Exercises

Exercise 1

Taking into account the following words over $\{0, 1\}$:

$$x = 00011$$
$$y = 011000$$
$$z = 01010$$

compute the following operations:

(a) $|x|,\ |y|_1$

Answer:

$|x| = 5$
$|y|_1 = 2$

(b) x^R, y^R, z^R

Answer:

$x^R = 11000$
$y^R = 000110$
$z^R = z$

(c) $xy,\ yz,\ z^2$

Answer:

$xy = 00011011000$
$yz = 011000010100$
$z^2 = 0101001010$

(d) $Pref(x),\ Suf(y),\ Seg(z),\ Pref(Pref(x)),\ Pref(Suf(z))$

Answer:

$Pref(x) = \{\lambda, 0, 00, 000, 0001, 00011\}$
$Suf(y) = \{\lambda, 0, 00, 000, 1000, 11000, 011000\}$
$Seg(z) = \{\lambda, 0, 1, 01, 10, 010, 101, 0101, 1010, 01010\}$
$Pref(Pref(x)) = Pref(x)$
$Pref(Suf(z)) = Seg(z)$

Exercise 2

Taking into account the following languages over $\{0, 1\}$:

$$L_1 = \{0, 01, 001\}$$
$$L_2 = \{\lambda, 01, 0011\}$$
$$L_3 = \{0x \ : \ x \in \{0, 1\}^*\}$$
$$L_4 = \{x0 \ : \ x \in \{0, 1\}^*\}$$
$$L_5 = \{x \in \{0, 1\}^* \ : \ |x|_0 = |x|_1\}$$

(a) Enumerate the first 10 words of L_3 in canonical order

> **Answer:**
>
> $\{0, 00, 01, 000, 001, 010, 011, 0000, 0001, 0010\}$

(b) Enumerate the first 10 words of L_4 in canonical order

> **Answer:**
>
> $\{0, 00, 10, 000, 010, 100, 110, 0000, 0010, 0100\}$

(c) Enumerate the first 10 words of L_5 in canonical order

> **Answer:**
>
> $\{\lambda, 01, 10, 0011, 0101, 0110, 1001, 1010, 1100, 000111\}$

Exercise 3

Taking into account the languages described in Exercise 2, give a description of the languages output by the following operations:

(a) $L_1 \cup L_2, L_1 \cup L_3, L_2 \cup L_3, L_3 \cup L_4$

> **Answer:**
>
> $L_1 \cup L_2 = \{\lambda, 0, 01, 001, 0011\}$
> $L_1 \cup L_3 = L_3$
> $L_2 \cup L_3 = \{x \in \{0, 1\}^* : 1 \notin Pref(x)\} = L_3 \cup \{\lambda\}$
> $L_3 \cup L_4 = \{x \in \{0, 1\}^* : 0 \in Pref(x) \cup Suf(x)\}$

(b) $L_1 \cap L_2, L_1 \cap L_3, L_1 \cap L_4, L_2 \cap L_4, L_3 \cap L_4$

> **Answer:**
>
> $L_1 \cap L_2 = \{01\}$

$L_1 \cap L_3 = L_1$
$L_1 \cap L_4 = \{0\}$
$L_2 \cap L_4 = \emptyset$
$L_3 \cap L_4 = \{x \in \{0,1\}^* : 0 \in Pref(x) \cap Suf(x)\} = \{0\} \cup \{0x0 \ : \ x \in \{0,1\}^*\}$

(c) $\overline{L_3},\ \overline{L_5}$

Answer:

$\overline{L_3} = \{x \in \{0,1\}^* \ : \ 0 \notin Pref(x)\} = \{\lambda\} \cup \{1x \ : \ x \in \{0,1\}^*\}$
$\overline{L_5} = \{x \in \{0,1\}^* \ : \ |x|_0 \neq |x|_1\}$

(d) $L_1 - L_2, L_2 - L_3, L_2 - L_4, L_3 - L_4$

Answer:

$L_1 - L_2 = \{0, 001\}$
$L_2 - L_3 = \{\lambda\}$
$L_2 - L_4 = L_2$
$L_3 - L_4 = \{0x1 : x \in \{0,1\}^*\}$

(e) $L_1 \triangle L_2, L_1 \triangle L_3, L_3 \triangle L_4$

Answer:

$L_1 \triangle L_2 = \{\lambda, 0, 001, 0011\}$
$L_1 \triangle L_3 = L_3 - L_1 = \{0x \ : \ x \in \{0,1\}^*\} - \{0, 01, 001\}$
$L_3 \triangle L_4 = \{0x1, 1x0 \ : \ x \in \{0,1\}^*\} = \{axb \ : \ x \in \{0,1\}^* \wedge a, b \in \{0,1\}, \ \ a \neq b\}$

(f) $L_1L_2, L_4L_3, L_2L_3, L_3L_4, L_1^2, L_5^2, L_2^3, L_3^5$

Answer:

$L_1L_2 = \{0, 01, 001, 0101, 00011, 00101, 010011, 0010011\}$
$L_4L_3 = \{x \in \{0,1\}^* : 00 \in Seg(x)\}$
$L_2L_3 = L_3$
$L_3L_4 = \{0x0 : x \in \{0,1\}^*\}$
$L_1^2 = \{00, 001, 010, 0001, 0010, 0101, 00101, 01001, 001001\}$
$L_5^2 = L_5$

$$L_2^3 = \left\{ \begin{array}{l} \lambda, 01, 0011, 0101, 001101, 010011, 010101, 00110011, \\ 00110101, 01001101, 01010011, 0011001101, 0011010011, \\ 0100110011, 001100110011 \end{array} \right\}$$

$L_3^5 = \{x \in \{0,1\}^* : 0 \in Pref(x) \wedge |x|_0 \geq 5\}$

(g) $L_1^*, L_4^*, L_1^+, L_3^+, L_5^*$

Answer:

$L_1^* = \{x \in \{0,1\}^* : 1 \notin Pref(x) \wedge 11 \notin Seg(x)\}$ (*)
$L_4^* = L_4 \cup \{\lambda\}$
$L_1^+ = \{x \in \{0,1\}^* : 0 \in Pref(x) \wedge 11 \notin Seg(x)\}$ (*)
$L_3^+ = L_3$
$L_5^* = L_5$

(h) L_2^R, L_3^R, L_5^R

Answer:

$L_2^R = \{\lambda, 10, 1100\}$
$L_3^R = L_4$
$L_5^R = L_5$

(i) $Pref(L_1), Pref(L_4), Pref(L_3), Seg(L_1), Seg(L_4), Suf(L_2)$

Answer:

$Pref(L_1) = \{\lambda, 0, 00, 01, 001\}$
$Pref(L_4) = \{0,1\}^*$
$Pref(L_3) = L_3 \cup \{\lambda\}$
$Seg(L_1) = \{\lambda, 0, 1, 00, 01, 001\}$
$Seg(L_4) = \{0,1\}^*$
$Suf(L_2) = \{\lambda, 1, 01, 11, 011, 0011\}$

(j) $0^{-1}L_1, 0^{-1}L_2, 0^{-1}L_3, 0^{-1}L_4, 1^{-1}L_1, 1^{-1}L_3, 1^{-1}L_4, (01)^{-1}L_1$

Answer:

$0^{-1}L_1 = \{\lambda, 1, 01\}$
$0^{-1}L_2 = \{1, 011\}$
$0^{-1}L_3 = \{0,1\}^*$
$0^{-1}L_4 = L_4 \cup \{\lambda\}$ (*)
$1^{-1}L_1 = \emptyset$
$1^{-1}L_3 = \emptyset$

$1^{-1}L_4 = L_4$ (*)
$(01)^{-1}L_1 = \{\lambda\}$

(k) $(01)^{-1}L_3, (01)^{-1}L_4$

Note that the languages L_3 and L_4 can be expressed as:

$$L_3 = \{0\}\{0,1\}^*$$
$$L_4 = \{0,1\}^*\{0\}$$

Hint: Consider the properties of the right quotient

Answer:

$$\begin{aligned}(01)^{-1}L_3 &= 1^{-1}(0^{-1}L_3) = 1^{-1}(0^{-1}\{0\}\{0,1\}^*) = 1^{-1}((0^{-1}\{0\})\{0,1\}^*) = \\ &= 1^{-1}\{\lambda\}\{0,1\}^* = 1^{-1}\{0,1\}^* = (1^{-1}\{0,1\})\{0,1\}^* = \{\lambda\}\{0,1\}^* \\ &= \{0,1\}^*\end{aligned}$$

$$\begin{aligned}(01)^{-1}L_4 &= 1^{-1}(0^{-1}L_4) = 1^{-1}(0^{-1}\{0,1\}^*\{0\}) = 1^{-1}((0^{-1}\{0,1\}^*)\{0\} \cup (0^{-1}\{0\})) = \\ &= 1^{-1}((0^{-1}\{0,1\})\{0,1\}^*\{0\} \cup \{\lambda\}) = 1^{-1}(\{\lambda\}\{0,1\}^*\{0\} \cup \{\lambda\}) = \\ &= (1^{-1}\{0,1\}^*\{0\}) \cup (1^{-1}\{\lambda\}) = (1^{-1}\{0,1\})\{0,1\}^*\{0\} \cup \emptyset = \\ &= \{\lambda\}\{0,1\}^*\{0\} = L_4\end{aligned}$$

Exercise 4

Consider the languages described in Exercise 2 and the following homomorphism:

$$h : \{0,1\} \to \{a,b,c\}^* \quad \begin{cases} h(0) = a \\ h(1) = bc \end{cases} \qquad g : \{a,b,c\} \to \{0,1\}^* \quad \begin{cases} g(a) = 01 \\ g(b) = 10 \\ g(c) = \lambda \end{cases} \qquad f : \{0,1\} \to \{0,1\}^* \quad \begin{cases} f(0) = 0 \\ f(1) = 011 \end{cases}$$

Give a description of the languages output by the following operations:

(a) $h(L_1), h(L_2), h(L_3), h(L_4)$

Answer:

$h(L_1) = \{a, abc, aabc\}$
$h(L_2) = \{\lambda, abc, aabcbc\}$
$h(L_3) = \{ax : x \in \{a, bc\}^*\}$
$h(L_4) = \{xa : x \in \{a, bc\}^*\}$

(b) $g^{-1}(L_1), g^{-1}(L_2), g^{-1}(L_3), g^{-1}(L_4)$

Answer:

$g^{-1}(L_1) = \{c^i a c^j, i, j \geq 0\}$
$g^{-1}(L_2) = \{c^i a c^j, i, j \geq 0\} \cup \{c\}^*$
$g^{-1}(L_3) = \{c^i a x \ : \ x \in \{a, b, c\}^*, i \geq 0\}$
$g^{-1}(L_4) = \{x b c^i \ : \ x \in \{a, b, c\}^*, i \geq 0\}$

(c) $f(L_1), f(L_2), f(L_3), f^{-1}(L_1), f^{-1}(L_2), f^{-1}(L_3), f^{-1}(L_4)$

Answer:

$f(L_1) = \{0, 0011, 00011\}$
$f(L_2) = \{\lambda, 0011, 00011011\}$
$f(L_3) = \{0x : x \in \{0, 011\}^*\}$
$f^{-1}(L_1) = \{0\}$
$f^{-1}(L_2) = \{\lambda, 01\}$
$f^{-1}(L_3) = \{0, 1\}^+$ (*)
$f^{-1}(L_4) = L_4$ (*)