A general note for the projects: Please play with the parameters and try e.g. different basis functions for the approximation of the conditional expectations. Find a good way to illustrate your results to be able to present it to your peers. This includes an explanation of the theory, numerics and your code. The idea is that you talk with me on a regular basis on your assigned project and get some feedback and help.

1 Optimal Stopping

Project 1.1 (Optimal Harvest Time). *Let W^1, W^2 be two correlated Brownian motions and consider the Schwartz-2-factor model*

$$\begin{aligned} dS_t &= (r - \delta_t)S_t dt + \sigma S_t dW_t^1, \quad S_0 = s_0 > 0 \\ d\delta_t &= (\kappa(\alpha - \delta_t) - \lambda)dt + \nu dW_t^2, \quad \delta_0 = d_0 \in \mathbb{R} \\ d\langle W^1, W^2 \rangle &= \rho dt, \end{aligned}$$

where $r, \sigma, \kappa, \alpha, \lambda, \nu > 0$ and $\rho \in (-1, 1)$. S_t will be interpreted as the current price of 1kg salmon. Notice that $X_t = (S_t, \delta_t)$ is Markovian, but not S_t alone!

In this task, we will model a pond of salmons, which grow over time and need to be fed. The question we try to answer is, when is the optimal time to harvest the salmons in terms of maximizing the revenue of the harvest.

So let us fix some functions to describe the growth and feeding costs of salmons:

1. *The number of fish in our pond is unfortunately decreasing over time due to disease, we model this as*

$$n(t) = n_0 \exp(-mt),$$

with $n_0 = 10000$ initial fish population and a mortality rate of $m = 0.1$;

2. *The growth over time in terms of biomass is given by*

$$w(t) := w_\infty (a - be^{-ct})^3,$$

with $w_\infty = 6$, $a = 1.113$, $b = 1.097$ and $c = 1.43$;

3. *The total biomass of the salmon farm is therefore simply*

$$B(t) = n(t)w(t).$$

4. *The harvesting costs per kg of salmon is given by*

$$H(t) = H_0 B(t),$$

where $H_0 = 3$ SEK/kg;

5. *The instantaneous feeding costs are given by*

$$f(t) = F_0 n(t)(\partial_t w)(t),$$

and discounted cumulative feeding costs are given by

$$F(t) = \int_0^t e^{-rs} f(s) ds,$$

where $F_0 = 7$ *SEK/kg.*

Now, the problem we try to solve can be formulated as an optimal stopping problem like follows

$$V_0 = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-r\tau}(S_\tau B(\tau) - H(\tau)) - F(\tau)\right]$$

Implement the LSMC algorithm for this and determine the optimal harvesting value.

Use the following values as a baseline:

1. *time:* $T = 3$, $N = 701$ *for the simulation of the Schwartz-2-factor model;*
2. *simulations:* $M = 10000$*;*
3. *parameters:* $r = 0.03$, $\sigma = 0.23$, $\nu = 0.75$, $\kappa = 2.6$, $\alpha = 0.02$, $\lambda = 0.2$, $\rho = 0.9$, $\delta_0 = 0.57$, $S_0 = 35$*;*
4. *LSMC: Use polynomials up to and including degree* 2 *and* $n = 71$ *time steps for the iteration, so every* 10*-th step of the simulated trajectories. Notice, that our basis functions need to have two arguments for* S_t *and* δ_t*!*

Hint: You need to change the LSMC algorithm to take the feeding costs correctly into account, you need to subtract $f(t_i)\Delta t$ *from the continuation value, since you have to pay the feed in this interval if you decide not to harvest. Also the exercise value is just* $S_t B(t) - H(t)$*.*

2 Forward Backward Stochastic Differential Equations (FBSDEs)

Project 2.1. *Let*

$$f(t, y, z) = -(yr + z\lambda - (y - \frac{z}{\sigma})^-(R - r)),$$
$$\Phi(x) = (S_T - K)^+$$

where $\lambda = \frac{\mu - r}{\sigma}$ *and* $x^- = -\min(x, 0)$, $K, \mu, \sigma > 0$, $R \geq r > 0$.

Consider the following FBSDE

$$S_t = S_0 + \mu \int_0^t S_s ds + \sigma \int_0^t S_s dW_s, \quad S_0 = s_0 > 0,$$
$$Y_t = \Phi(S_T) + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.$$

Compute $\mathbb{E}[Y_0]$*. For this proceed as follows:*

1. *Look at the structure of the FBSDE, can you maybe simulate one of the processes before*

the other?

2. *For the BSDE we need to develope a numerical scheme and we will proceed similar as in the LSMC algorithm:*
   a) *At $t = T$ we know the payoff, so this is where we start the algorithm;*
   b) *Now, choose a homogeneous time-grid and consider the difference $Y_{t_{i+1}} - Y_{t_i}$*
   c) *Apply the conditional expectation with respect to $\mathcal{F}_{t_i}$ to both sides of the equation. Can you make assumptions, which simplifies the equation?*
   d) *You will notice that we now need to evaluate conditional expectations. Use the approximation we used for the LSMC algorithm, i.e., use the polynomial basis functions up to degree 2.*
   e) *We still need an approximation for Z for this consider the following:*

$$Y_{t_i} = Y_{t_{i+1}} + f(t_i, Y_{t_i}, Z_{t_i})\Delta t - Z_{t_i}\Delta W_{t_i}.$$

   Multiply both sides with ΔW_{t_i} and take the conditional expectation with respect to $\mathcal{F}_{t_i}$.

Use the following values as a baseline:

1. *time: $T = 0.5$, $N = 100$ for both simulation and BSDE solver;*
2. *simulations: $M = 10^6$;*
3. *parameters: $\mu = 0.06$, $r = 0.04$, $R = 0.06$, $K = 100$, $s_0 = 100$;*

3 Kalman Filtering

Project 3.1. *Study* Oksendal *(1998, pp. 81 ff. Chapter 6). Explain the problem and implement the Kalman filter.*

Warning: This is a difficult topic.

4 Optimal Control

Project 4.1. *Study* Oksendal *(1998, pp. 224 ff. Chapter 11). Explain the problem and implement the Linear-Quadratic Regulator.*

Warning: This is a difficult topic.

5 Suggest your own topic

If you want, you can suggest your own topic but you need approval from me, e.g., if you are interested there are a lot of opportunities for Deep Learning.