Exercises

Exercise 1

Let consider the following languages over $\{0,1\}$:

$$
\begin{aligned}
L_1 &= \{0x \ : \ x \in \{0,1\}^*\} \\
L_2 &= \{x1 \ : \ x \in \{0,1\}^*\} \\
L_3 &= \{0x1 \ : \ x \in \{0,1\}^*\} \\
L_4 &= \{x \in \{0,1\}^* \ : \ |x|_0 = 2\} \\
L_5 &= \{x \in \{0,1\}^* \ : \ |x|_0 \ mod\ 2 = 0\} \\
L_6 &= \{x \in \{0,1\}^* \ : \ 001 \in Suf(x)\} \\
L_7 &= \{x \in \{0,1\}^* \ : \ 001 \in Seg(x)\}
\end{aligned}
$$

(a) Taking into account the equivalence relation of L_1, decide whether the following string pairs are equivalent or not: $(001, 10)$, $(000, 0)$, $(11101001, 10)$, $(\lambda, 001)$, $(\lambda, 1001)$

Answer:

$001 \not\equiv_{R_{L_1}} 10$
$000 \equiv_{R_{L_1}} 0$
$11101001 \equiv_{R_{L_1}} 10$
$\lambda \not\equiv_{R_{L_1}} 001$
$\lambda \not\equiv_{R_{L_1}} 1001$

(b) Taking into account the equivalence relation of L_2, describe each class of the equivalence relation R_{L_2} and which is the first string in canonical order of each class.

Answer:

- strings that do not end with 1

  first string: λ

- strings that end with 1

  first string: 1

(c) Taking into account the equivalence relation of L_2, decide whether the following string pairs are equivalent or not: $(001, 10)$, $(000, 0)$, $(11101001, 10)$, $(\lambda, 001)$, $(\lambda, 10010)$

Answer:

$001 \not\equiv_{R_{L_2}} 10$
$000 \equiv_{R_{L_2}} 0$
$11101001 \not\equiv_{R_{L_2}} 10$

$\lambda \not\equiv_{R_{L_2}} 001$
$\lambda \equiv_{R_{L_2}} 10010$

(d) Taking into account the equivalence relation of L_3, provide examples of three pairs of equivalent strings, and three pairs of non-equivalent strings.

Answer:

$0 \equiv_{R_{L_3}} 0000$
$010 \equiv_{R_{L_3}} 0$
$11 \equiv_{R_{L_3}} 10100$

$\lambda \not\equiv_{R_{L_3}} 100$
$010 \not\equiv_{R_{L_3}} 0011$
$11 \not\equiv_{R_{L_3}} 0001$

(e) Taking into account the equivalence relation of L_3, provide examples of three pairs of equivalent strings, and three pairs of non-equivalent strings.

Answer:

$111 \equiv_{R_{L_4}} \lambda$
$010 \equiv_{R_{L_4}} 1100$
$11011000 \equiv_{R_{L_4}} 00011$

$\lambda \not\equiv_{R_{L_4}} 100$
$01010 \not\equiv_{R_{L_4}} 0011$
$11 \not\equiv_{R_{L_4}} 0001$

(f) Taking into account the equivalence relation of L_5, describe each class of the equivalence relation R_{L_5} and which is the first string in canonical order of each class.

Answer:

- strings x such that $|x|_0 \ mod\ 2 = 0$
  first string: λ
- strings x such that $|x|_0 \ mod\ 2 = 1$:
  first string: 0

(g) Taking into account the equivalence relation of L_6, describe each class of the equivalence relation R_{L_6} and which is the first string in canonical order of each class.

Answer:

- strings of the form $x0$ where $x \in \{\{0,1\}^*1\} \cup \{\lambda\}$
  first string: 0
- strings of the form $x00$ where $x \in \{0,1\}^*$
  first string: 00
- strings of the form $x001$ where $x \in \{0,1\}^*$
  first string: 001
- strings x such that $0, 00, 001 \notin Suf(x)$
  first string: λ

(h) Taking into account the equivalence relation of L_6, provide examples of three pairs of equivalent strings, and three pairs of non-equivalent strings.

Answer:

$001 \equiv_{R_{L_6}} 1001$
$010 \equiv_{R_{L_6}} 0$
$\lambda \equiv_{R_{L_6}} 01$

$001 \not\equiv_{R_{L_6}} 100$
$1010 \not\equiv_{R_{L_6}} 00$
$\lambda \not\equiv_{R_{L_6}} 000$

(i) Provide three examples of strings in each of the following languages: $(11)^{-1}L_7$, $(001)^{-1}L_7$, $(1001)^{-1}L_7$, $(1100)^{-1}L_7$

Answer:

$(11)^{-1}L_7 = L_7$, therefore, $001, 1001, 01001 \in (11)^{-1}L_7$
$(001)^{-1}L_7 = \Sigma^*$, thus, $\lambda, 01, 00, 01001 \in (001)^{-1}L_7$
$(1001)^{-1}L_7 = \Sigma^*$, therefore, $\lambda, 01, 00, 01001 \in (001)^{-1}L_7$
$(1100)^{-1}L_7 = \{x \in \{0,1\}^* : |x|_1 \geq 1\}$, thus, $10001, 001, 1111 \in (1100)^{-1}L_7$

Exercise 2

Decide whether the following languages are regular or not.

(a) $L = \{x \in \{0,1\}^* \ : \ x = x^r\}$

Answer:

Let the infinite family of words $\{0^i1 \; : \; i \geq 0\}$, it can be seen that, given whichever pair of words of the family 0^i1 and 0^j1 where $i \neq j$, there exists a word 0^i such that:

$$0^i10^i \in L$$
$$0^j10^i \notin L$$

Therefore, R_L is a equivalence relation of infinite order and L is not regular.

(b) $L = \{x \in \{0,1\}^* \; : \; |x|_0 = |x|_1\}$

Answer:

Let consider the infinite family of words $\{0^i \; : \; i \geq 0\}$. Given any word 0^i of the family, it can be seen that, when concatenated to another word of the form 1^i, a word of the language is obtained. Taking into account that, for each i, j such that $i \neq j$, it is hold that:

$$0^i1^i \in L$$
$$0^j1^i \notin L$$

therefore it is possible to conclude that R_L is of infinite index and L is not regular.

(c) $L = \{x \in \{0,1,2\}^* \; : \; 2|x|_0 = |x|\}$

Answer:

Taking into account words of the form $\{1^i \; : \; i \geq 0\}$ it is possible to obtain an infinite family.
Given any two words 1^i and 1^j of the family, where $i \neq j$, it is possible to find two words 0^i and 0^j such that the following is fulfilled:

$$1^i0^i \in L$$
$$1^j0^i \notin L$$

which implies that R_L is of infinite order and L is not regular.

(d) $L = \{x \in \{0,1,2\}^* \; : \; |x|_0 = |x|_1 \times |x|_2\}$

Answer:

Let us consider the infinite family of words of the form $\{0^i1 \; : \; i \geq 0\}$. It is possible to see that, given any two words 0^i1 and 0^j1 of the family, where $i \neq j$:

$$0^i12^i \in L$$
$$0^j12^i \notin L$$

thus, the equivalence relation R_L is of infinite index and L is not regular.

(e) $L = \{x \in \{0,1,2\}^* \ : \ |x|_2 = |x|_0 + |x|_1\}$

Answer:

Let us consider the infinite family of words of the form $\{2^{2i} \ : \ i \geq 0\}$. Please note that it is possible to obtain words in L when each word in the family is concatenated with another word of the form 0^i1^i.
Note that, when two distinct words 2^{2i} and 2^{2j} are considered, the following holds:

$$2^{2i}0^i1^i \in L$$
$$2^{2j}0^i1^i \notin L$$

thus, it is possible to conclude that R_L is of infinite indes and L is not regular.

(f) $L = \{xx \ : \ x \in \{0,1\}^*\}$

Answer:

Let us consider the infinite family of words $\{10^i \ : \ i \geq 0\}$. Given any two words of the family 10^i and 10^j, where $i \neq j$, it can be seen that, in order to obtain words of the language, there exists at least a word that is a valid suffix for one word but it is not a valid suffix for the other one:

$$10^i10^i \in L$$
$$10^j10^i \notin L$$

therefore, R_L is of infinite index and L is not regular.

(g) Let L be the language over $\{0,1\}^*$ that contains the words such that the longest segment of symbols 0 has the same length than the longest segments of symbols 1.

Answer:

Let the infinite family of words of the form $\{0^i \ : \ i \geq 0\}$. Note that, given any two distinct words 0^i and 0^j of the family, there exist another word 1^i such that:

$$0^i1^i \in L$$
$$0^j1^i \notin L$$

therefore R_L is of infinite index and L is not regular.

(h) $L = \{a^p b^q c^r d^s \ : \ p = r \vee q = s\}$

Answer:

Let us consider the set of words of the form $\{a^i b \ : \ i \geq 0\}$. for each word $a^i b$ in this infinite family of words, it is possible to obtain a word in the language by concatenating a word of the form c^i. Taking into account that, for every pair of distinct words $a^i b$ and $a^j b$ the following is fulfilled:

$$a^i b c^i \in L$$
$$a^j b c^i \notin L$$

it is possible to conclude that R_L is of finite index and L is not regular.

(i) Let L be the language over $\{0,1\}^*$ that contains the words such that the longest segment of 0 symbols has odd length.

Answer:

Let the infinite family of words $\{0^{2i}1 \ : \ i \geq 0\}$, it can be seen that, for each word of the family, it is possible to obtain a word that belongs to L by concatenating a suffix of the form 0^{2i+1}.
Taking into account that, for every i, j such that $i < j$, the following holds:

$$0^{2i}10^{2i+1} \in L$$
$$0^{2j}10^{2i+1} \notin L$$

it is possible to conclude that R_L is of finite index and L is not regular.

(j) Let L be the language of the words such that the number of 0 symbols in even and odd positions is the same.

Answer:

Let us consider the following infinite family of words $\{(01)^i \ : \ i \geq 0\}$. It is possible to obtain words of the language by concatenating any word $(01)^i$ of the family with another word of the form $(10)^i$ Taking into account two distinct integers i, j, the following holds:

$$(01)^i(10)^i \in L$$
$$(01)^j(10)^i \notin L$$

thus, it is possible to conclude that R_L is of finite index and L is not regular.