

1. Let f and g be two functions taking numbers to numbers. We define \mathbf{m}_{fg} to be the function so that

$$\mathbf{m}_{fg}(x) = \text{the larger of } f(x) \text{ and } g(x).$$

For example, if $f(x) = x$ and $g(x) = -x$ then \mathbf{m}_{fg} is the absolute value function. Define a function \mathbf{max} which takes a *pair* of functions, f and g , as an argument and returns the function \mathbf{m}_{fg} as its value. (So the return value is a *function*: the function which, at every point x , returns the larger value of $f(x)$ and $g(x)$.)

```
(define (max f g)
  (define (int-max x y)
    (if (> x y) x y))
  (lambda (x) (int-max (f x) (g x))))
```

2. Define a SCHEME function, \mathbf{zip} , which takes as arguments two lists $(a_1 \dots a_n)$ and $(b_1 \dots b_n)$ of the same length, and produces the list of pairs $((a_1.b_1) \dots (a_n.b_n))$.

```
(define (zip lista listb)
  (if (or (null? lista)
           (null? listb))
      '()
      (cons (cons (car lista)
                  (car listb))
            (zip (cdr lista)
                  (cdr listb)))))
```

3. Define a SCHEME function, \mathbf{unzip} , which takes a list of pairs $((a_1.b_1) \dots (a_n.b_n))$ and returns a *pair* consisting of the two lists $(a_1 \dots a_n)$ and $(b_1 \dots b_n)$.

(Note that these functions are not exactly inverses of each other, since \mathbf{zip} takes its lists as a two arguments, while \mathbf{unzip} produces a pair of lists.)

```
(define (unzip plist)
  (if (null? plist)
      '(() ())
      (let* ((rest (unzip (cdr plist)))
             (first (car rest))
             (second (cdr rest)))
          (cons (cons (caar plist) first)
                (cons (cdar plist) second)))))
```

4. Consider the problem of *making change*. Given a list of “denominations” $(d_1 d_2 \dots d_k)$ and a positive integer n , the problem is to determine the number of ways that n can be written as a sum of the d_i . For example, if we consider the denominations used for US coins, $(1 \ 5 \ 10 \ 25)$, the number 11 can be written in 4 ways:

$$10 + 1, \ 5 + 5 + 1, \ 5 + \underbrace{1 + \dots + 1}_6, \ \text{ and } \underbrace{1 + \dots + 1}_{11}.$$

Note that we do not consider $10 + 1$ and $1 + 10$ as “different” ways to write 11: all that matters is the number of occurrences of each denomination, not their order.

Write a SCHEME function (`change k ℓ`) that returns the number of ways to express k as a sum of the denominations appearing in the list ℓ .

(Hint: There is a nice recursive decomposition of this problem: Let $C(n, \ell)$ denote the number of ways to express n as a sum of elements from the list $\ell = (\ell_1 \ell_2 \dots \ell_k)$. We can mentally divide the different ways to do this into two types: those that do not use the denomination ℓ_1 and those that do. There are $C(n, (\ell_2 \dots, \ell_k))$ ways that do not use the denomination ℓ_1 . There are $C(n - \ell_1, \ell)$ ways that involve at least one ℓ_1 . It follows that

$$C(n, \ell) = C(n, (\ell_2 \dots, \ell_k)) + C(n - \ell_1, \ell),$$

which you can use in your definition. (To do this, make sure you understand what happens, e.g., when n is smaller than all of the elements in ℓ .)

```
(define (change n denominations)
  (cond ((< n 0) 0)
        ((= n 0) 1)
        ((null? denominations) 0)
        (else (+ (change n (cdr denominations))
                  (change (- n (car denominations))
                          denominations))))))
```

5. **(The Cantor pairing function.)** A *pairing function* p is a function that places the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ into one-to-one correspondence with the set of all *pairs* of natural numbers (usually denoted $\mathbb{N} \times \mathbb{N}$). It is somewhat surprising that such a function should exist at all: it shows that the set of natural numbers has the same “size” as the set of all *pairs* of natural numbers. To be more precise, a pairing function p takes two natural numbers x and y as arguments and returns a single number z with the property that the original pair can always be determined exactly from the value z . (In particular, the function maps no more than one pair to any particular natural number.)

One famous pairing function is the following:

$$p(x, y) = \frac{1}{2}(x + y)(x + y + 1) + y.$$

- (a) Write a SCHEME function `encode` that computes the pairing function above. (It should take a pair of numbers as arguments and return a single number.)
- (b) As mentioned above, this function has the property that if $(x, y) \neq (x', y')$ then $p(x, y) \neq p(x', y')$: it follows that, in principle, the pair (x, y) can be reconstructed from the single value $z = p(x, y)$. In fact, the values x and y can be reconstructed from $z = p(x, y)$ by first computing the quantities

$$w = \left\lfloor \frac{\sqrt{8z + 1} - 1}{2} \right\rfloor, \quad \text{and}$$

$$t = \frac{w^2 + w}{2}.$$

Then $y = z - t$ and $x = w - y$.

Write a SCHEME function `decode` that takes as an argument an integer z and produces the pair (x, y) for which $p(x, y) = z$. You’ll need the `floor` function: `(floor x)` returns the largest integer less than or equal to x (that is, it rounds x down to the nearest integer).

Hint: You may wish to use the `let*` form for this problem. `let*` has the form

```
(let* ((x1 <expr1>)
      (x2 <expr2>)
      ...
      (xk <exprk>))
  <let-expr>)
```

The form is a simple method for writing “nested lets.” It is evaluated, informally, as follows. Starting with the external environment, `expr1` is evaluated and the variable `x1` is immediately bound to the resulting value. Following this, `<expr2>` is evaluated in the resulting environment and the variable `x2` is bound to the value. This continues for the remaining variable/expression pairs. Finally, `<let-expr>` is evaluated in the resulting environment and its value is returned. Note, for example, that `x1` may appear in `<expr2>`. (Recall that in a regular `let` expression, the `<expri>` are all evaluated in the external environment.)

```
(define (encode p)
  (let ((sum (+ (car p) (cdr p))))
    (+ (* sum
           (+ sum 1)
           (/ 1 2))
       (cdr p)))

(define (decode z)
  (let* ((w (floor (/ (- (sqrt (* 8 z)) 1) 2)))
         (t (/ (+ (* w w) w) 2))
         (y (- z t))
         (x (- w y)))
    (cons x y)))
```

6. Write a SCHEME function `positives` which takes a list—call it ℓ —as an argument and returns a list consisting of all elements of ℓ that are positive. In particular, once you have `positives` defined correctly, you should be able to reproduce the following behavior.

```
> (positives (list -2 -1 0 1 2))
'(1 2)
> (positives (list 2 1 0 -2 -1))
'(2 1)
> (positives '(3 1 -1 1 -1))
'(3 1 1)
```

To keep things simple, it’s fine if your function just removes from the list all numbers that are zero or less (leaving duplicates in the remaining list, as shown above).

```
(define (positives numbers)
  (cond ((null? numbers) '())
        ((< (car numbers) 0) (positives (cdr numbers)))
        ((else (cons (car positives) (positives (cdr numbers)))))))
```

7. Write a SCHEME function that removes all duplicates from a list. (Hint: you might start by defining a function which removes all duplicates of a particular given value v from a list; then what?)

```
(define (remove-duplicates numbers)
  (define (remove v elements)
    (cond ((null? elements) '())
          ((= v (car elements)) (remove v (cdr elements)))
          (else (cons (car elements) (remove v (cdr elements)))))))
  (if (null? numbers) '()
      (cons (car numbers)
            (remove (car numbers)
                  (remove-duplicates (cdr numbers))))))
```