Irreducible Representation of $SU(2)$ and $SO(3)$

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Abstract

$SU(2)$ and $SO(3)$ are two simple but important compact Lie group. The problem that this paper mainly discusses is the classfication and construction of all irreducible representations of these two groups, and the main approaches adpoted in this paper to solve it are Lie algebra representation and weight space decomposition. The classfication of $SU(2)$ can be obtained directly by weight space decomposition(but not be proved here), then the double cover map induces that of $SO(3)$. As for construction, weight vectors and weight spaces, which can be calculate by solve differential equations, determin the irreducible representation. Finally, Casimir operator of $SO(3)$ and some propositions of that are discussesed.

Keywords: $SU(2)$, $SO(3)$, irreducible representation, weight, weight space decomposition, raisng operator, lowering operator, Casimir operator

Contents

1 Representations of $SU(2)$

The Lie algebra of $SU(2)$ is $\mathfrak{su}(2)$, and the $\mathfrak{sl}_2$-triple of $\mathfrak{su}(2)$ are s_+, s_-, s_0, which satisfy commutation relations

$$[s_0, s_+] = 2s_+, \quad [s_0, s_-] = -2s_-, \quad [s_-, s_+] = s_0 \tag{1.1}$$

Defination 1.1. *Given a finite dimension representation (π, V) of $SU(2)$, there is a Lie algebra representation (π', V) of $\mathfrak{su}(2)$. Let*

$$S_+ = \pi'(s_+), \quad S_- = \pi'(s_-), \quad S_0 = \pi'(s_0)$$

*. We call S_+ as **raisng operator**, S_- as **lowering operator**.*

These three operators' commutation relations are just the same as $\mathfrak{sl}_2$-triple

$$[S_0, S_+] = 2S_+, \quad [S_0, S_-] = -2S_-, \quad [S_-, S_+] = S_0 \tag{1.2}$$

Defination 1.2. *The S_0-eigensubspace of V with eigenvalue $\lambda \in \mathbb{C}$*

$$V^\lambda = \{v \in V \ | S_0 v = \lambda v\}$$

*λ called **weight**, and V^λ called **weight space** with weight λ.*

Proposition 1.1. $S_+ V^\lambda \subset V^{\lambda+2}, S_- V^\lambda \subset V^{\lambda-2}$

Proof. By the commutation relations, $\forall v \in V^\lambda$

$$\begin{aligned} S_0(S_+v) &= [S_0, S_+]v + S_+(S_0 v) = 2S_+v + S_+(\lambda v) = (\lambda+2)S_+v \\ S_0(S_-v) &= [S_0, S_-]v + S_-(S_0 v) = -2S_-v + S_-(\lambda v) = (\lambda-2)S_-v \end{aligned}$$

so $S_+v \in V^{\lambda+2}$, $S_-v \in V^{\lambda-2}$. □

Defination 1.3. *$v \in V^\lambda \subset V$, if $v \neq 0$ and*

$$S_+ v = 0, \quad S_0 v = \lambda v$$

*then v is called **the highest weight vector**, and V^λ is called **the highest weight space***

Obviously, any finite dimension representation (π, V) of $SU(2)$, there must be the highest weight vector and the highest weight space. With lowering operator, we can get all the other weight spaces, with gives a decomposition of π, called **weight space decomposition**. If there is another representation ρ have the same weight space decomposition as π, then $\pi \cong \rho$[2].

What's more, following theorem gives the classfication of all irreducible representations of $SU(2)$.

Theorem 1.1. *Given finite dimension irreducible representation (π, V) of $SU(2)$, it have weight in form*

$$-n, -n+2, \cdots, n-2, n$$

n is a non-negative integer, and $n = \dim \pi - 1$.

The proof of Theorem 1.1 is not the main content of this paper, you can find it in [6].

Now, we are going to construct all irreducible representations of $SU(2)$ using the powerful tool introduced above.

Given irreducible representation (π_n, V^n) of $SU(2)$, the highest weight of π is n. Actually, V^n could be any space whose dimension is $n+1$, but here we chose V^n as homogeneous polynomials space of degree n for convenience

$$V^n = \{f : \mathbb{C}^2 \to \mathbb{C} | f(z_1, z_2) = \sum_{i=0}^{n} a_i z_1^i z_2^{n-i},\ a_i \in \mathbb{C}\} \tag{1.3}$$

Obviously, $\dim \pi = \dim V^n = n+1$.

$\forall g \in SU(2)$, $g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$, and $g^{-1} = \begin{pmatrix} \bar{\alpha} & -\beta \\ \bar{\beta} & \alpha \end{pmatrix}$. $\forall g \in SU(2), f \in V^n$, the action of π is

$$\pi_n(g)(f)(z_1, z_2) = f(g^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}) = \sum_{i=0}^{n} a_i (\bar{\alpha} z_1 - \beta z_2)^i (\bar{\beta} z_1 + \alpha z_2)^{n-i} \tag{1.4}$$

and $\forall g = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathfrak{su}(2)$, the Lie algebra representation π_n' defined as[6]

$$\pi_n'(g)(f)(z_1, z_2) = \frac{d}{dt} \pi_n(e^{tg}) f(z_1, z_2)|_{t=0} = -\left((a_{11} z_1 + a_{12} z_2) \frac{\partial}{\partial_{z_1}} + (a_{21} z_1 + a_{22} z_2) \frac{\partial}{\partial_{z_2}} \right) f \tag{1.5}$$

The $\mathfrak{sl}_2$-triple s_+, s_-, s_0 are

$$s_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad s_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad s_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.6}$$

so

$$S_+ = -z_2 \partial_{z_1}, \quad S_- = -z_1 \partial_{z_2}, \quad S_0 = -z_1 \partial_{z_1} + z_2 \partial_{z_2} \tag{1.7}$$

Assume the highest weight vector is $F_n^n(z_1, z_2) = \sum_{i=0}^{n} a_i z_1^i z_2^{n-i}$, by the defination, we have

$$\begin{aligned} S_+ F_n^n(z_1, z_2) &= -z_2 \partial_{z_1} F_n^n(z_1, z_2) = 0 \\ S_0 F_n^n(z_1, z_2) &= (-z_1 \partial_{z_1} + z_2 \partial_{z_2}) F_n^n(z_1, z_2) = n F_n^n(z_1, z_2) \end{aligned} \tag{1.8}$$

From the first differential equation of (1.8), we know that $\partial_{z_1} F_n^n = 0$ and

$$F_n^n(z_1, z_2) = C_n z_2^k \tag{1.9}$$

where C_n is a constant. Put it into the second differential equation of (1.8)

$$k C_n z_2^k = n C_n z_2^k \tag{1.10}$$

so $k = n$, which means

$$F_n^n(z_1, z_2) = C_n z_2^n \tag{1.11}$$

Let

$$F_k^n = (S_-)^{n-k} F_n^n = (-1)^{n-k} \frac{n!}{k!} C_n z_1^{n-k} z_2^k \tag{1.12}$$

$k = 0, 1, \cdots, n-1, n$. These $\{F_0^n, F_1^n, \cdots, F_{n-1}^n, F_n^n\}$ are the basis of V^n, which determin the weight space decomposition and the π_n.

2 Relation between $SU(2)$ and $SO(3)$

There exists a double cover homomorphism $\Phi : SU(2) \to SO(3)$ with $\ker\Phi = \{\pm 1\}$, and Φ is a surjection.[1] For any representation (ρ, V) of $SO(3)$, we can get a representation of $SU(2)$

$$\pi = \rho \circ \Phi$$

It is obvious that any finite dimension representation (ρ, V) of $SO(3)$, if ρ is irreducible, $\pi = \rho\circ\Phi$ is irreducible representation of $SU(2)$. Otherwise, π is reducible, $\exists V' \subset V$, $V' \neq V$, $\forall X \in SU(2), v \in V'$, $\pi(X)(v) = \rho(\Phi(X))(v) \in V'$. Since Φ is surjection, $\forall Y \in SO(3), v \in V'$, $\rho(Y)(v) \in V'$, which is contradictory to the assumption that ρ is irreducible.[3]

Since we have already known all irreducible representations of $SU(2)$, $\pi_n (\forall n \in \mathbb{N}^+)$, following theorem gives all irreducible representation of $SO(3)$.

What's more, as the result (1.4) in last section

$$\pi_n(-1)f(z_1,z_2) = \pi_n \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} f(z_1,z_2) = \sum_{i=0}^{n} a_i(-z_1)^i(-z_2)^{n-i} = (-1)^n f(z_1,z_2) \tag{2.1}$$

so $\pi_n(-1) = \begin{pmatrix} (-1)^n & & \\ & \ddots & \\ & & (-1)^n \end{pmatrix} = (-1)^n$. If π_n can be divided in form $\pi_n = \rho_n \circ \Phi$, then $\pi_n(-1) = \rho_n \circ \Phi(-1) = \rho_n(1) = 1 = (-1)^n$, which means n is even.

So for all $n \in \mathbb{N}^+$, (ρ_{2n}, V^{2n}), which satisfy $\pi_{2n} = \rho_{2n}\circ\Phi$, are all irreducible representations of $SO(3)$.

3 Representations of $SO(3)$

In this section, we will calculate the irreducible representation of $SO(3)$.

(ρ, V) is a finite dimension representation of $SO(3)$, V is a subspace of all functions on $\mathbb{R}^3$

$$\rho(g)(f)(x_1,x_2,x_3) = f(g^{-1}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}), \quad \forall g \in SO(3), f \in V$$

The Lie algebra representation ρ' defined as

$$\rho'(g)(f)(x_1,x_2,x_3) = \frac{d}{dt}\rho(e^{tg})f(x_1,x_2,x_3)|_{t=0} = \frac{d}{dt}f(e^{-tg}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix})|_{t=0}$$

Let

$$l_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad l_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad l_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$l_1, l_2, l_3 \in \mathfrak{so}(3)$, and the commutation relations between them are

$$[l_1,l_2] = l_3, \quad [l_2,l_3] = l_1, \quad [l_3,l_1] = l_2 \tag{3.1}$$

Then calculate $L_k = i\rho'(l_k)$

$$\begin{aligned}\rho'(l_1)f(x_1,x_2,x_3) =& \frac{d}{dt}\rho(e^{tl_1})(f)(x_1,x_2,x_3)|_{t=0}\\ =& \frac{d}{dt}f(\begin{pmatrix}0 & 0 & 0\\ 0 & \cos t & \sin t\\ 0 & -\sin t & \cos t\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix})|_{t=0}\\ =& (\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2},\frac{\partial f}{\partial x_3})\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix})|_{t=0}\\ =& \left(x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3}\right)f\end{aligned} \tag{3.2}$$

similarly, we can get $\rho'(l_2)$ and $\rho'(l_3)$, so

$$\begin{aligned}L_1 &= ix_3\partial_{x_2} - ix_2\partial_{x_3}\\ L_2 &= ix_1\partial_{x_3} - ix_3\partial_{x_1}\\ L_3 &= ix_2\partial_{x_1} - ix_1\partial_{x_2}\end{aligned} \tag{3.3}$$

We define three operators of $SO(3)$ with L_i

$$L_+ = L_1 + iL_2, \quad L_- = L_1 - iL_2, \quad L_0 = 2L_3 \tag{3.4}$$

More specificly, these two operators in are form

$$\begin{aligned}L_+ &= ix_3\left(\partial_{x_2} + \partial_{x_1}\right) - \left(+x_1 + x_2 i\right)\partial_{x_3}\\ L_- &= ix_3\left(\partial_{x_2} - \partial_{x_1}\right) - \left(-x_1 + x_2 i\right)\partial_{x_3}\\ L_0 &= 2i(x_2\partial_{x_1} - x_1\partial_{x_2})\end{aligned} \tag{3.5}$$

their commutation relations are

$$[L_0, L_+] = 2L_+, \quad [L_0, L_-] = -2L_-, \quad [L_-, L_+] = L_0 \tag{3.6}$$

so they are $\mathfrak{sl}_2$-triple, and L_+ is raisng operator, L_- is lowering operator.

Notice that in spherical coordinate (r,θ,ϕ), $\forall g \in SO(3)$, g keeps r invariant, so we can just foucs on functions on unit sphere, notes as $f(\theta,\phi)$, $\theta \in [0,2\pi)$, $\phi \in [0,\pi)$.[6]

As we all know, the transformation from spherical coordinate (r,θ,ϕ) to cartesian coordinate (x_1,x_2,x_3) is

$$\begin{aligned}x_1 &= r\sin\theta\cos\phi\\ x_2 &= r\sin\theta\sin\phi\\ x_3 &= r\cos\phi\end{aligned} \tag{3.7}$$

we find

$$\begin{pmatrix}\partial_r\\ \partial_\theta\\ \partial_\phi\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta\\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0\end{pmatrix}\begin{pmatrix}\partial_{x_1}\\ \partial_{x_2}\\ \partial_{x_3}\end{pmatrix} \tag{3.8}$$

then

$$\begin{pmatrix}\partial_r\\ \frac{1}{r}\partial_\theta\\ \frac{1}{r\sin\theta}\partial_\phi\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{pmatrix}\begin{pmatrix}\partial_{x_1}\\ \partial_{x_2}\\ \partial_{x_3}\end{pmatrix} \tag{3.9}$$

Notes the matrix as A, it is easy to check that A is orthogonal, so $A^{-1} = A^\top$, and then

$$\begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \frac{1}{r}\sin\theta\sin\phi & \frac{1}{r}\cos\theta\sin\phi & \frac{1}{r}\cos\phi \\ \frac{1}{r\sin\theta}\cos\theta & -\frac{1}{r\sin\theta}\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta \\ \partial_\phi \end{pmatrix} \tag{3.10}$$

In unit sphere $r = 1$, then we can calculate the L_+, L_-, L_3 in spherical coordinate

$$\begin{aligned} L_0 &= 2i(x_2\partial_{x_1} - ix_1\partial_{x_2}) = -2i\partial_\phi \\ L_+ &= ix_3\left(\partial_{x_2} + \partial_{x_1}\right) - \left(-x_1 + x_2 i\right)\partial_{x_3} = e^{i\phi}\left(\partial_\theta + i\cos\theta\partial_\phi\right) \\ L_- &= ix_3\left(\partial_{x_2} - \partial_{x_1}\right) - \left(-x_1 - x_2 i\right)\partial_{x_3} = e^{i\phi}\left(\partial_\theta + i\cos\theta\partial_\phi\right) \end{aligned} \tag{3.11}$$

In last section, we know the classfication of all irreducible representations of $SO(3)$ and notes them as (ρ_{2n}, V^{2n}). By explicitly constructing a function Y_n^n, the highest weight vector of weight $2n$, We can construct irreducible Lie algebra representations ρ'_{2n}, and then ρ_{2n} is achieved naturally.[6]

By the defination of the highest weight vector, we have

$$\begin{aligned} L_0 Y_n^n &= -2i\partial_\phi Y_n^n = 2nY_n^n \\ L_+ Y_n^n &= e^{i\phi}\left(\partial_\theta + i\cos\theta\partial_\phi\right) Y_n^n = 0 \end{aligned} \tag{3.12}$$

from the first differential equation of (3.12) we get

$$Y_n^n(\theta, \phi) = e^{in\phi}G(\theta) \tag{3.13}$$

put this into the second differential equation of (3.12) and get

$$\partial_\theta G_n^n(\theta) = n\cot\theta G_n^n(\theta) \tag{3.14}$$

so $G_n^n(\theta) = C'_n \sin^n\theta$, and

$$Y_n^n(\theta, \phi) = C_n e^{in\phi}\sin^n\theta \tag{3.15}$$

where C_n, C'_n are constant.

Let

$$Y_k^n = (L_-)^{n-k} Y_n^n \tag{3.16}$$

where $k = -n, -n+1, \cdots, n-1, n$. This kind of function Y_k^n called ***Spherical harmonics***[5].

$\{Y_{-n}^n, \cdots, Y_n^n\}$ are basis of V^{2n+1}, which give the weight decomposition and determin the irreducible representation ρ'_{2n} and ρ_{2n}

4 Casimir Operator of $SO(3)$

Let

$$C = L_1^2 + L_2^2 + L_3^2 \tag{4.1}$$

where L_1, L_2, L_3 are defined in (3.3). C is the **Casimir operator** of $SO(3)$, and have many good propositions.

Proposition 4.1. *C is commutative with any Lie algebra representation of $\mathfrak{so}(3)$*

Proof. Any Lie algebra representation $\rho'(X)$ of $\mathfrak{so}(3)$, it can be writen in form $\rho'(X) = a_1L_1 + a_2L_2 + a_3L_3$. So it is sufficient to prove C is commutative with L_1, L_2 and L_3.

$$\begin{aligned}
&[L_1^2, L_1] = 0\\
&[L_2^2, L_1] = L_2^2L_1 - L_1L_2^2 = L_2([L_2, L_1] + L_1L_2) - ([L_1, L_2] + L_2L_1)L_2 = -iL_2L_3 - iL_3L_2\\
&[L_3^2, L_1] = L_3^2L_1 - L_1L_3^2 = L_3([L_3, L_1] + L_1L_3) - ([L_1, L_3] + L_3L_1)L_3 = iL_2L_3 + iL_3L_2
\end{aligned}$$

so

$$[C, L_1] = 0 + iL_2L_3 + iL_3L_2 - iL_2L_3 - iL_3L_2 = 0$$

Similarly, $[C, L_2] = 0$, $[C, L_3] = 0$. □

Proposition (4.1) shows that there must be a basis of V, in which Casimir operator C can be writen as diagonal matrix. So Casimir operator can help us to find decomposition of representation (ρ, V).

Proposition 4.2. *$C = -x^2\Delta + L^2 + L$, where $x = (x_1, x_2, X_3)$, $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$ is Laplace operator, $L = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}$ is Euler operator.*

Proof.

$$\begin{aligned}
L_1^2 &= ix_3\partial_{x_2}L_2 - ix_2\partial_{x_3}L_2\\
&= -x_2^2\partial_{x_3^2}^2 + 2x_2x_3\partial_{x_2x_3}^2 - x_3^2\partial_{x_2^2}^2 + x_2\partial_{x_2} + x_3\partial_{x_3}\\
L_2^2 &= -x_1^2\partial_{x_3^2}^2 + 2x_1x_3\partial_{x_1x_3}^2 - x_3^2\partial_{x_1^2}^2 + x_1\partial_{x_1} + x_3\partial_{x_3}\\
L_3^2 &= -x_1^2\partial_{x_2^2}^2 + 2x_1x_2\partial_{x_1x_2}^2 - x_2^2\partial_{x_1^2}^2 + x_1\partial_{x_1} + x_2\partial_{x_2}
\end{aligned}$$

and

$$\begin{aligned}
\partial_{x_1}(L) &= \partial_{x_1}(x_1\partial_{x_1}) + \partial_{x_1}(x_2\partial_{x_2}) + \partial_{x_1}(x_3\partial_{x_3})\\
&= \partial_{x_1} + x_1\partial_{x_1^2}^2 + x_2\partial_{x_1x_2}^2 + x_3\partial_{x_1x_3}^2\\
\partial_{x_2}(L) &= \partial_{x_2} + x_2\partial_{x_2^2}^2 + x_1\partial_{x_1x_2}^2 + x_3\partial_{x_2x_3}^2\\
\partial_{x_3}(L) &= \partial_{x_3} + x_3\partial_{x_3^2}^2 + x_1\partial_{x_1x_3}^2 + x_2\partial_{x_2x_3}^2\\
L^2 &= x_1\partial_{x_1}(L) + x_2\partial_{x_1}(L) + x_3\partial_{x_1}(L)\\
&= x_1^2\partial_{x_1^2}^2 + x_2^2\partial_{x_2^2}^2 + x_3^2\partial_{x_3^2}^2 + 2x_1x_2\partial_{x_1x_2}^2 + 2x_1x_3\partial_{x_1x_3}^2 + 2x_3x_2\partial_{x_3x_2}^2 + L
\end{aligned}$$

so

$$\begin{aligned}
C =& L_1^2 + L_2^2 + L_3^2\\
=& -\left((x_1^2 + x_2^2)\partial_{x_3^2}^2 + (x_1^2 + x_3^2)\partial_{x_2^2}^2 + (x_3^2 + x_2^2)\partial_{x_1^2}^2\right)\\
&+ 2x_1x_2\partial_{x_1x_2}^2 + 2x_1x_3\partial_{x_1x_3}^2 + 2x_3x_2\partial_{x_3x_2}^2\\
&+ 2x_1\partial_{x_1} + 2x_2\partial_{x_2} + 2x_3\partial_{x_3}\\
=& -x^2\Delta + x_1^2\partial_{x_1^2}^2 + x_2^2\partial_{x_2^2}^2 + x_3^2\partial_{x_3^2}^2 + 2x_1x_2\partial_{x_1x_2}^2 + 2x_1x_3\partial_{x_1x_3}^2 + 2x_3x_2\partial_{x_3x_2}^2 + 2L\\
=& -x^2\Delta + L^2 + L
\end{aligned}$$

□

Proposition (4.2) indicates that there are strong relations between Casimir operator and Laplace operator.

Proposition 4.3. *f_n is the highest weight vector with weight $2n$, then*

$$Cf_n = n(n+1)f_n$$

Proof.

$$L_-L_+ = (L_1 - iL_2)(L_1 + iL_2) = L_1^2 + L_2^2 + i[L_1, L_2] = L_1^2 + L_2^2 - L_3$$

so

$$C = L_1^2 + L_2^2 + L_3^2 = L_-L_+ + L_3^2 + L_3$$

f_n is highest weight vector, $L_-L_+f_n = L_-0 = 0$ and $L_3 f_n = \frac{1}{2}L_0 f_n = nf_n$, so

$$Cf_n = 0 + n^2 f_n + nf_n = n(n+1)f_n$$

□

Proposition (4.3) show that we can also solve differential equations

$$\begin{aligned} CY_n^n &= n(n+1)Y_n^n \\ L_+Y_n^n &= 0 \end{aligned} \tag{4.2}$$

to get the highest vector Y_n^n.

Of course, Casimir operator not only exists in $SO(3)$, but in other Lie groups[4]. It is a useful operator in studying the representation of Lie group.

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