q = 1$ 等式成立, 故通解为

$$u(x, y) = c_1e^{px+py} + c_2e^{px+y}$$

其中 c_1, c_2, p 是任意常数.

2.7 原方程的特征形式为

$$2\xi_1^2 + \xi_1\xi_2 = 0$$

a) 方程是双曲型的.

b) 特征为 $y = c$ 或 $x - 2y = c$, $\forall c \in \mathbb{R}$.

c) 设 $u(x,y) = e^{px+qy}$, 则原方程化为

$$(2p^2 + pq)u(x,y) = 0$$

那么 $p = 0$ 或 $p = -\frac{1}{2}q$ 时等式成立, 故通解为

$$u(x,y) = c_1 e^{qy} + c_2 e^{-\frac{1}{2}q(x-2y)}$$

其中 c_1, c_2, q 是任意常数.

2.8 原方程的特征形式为

$$\xi_1^2 - 2\alpha\xi_1\xi_2 - 3\alpha^2\xi_2^2 = (\xi_1 + \alpha\xi_2)(\xi_1 - 3\alpha\xi_2) = 0$$

a) 当 $\alpha \neq 0$, 方程是双曲型的, 当 $\alpha = 0$, 方程是抛物型的.

b) 当 $\alpha \neq 0$, 特征为 $\alpha x - y = c$ 或 $3\alpha x + y = c$, 故设变量 $x' = \alpha x - y$, $y' = \alpha x + y$

$$\begin{aligned}
u_x &= \alpha u_{x'} + 3\alpha u_{y'}, \quad u_y = -u_{x'} + u_{y'} \\
u_{xx} &= \alpha^2 u_{x'x'} + 6\alpha^2 u_{x'y'} + 9\alpha^2 u_{y'y'} \\
u_{xy} &= -\alpha u_{x'x'} - 2\alpha u_{x'y'} + 3\alpha u_{y'y'} \\
u_{yy} &= u_{x'x'} - 2u_{x'y'} + u_{y'y'}
\end{aligned}$$

则转化为标准型为 $u_{xx} - 2\alpha u_{xy} - 3\alpha^2 u_{yy} + \alpha u_y + u_x = 4\alpha^2 u_{x'y'} + 4\alpha u_{y'} = 0$

当 $\alpha = 0$, 原式就是标准型 $u_{xx} + u_x = 0$

c) 当 $\alpha \neq 0$, 设 $u(x,y) = e^{px'+qy'}$ (第二小题中的 x', y'), 则原方程化为

$$4\alpha q(p+1)u(x',y') = 0$$

当 $q = 0$ 或 $p = -1$ 等式成立, 故通解为

$$u(x',y') = c_1 e^{px'} + c_2 e^{-x'+qy'} = c_1 e^{\alpha px - py} + c_2 e^{(q-1)\alpha x + (q+1)y}$$

其中 c_1, c_2, p, q 是任意常数

当 $\alpha = 0$, 设 $u(x,y) = e^{px+qy}$, 则原方程化为

$$p(p+1)u(x,y) = 0$$

当 $p = 0$ 或 $p = -1$ 等式成立, 故通解为

$$u(x,y) = c_1 e^{qy} + c_2 e^{-x+qy}$$

其中 c_1, c_2, q 是任意常数