偏微分方程第2周作业

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1　对 $\forall\varphi\in C_0^\infty(\mathbb{R}^2)$,

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\begin{aligned}
\langle u,\frac{\partial^2\varphi}{\partial x\partial y}\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(x,y)\frac{\partial^2\varphi}{\partial x\partial y}dxdy\\
&= \int_{-1}^{1}\int_{-1}^{1}\frac{\partial^2\varphi}{\partial x\partial y}dxdy = \int_{-1}^{1}\int_{-1}^{1}\frac{\partial}{\partial x}(\frac{\partial\varphi}{\partial y})dxdy\\
&= \int_{-1}^{1}\frac{\partial\varphi(1,y)}{\partial y} - \int_{-1}^{1}\frac{\partial\varphi(-1,y)}{\partial y}dy\\
&= \varphi(1,1)-\varphi(1,-1)-\varphi(-1,1)+\varphi(-1,-1)\\
&= \langle\delta_{(1,1)},\varphi\rangle-\langle\delta_{(1,-1)},\varphi\rangle-\langle\delta_{(-1,1)},\varphi\rangle+\langle\delta_{(-1,-1)},\varphi\rangle\\
&= \langle\delta_{(1,1)}-\delta_{(1,-1)}-\delta_{(-1,1)}+\delta_{(-1,-1)},\varphi\rangle
\end{aligned}
$$

故 u 的广义二阶导数 $\frac{\partial^2 u}{\partial x\partial y}=\delta_{(1,1)}-\delta_{(1,-1)}-\delta_{(-1,1)}+\delta_{(-1,-1)}$.

2　设 $u_t(x)=(4\pi t)^{-\frac{1}{2}}e^{-\frac{x^2}{4t}}$, 这是一个正态分布的密度函数, $\int_{-\infty}^{\infty}u_t(x)dx=1$. $\forall\varepsilon>0$, 当 t 给定, $\exists c_t>0$, s.t. $\int_{-\infty}^{-c_t}u_t(x)dx=\int_{c_t}^{\infty}u_t(x)dx<\varepsilon$. 这样的 c_t 的存在性是显然的, 且 $\lim_{t\to0}c_t=0$.

$\forall\varphi\in C_0^\infty(\mathbb{R})$,

$$\int_{-\infty}^{\infty}u_t(x)\varphi(x)dx=\int_{-\infty}^{-c_t}u_t(x)\varphi(x)dx+\int_{-c_t}^{c_t}u_t(x)\varphi(x)dx+\int_{c_t}^{\infty}u_t(x)\varphi(x)dx$$

由 $\varphi\in C_0^\infty(\mathbb{R})$, $\exists M>0$, s.t. $|\varphi(x)|<M$, 则

$$\left|\int_{-\infty}^{-c_t}u_t(x)\varphi(x)dx\right|<M\left|\int_{-\infty}^{-c_t}u_t(x)dx\right|<M\varepsilon$$

同理 $|\int_{c_t}^{\infty}u_t(x)\varphi(x)|<M\varepsilon$.

同时, 由 φ 连续, 对于这个 ε , $\exists c>0$, s.t. $\forall xin(-c,c)$, $|\varphi(x)-\varphi(0)|<\varepsilon$. 当取足够小的 t 令 $c_t<c$, 我们可以得到

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} u_t(x)\varphi(x)dx-\varphi(0)\right| &< \left|\int_{-c_t}^{c_t} u_t(x)\varphi(x)dx-\varphi(0)\right|+2M\varepsilon \\
&= \left|\int_{-c_t}^{c_t} u_t(x)\big(\varphi(x)-\varphi(0)\big)dx-2\varepsilon\varphi(0)\right|+2M\varepsilon \\
&< \varepsilon\left|\int_{-c_t}^{c_t} u_t(x)dx\right|+2|\varphi(0)|\varepsilon+2M\varepsilon \\
&< (1+2M+|\varphi(0)|)\varepsilon
\end{aligned}$$

由 ε 充分小, 则 $\lim_{t\to 0}\langle u_t,\varphi\rangle=\varphi(0)=\langle\delta,\varphi\rangle$, 即 $u_t\to\delta$

3 考虑 $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-|x|^2}e^{i(x,\xi)}dx &= \underset{\mathbb{R}^n}{\int\cdots\int} e^{\sum_{k=1}^n(-x_k^2+ix_k\xi_k)}dx_1\cdots dx_n \\
&= \prod_{k=1}^n\int_{-\infty}^{\infty} e^{-x_k^2+ix_k\xi_k}dx_k \\
&= \prod_{k=1}^n\int_{-\infty}^{\infty} e^{-(x_k+\frac{i\xi_k}{2})^2}e^{-\frac{\xi_k^2}{4}}dx_k \\
&= \prod_{k=1}^n\sqrt{\pi}e^{-\frac{\xi_k^2}{4}} = \pi^{\frac{n}{2}}e^{-\frac{|\xi|^2}{4}}
\end{aligned}$$

4 (1) $\Rightarrow$ (2) 首先, $\forall f\in L^2$

$$\begin{aligned}
\int_{\mathbb{R}^n}\hat{f}^2(x)dx &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} f(y)e^{i(y,x)}dy\right)^2dx \\
&\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)^2e^{2i(y,x)}dydx \\
&= \int_{\mathbb{R}^n} f(y)^2\left(\int_{\mathbb{R}^n} e^{2i(y,x)}dx\right)dy
\end{aligned}$$

分析可知

$$\int_{\mathbb{R}^n} e^{2i(y,x)}dx=\prod_{k=1}^n\int_{-\infty}^{\infty} e^{2iy_kx_k}dx_k\le 2^n$$

则

$$\int_{\mathbb{R}^n}\hat{f}^2(x)dx\le 2^n\int_{\mathbb{R}^n} f^2(x)dx<\infty$$

即 $\hat{f}\in L^2$

已知 $\mathcal{D}_x^\alpha f\in L^2$, 则 $\widehat{\mathcal{D}_x^\alpha f}\in L^2$, 由定理可知 $\xi^\alpha\hat{f}(\xi)=\widehat{\mathcal{D}_x^\alpha f}$, 故 $\xi^\alpha\hat{f}(\xi)\in L^2$.

(2) $\Rightarrow$ (1) 类似的, $\forall f\in L^2$, 记 f 的傅里叶逆变换为 $\check{f}$, 可以证明 $\check{f}\in L^2$. 已知 $\widehat{\mathcal{D}_x^\alpha f}=\xi^\alpha\hat{f}(\xi)\in L^2$, 则其(精确到相差一个常系数意义下的)逆变换 $\mathcal{D}_x^\alpha f\in L^2$

(2) $\Leftrightarrow$ (3) L^2 对线性运算法是封闭的, 故从 (2) 得到 (3) , 而同时 (3) 是包含 (2) 的, 这即证明了等价.

(2) $\Leftrightarrow$ (4)

$$\begin{aligned}
\int_{\mathbb{R}^n} \left((1+|\xi|^2)^{\frac{m}{2}} \hat{f}(\xi)\right)^2 d\xi &= \int_{\mathbb{R}^n} (1+|\xi|^2)^m \hat{f}^2(\xi) d\xi \\
&= \sum_{k=1}^{m} \binom{n}{k} \int_{\mathbb{R}^n} |\xi|^{2k} \hat{f}^2(\xi) d\xi \\
&= \sum_{k=1}^{m} \binom{n}{k} \int_{\mathbb{R}^n} \left(|\xi|^k \hat{f}(\xi)\right)^2 d\xi
\end{aligned}$$

当 (2) 成立, 则单独每项都是小于无穷的, 且项数也是有限的, 故和小于无穷, (4) 成立.

当 (2) 不成立, 则至少有一项是无穷, 由于每项都是非负的, 故和也是无穷, (4) 不成立. 这即证明了等价.