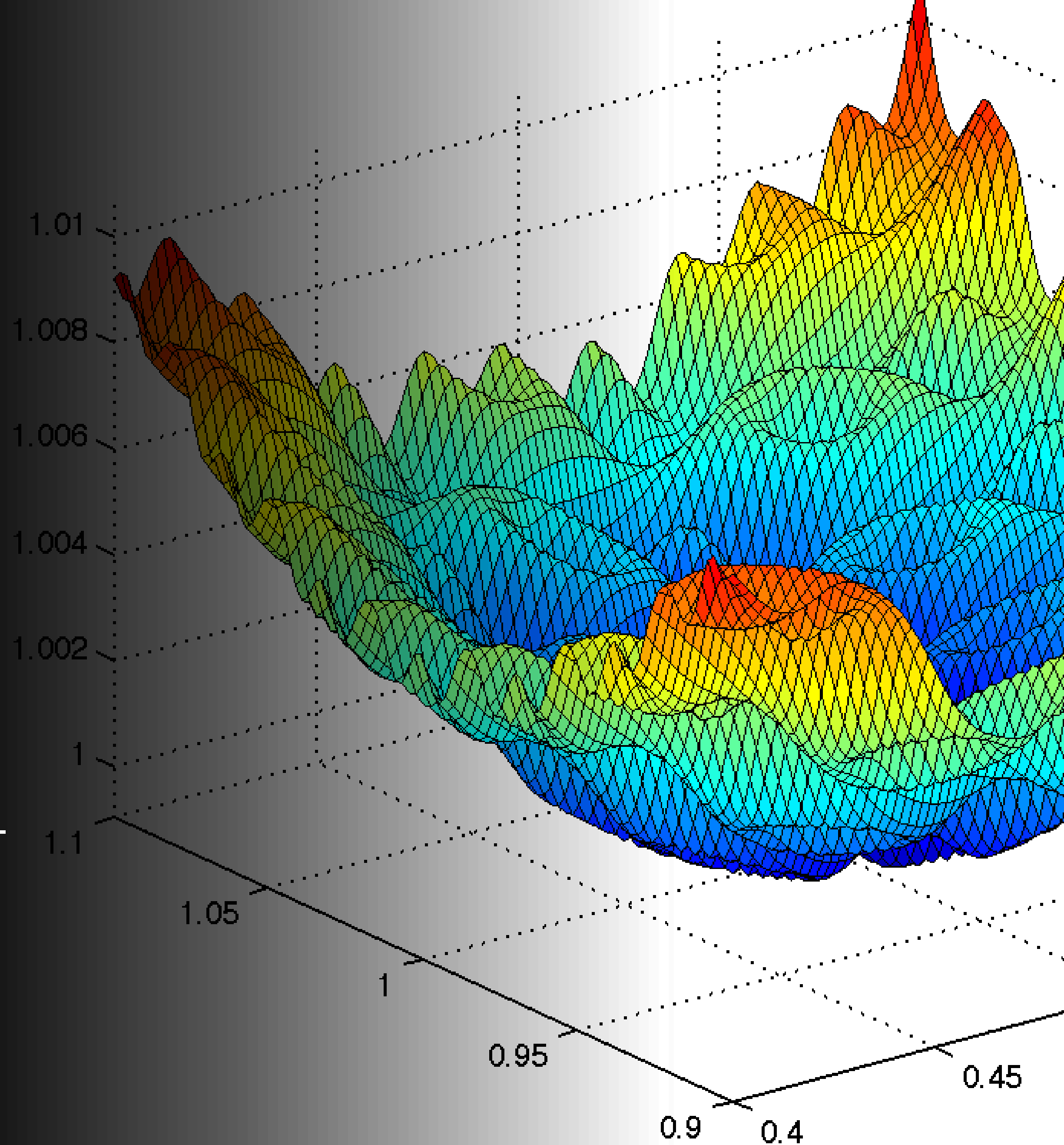
CS 5/7320
Artificial Intelligence

Local Search
AIMA Chapters 4.1 & 4.2

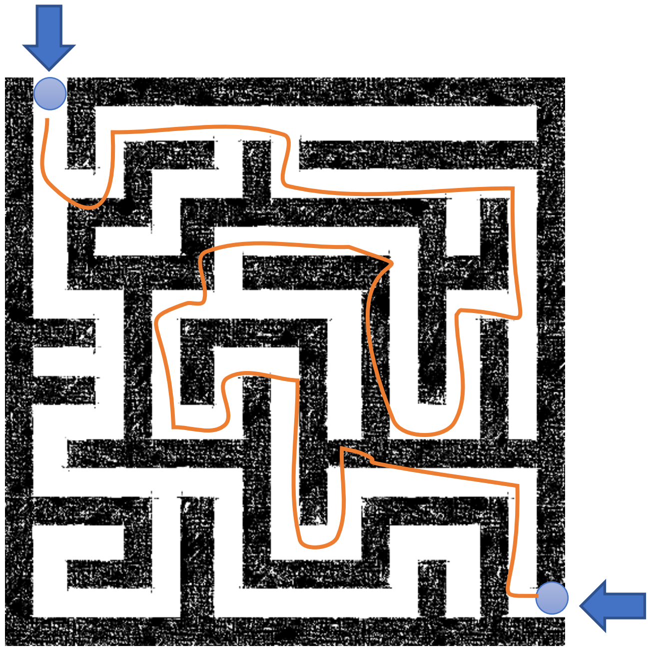


Slides by Michael Hahsler
based on slides by Svetlana Lazepnik
with figures from the AIMA textbook.

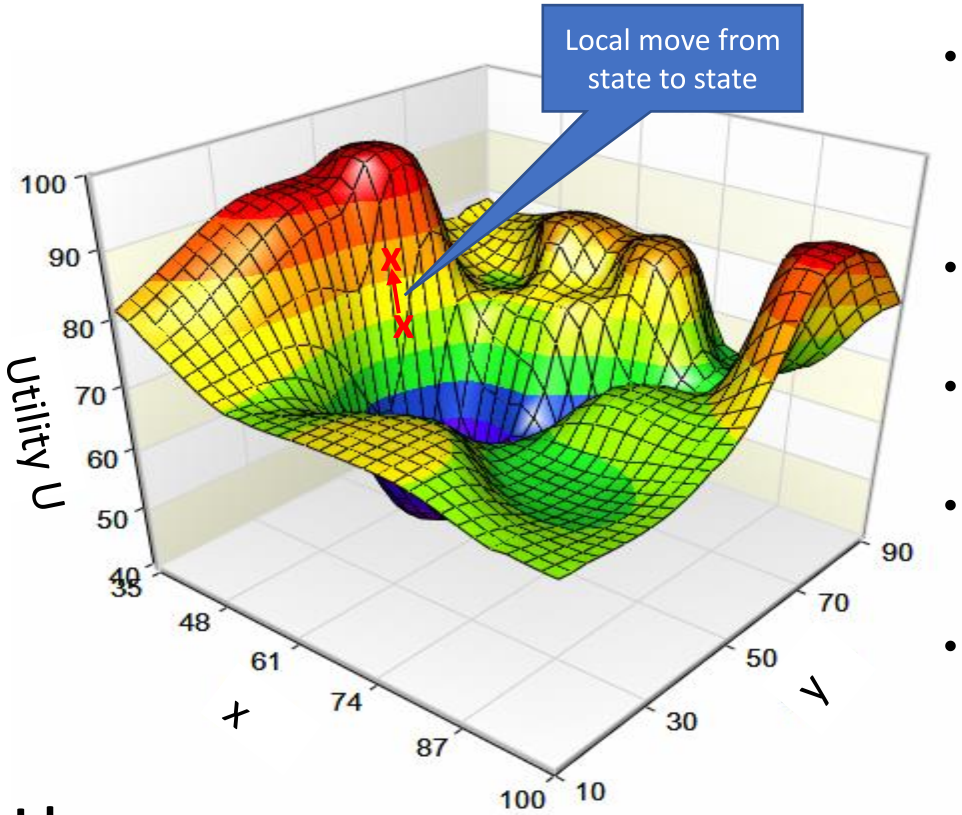
Recap: Uninformed and Informed Search

Tries to plan the
best path
from a
given initial state
to a
given goal state.

- Typically searches a large portion of the search space (needs time and memory).
- Often comes with optimality guarantees.



Local Search Algorithms



- What if we do not know the goal state, but the utility of different states is given by a utility function
$$U = u(s)?$$
- We use a factored state description. Here $s = (x, y)$
- We could try to identify the best or a at least a "good" state?
- This is the **optimization problem**:
$$s^* = \underset{s \in S}{\operatorname{argmax}}\, u(s)$$
- We need a fast and memory-efficient way to find the best/a good state.

Idea:

Start with a current solution (a state) and improve the solution by moving from the current state to a "neighboring" better state (a.k.a. performing a series of **local moves**).

Local Search Algorithms

Difference to search from the previous chapter:

a) **Goal state is unknown,** but we know or can calculate the utility for each state. We want to identify the state with the highest utility.

b) Often no explicit initial state + **path to goal and path cost are not important.**

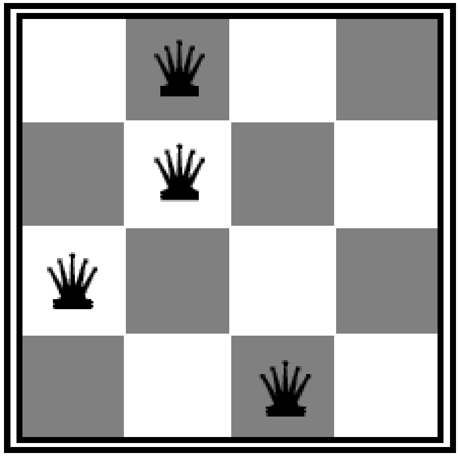
c) **No search tree**. Just stores the current state and move to a “better” state if possible.

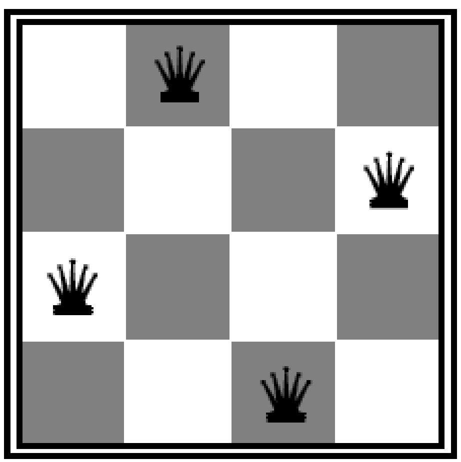
Use in AI

- **Goal-based agent**: Identify a good goal state with a good utility before planning the path to that state.
- **Utility-based agent**: Always move to neighboring higher utility states. A simple greedy method used for complicated/large state spaces or online search.
- **General optimization**: $u(s)$ can be replaced by a general objective function. Local search is an effective heuristic to find good solutions in large or continuous search spaces. E.g., gradient descend to train neural networks.

Example: *n*-Queens Problem

states





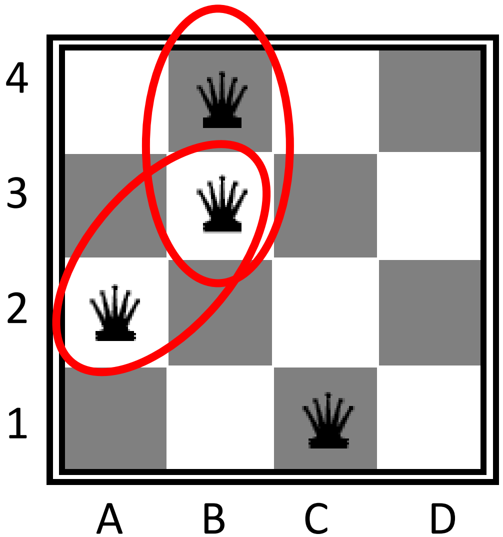
Goal: Put *n* queens on an $n \times n$ board with no two queens on the same row, column, or diagonal.

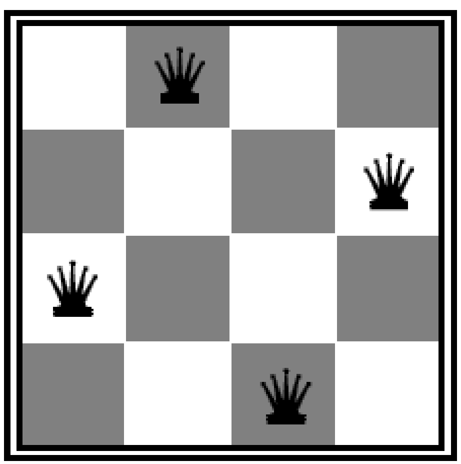
Defining the search problem:

- **State space:** All possible *n*-queen configurations. How many are there?
- **State representation:** How do we define a structured representation?
- **Objective function**: What is a possible utility function given the state representation?
- **Local neighborhood:** What states are close to each other?

Example: *n*-Queens Problem

2 conflicts = utility of -2





0 conflicts = utility of 0

Defining the search problem:

- **State space:** All possible *n*-queen configurations. How many are there?
  4-queens problem: $\binom{16}{4} = 1820$
- **State representation:** How do we define a structured representation?
  E.g. $(A2, B3, B4, C1)$
- **Objective function**: What is a possible utility function given the state representation?
  Maximizing utility means minimize the number of pairwise conflicts based on the state representation.
- **Local neighborhood:** What states are close to each other?
  Move a single queen.

Example: Traveling salesman problem

- **Goal**: Find the shortest tour connecting a given set of cities
- **State space:** all possible tours (states are not individual cities!)
- **State representation:** Order of cities in the tour.
- **Objective function:** minimize the length of the tour
- **Local neighborhood:** Change the order of visiting a few cities.

Note: We have solved a different problem with uninformed/informed search! Each city was defined as a state and the path was the solution.

Hill-Climbing Search aka Greedy Local Search

Idea: keep a single "current" state and try to find better neighboring states.

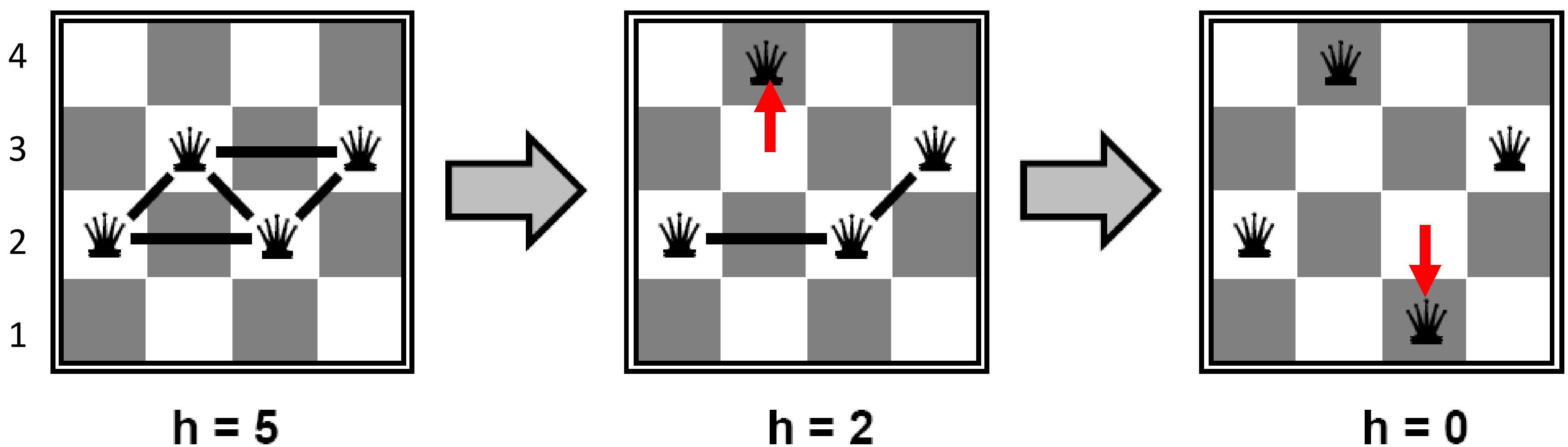
Example: *n*-Queens Problem

- **Goal:** Put *n* queens on an *n* × *n* board with no two queens on the same row, column, or diagonal.
- **State space:** all possible *n*-queen configurations. We can restrict the state space: Only one queen per column.
- **State representation:** row position of each queen in its column (e.g., 2, 3, 2, 3)
- **Objective function:** minimize the number of pairwise conflicts.
- **Local neighborhood:** Move one queen anywhere in its column.

State space is reduced from 1820 to $4^4 = 256$

Improvement strategy

- Find a local neighboring state (move one queen within its column) to reduce conflicts



Example: *n*-Queens Problem

To find the best local move, we must evaluate all local neighbors (moving a queen in its column) and calculate the objective function.

Objective value after moving the queen to this square

Current objective value: $h = 17$
best local improvement has $h = 12$

Notes:

- There are many options with $h = 12$. We must choose one!
- Calculating all the objective values may be expensive!

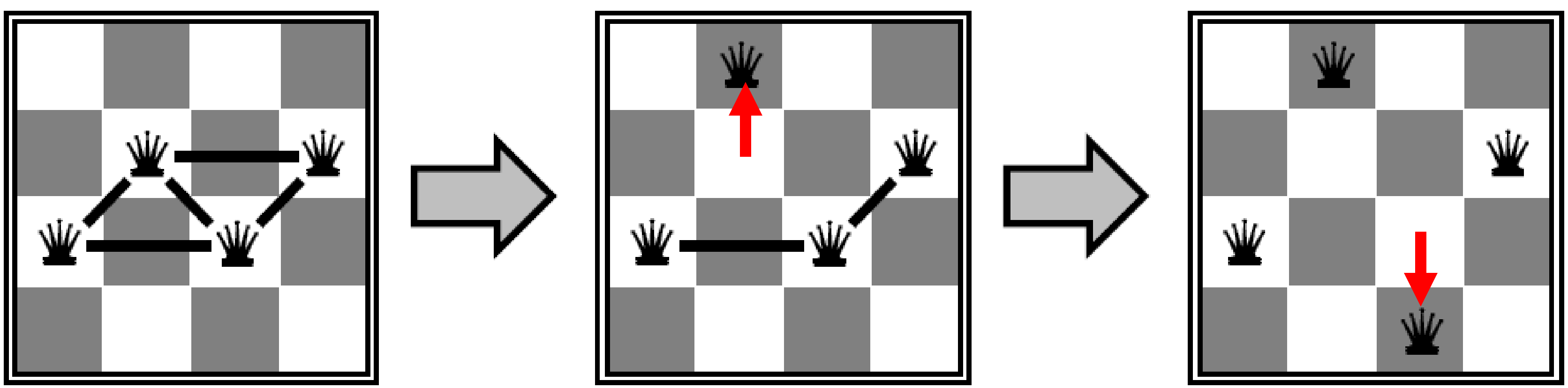
Example: *n*-Queens Problem

Formulation as an optimization problem:
Find the best state s^* representing an arrangement of queens.

$$s^* = \text{argmin}_{s \in S}\ \text{conflicts}(s)$$

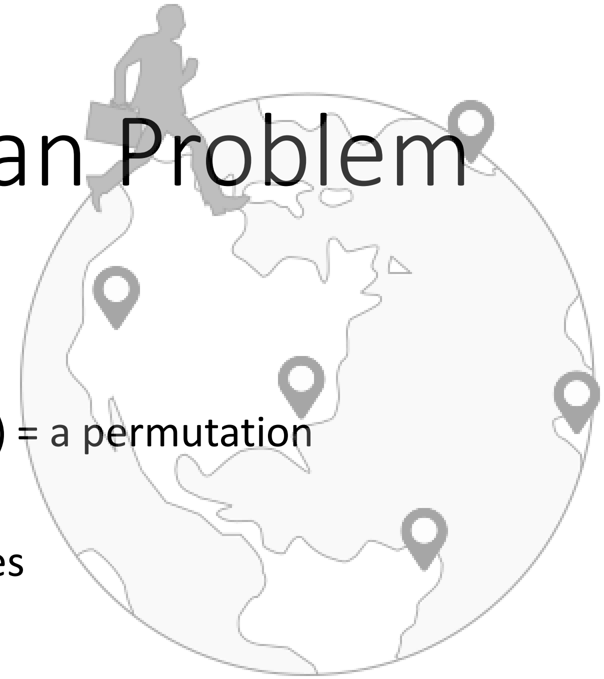
subject to: s has one queen per column

Remember: This makes the problem a lot easier.



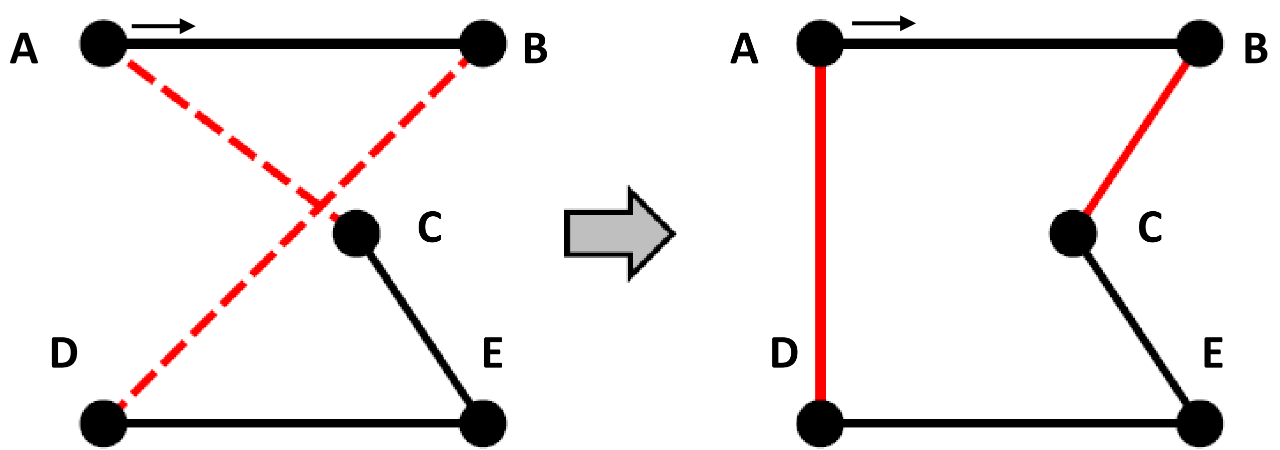
h = 5 h = 2 h = 0

Example: Traveling Salesman Problem



- **Goal:** Find the shortest tour connecting n cities
- **State space:** all possible tours
- **State representation:** tour (order in which to visit the cities) = a permutation
- **Objective function:** length of tour
- **Local neighborhood:** reverse the order of visiting a few cities

Local move to reverse the order of cities C, E and D:



State representation: **ABDEC** **ABCED**

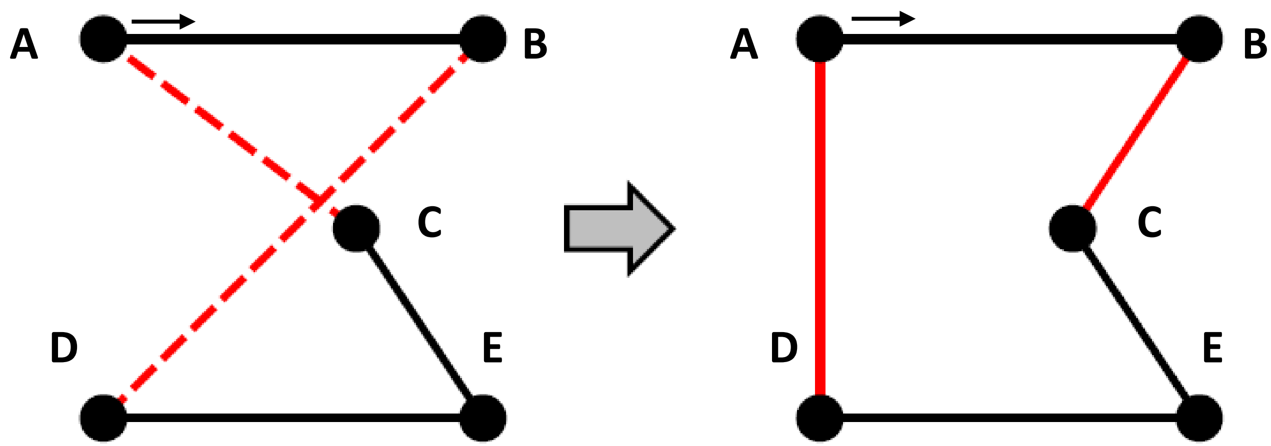
Example: Traveling Salesman Problem

Formulation as an optimization problem:
Find the best tour π

$$\pi^* = \operatorname{argmin}_{\pi} \ \ \text{tourLength}(\pi)$$

s.t. π is a valid permutation (i.e., sub-tour elimination)

Local move to reverse the order of cities C, E and D:



State representation: **ABDEC** **ABCED**

Hill-Climbing Search (= Greedy Local Search)

function HILL-CLIMBING($problem$) **returns** a state that is a local maximum
  $current \leftarrow problem$.INITIAL
  while $true$ **do**
    $neighbor \leftarrow$ a highest-valued successor state of $current$
    if VALUE($neighbor$) $\leq$ VALUE($current$) **then return** $current$
    $current \leftarrow neighbor$

Typically, we start with a random state

Variants:

Steepest-ascend hill climbing

- Check all possible successors and choose the highest-valued successors.

Stochastic hill climbing

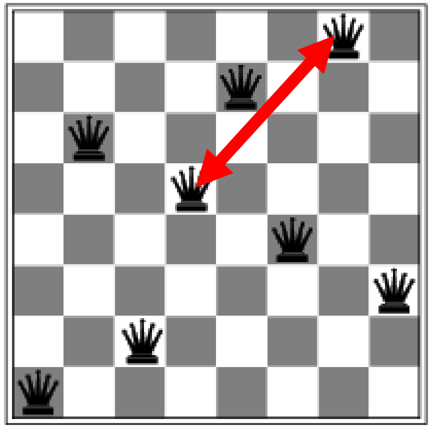
- choose randomly among all uphill moves, or
- generate randomly one new successor at a time until a better one is found = first-choice hill climbing – **the most popular variant**, this is what people often mean when they say "stochastic hill climbing"

Local Optima

Hill-climbing search is like greedy best-first search with the objective function as a (maybe not admissible) heuristic and no frontier (just stops in a dead end).

Is it complete/optimal?

- No – can get stuck in local optima



$h = 1$

Example: local optimum for the 8-queens problem. No single queen can be moved within its column to improve the objective function.

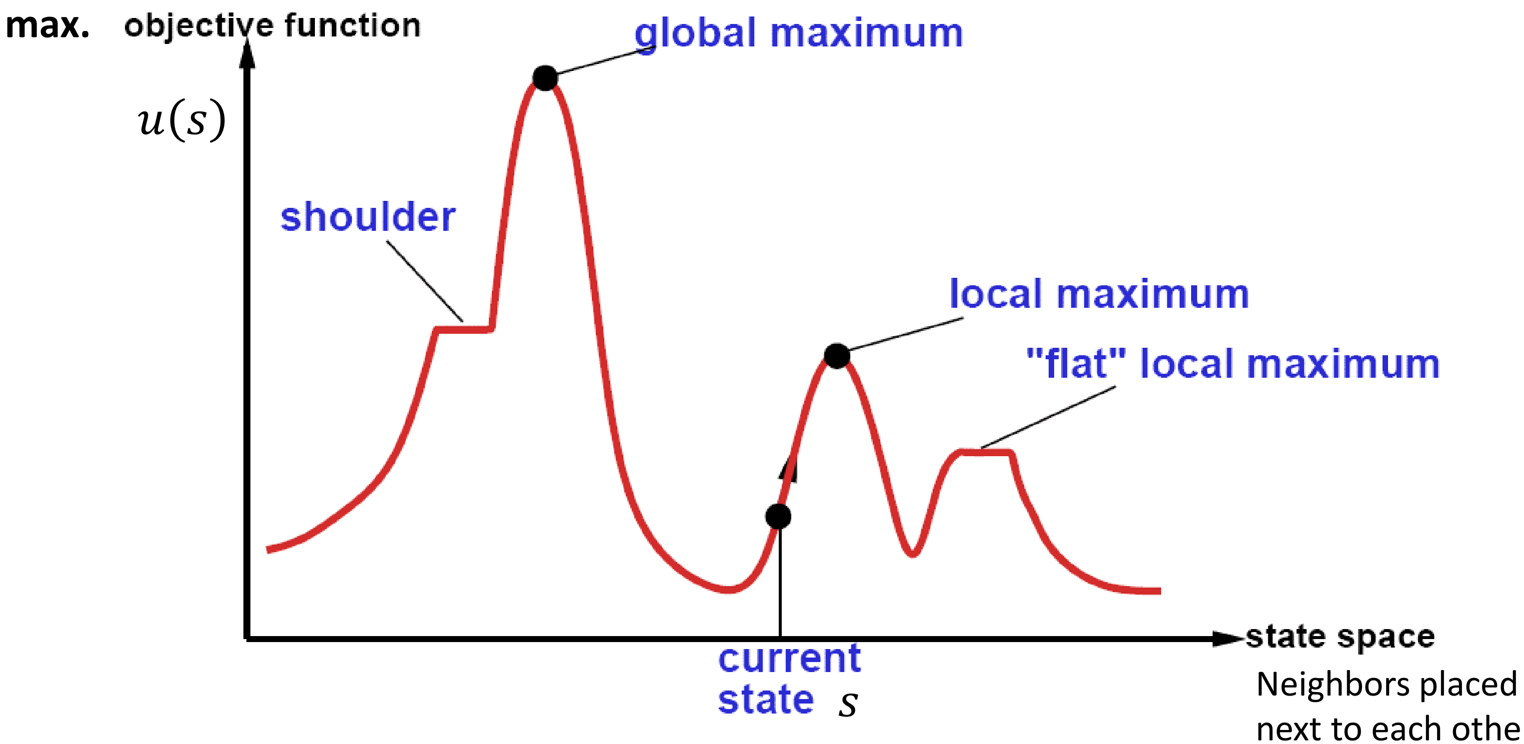
Simple approach that can help with local optima:

Random-restart hill climbing

- Restart hill-climbing many times with random initial states and return the best solution.

The State Space “Landscape”

We can get the utility (objective function value) from the state description using $U = u(s)$.



How to escape local maxima?

→ Random restart hill-climbing can help.

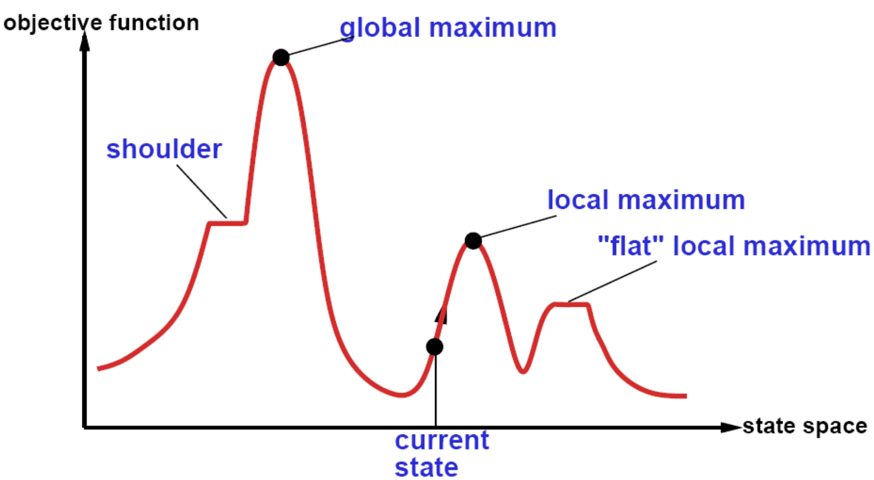
What about “shoulders” (called “ridges” in higher dimensional space)?

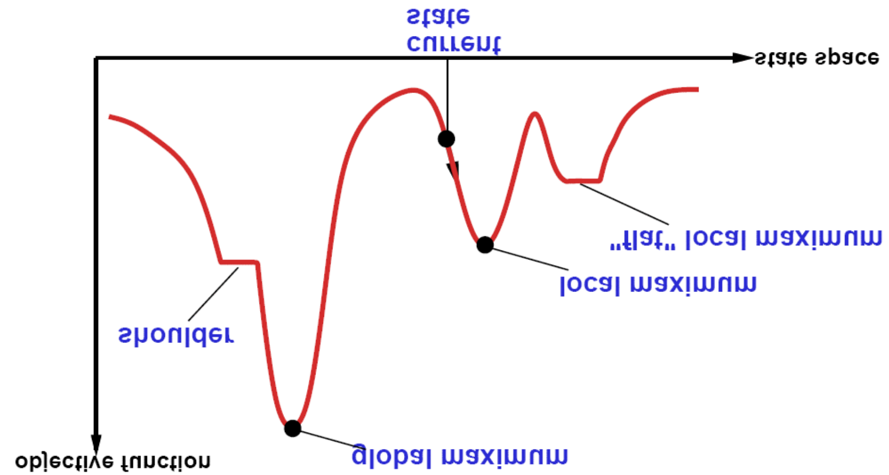
→ Hill-climbing that allows sideways moves and uses momentum.

Minimization vs. Maximization

- The name hill climbing implies **maximizing a function**.
- Optimizers like to state problems as **minimization problems** and call hill climbing gradient descent instead.
- Both types of problems are equivalent:

$$\max(f(x)) \quad \Leftrightarrow \quad \min(-f(x))$$

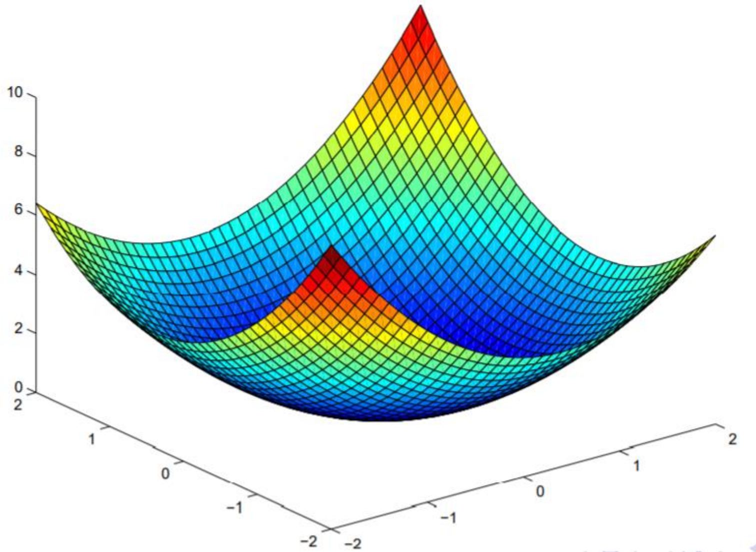




Convex vs. Non-Convex Optimization Problems

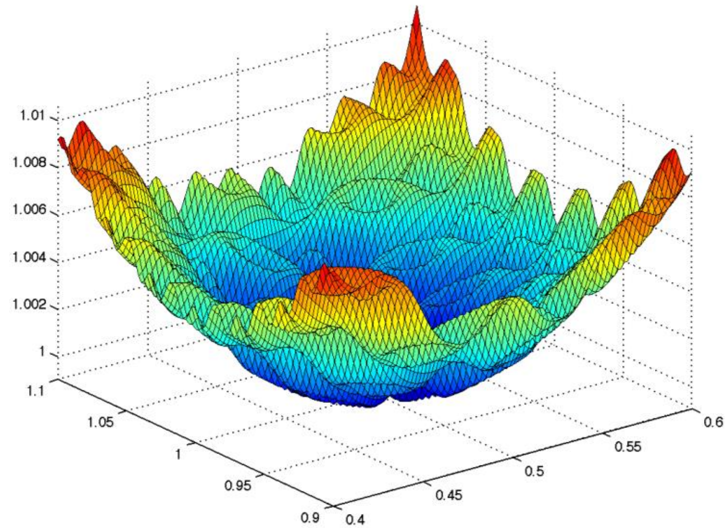
Minimization problems

Convex Problem



One global optimum + smooth function → calculus makes it easy

Non-convex Problem



Many local optima → hard

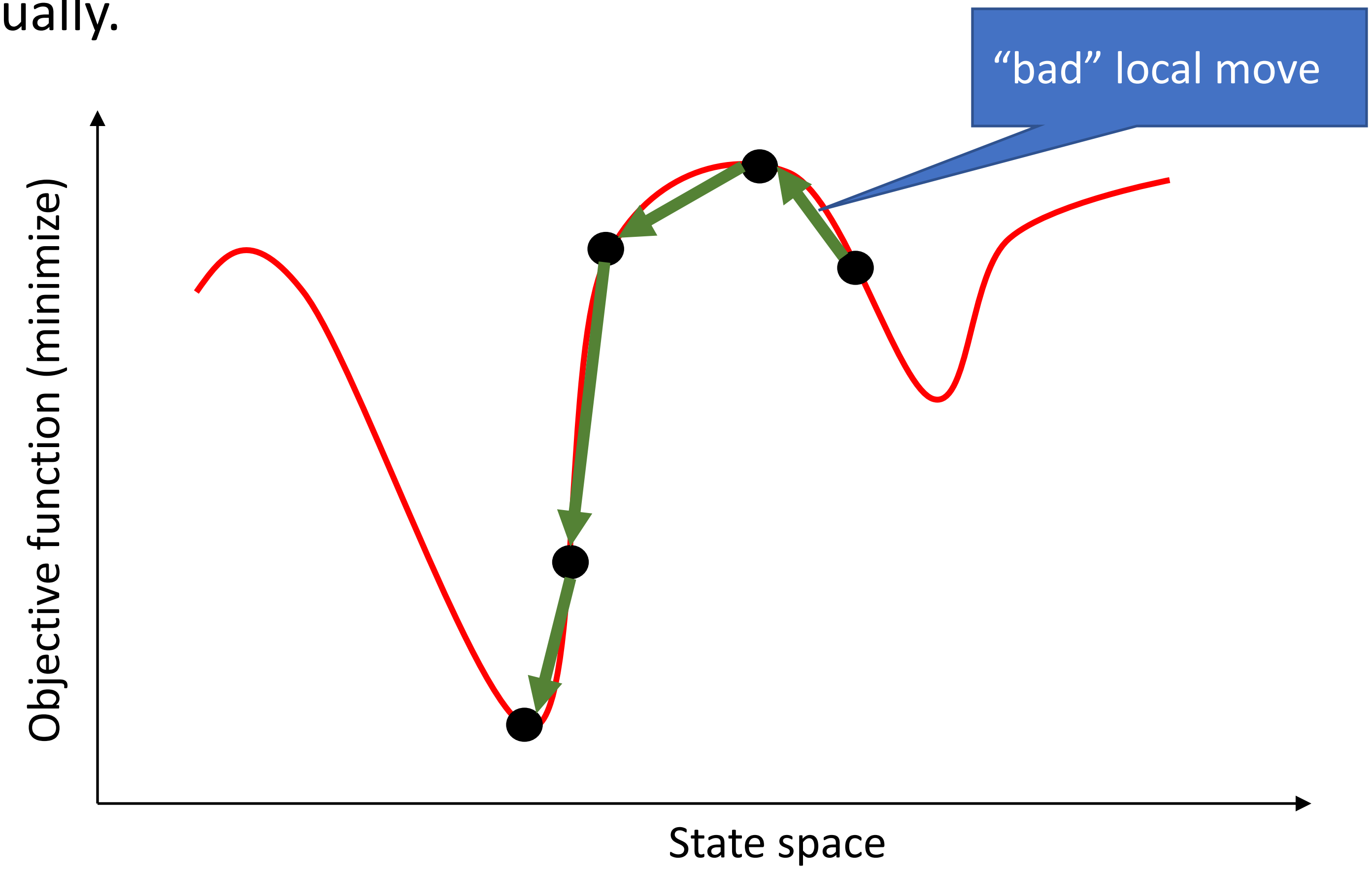
Many discrete optimization problems are like this.

Simulated Annealing

Use heat to escape local optima…

Simulated Annealing

- **Idea**: First-choice stochastic hill climbing + escape local minima by **allowing some "bad" moves** but gradually decrease their frequency.
- Inspired by the process of tempering or hardening metals by decreasing the temperature (chance of accepting bad moves) gradually.



Simulated Annealing

- **Idea**: First-choice stochastic hill climbing + escape local minima by allowing some "bad" moves but gradually decreasing their frequency as we get closer to the solution.
- The probability of accepting "bad" moves follows **an annealing schedule** that reduces the temperature T over time t.

```
function SIMULATED-ANNEALING(problem, schedule) returns a solution state
    current ← problem.INITIAL
    for t = 1 to ∞ do
        T ← schedule(t)
        if T = 0 then return current
        next ← a randomly selected successor of current
        ΔE ← VALUE(next) – Value(current)
        if ΔE > 0 then current ← next
        else current ← next only with probability e^(−ΔE/T)
```

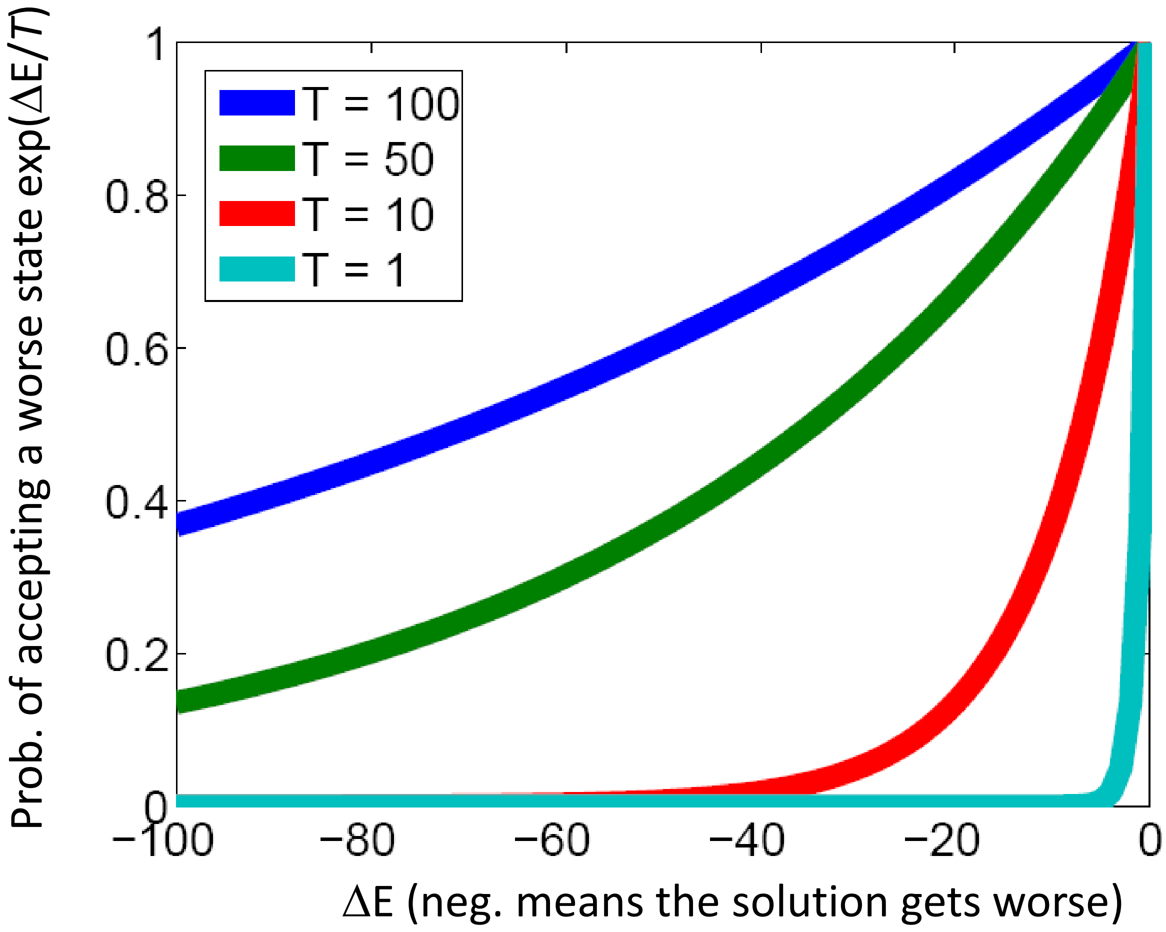
Typically, we start with a random state

Always do good moves

Uses the Metropolis acceptance criterion to accept "bad" moves

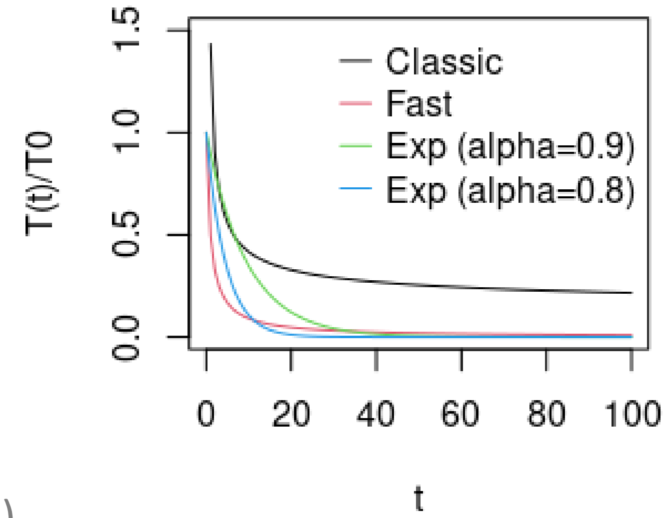
Note: Use VALUE(current) – VALUE(next) for minimization

The Effect of Temperature



The lower the temperature, the less likely the algorithm will accept a worse state.

Cooling Schedule



The cooling schedule is very important.
Popular schedules for the temperature at time t:

- **Classic simulated annealing:** $T_t = T_0 \frac{1}{\log(1+t)}$
- **Fast simulated annealing** (Szy and Hartley; 1987)

$$T_t = T_0 \frac{1}{1+t}$$

- **Exponential cooling** (Kirkpatrick, Gelatt and Vecchi; 1983)

$$T_t = T_0 \alpha^t \quad \text{for} \quad 0.8 < \alpha < 1$$

Notes:

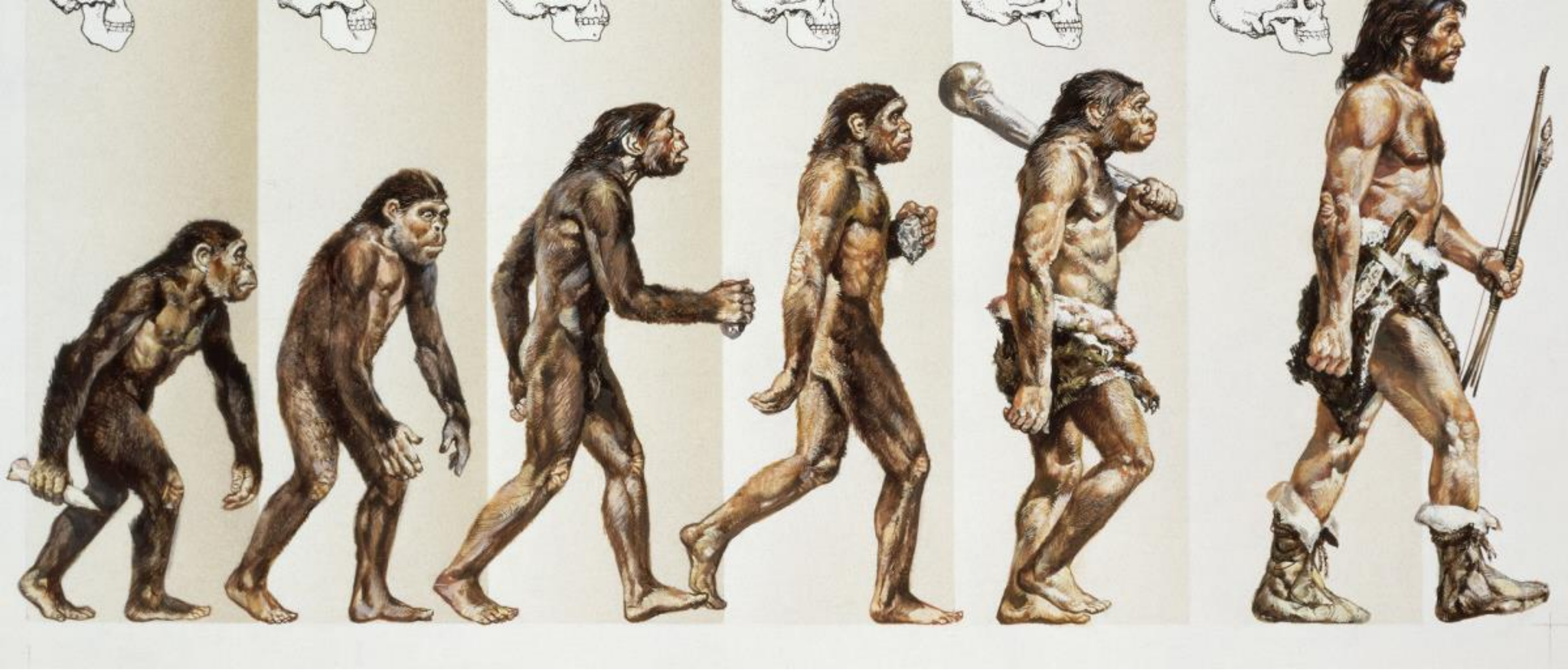
- The best schedule is typically determined by trial-and-error.
- Choose T_0 to provide a high probability that any move will be accepted at time $t = 0$.
- T_t will not become 0 but very small. Stop when $T < \epsilon$ (ϵ is a very small constant).

Simulated Annealing Search

Guarantee: If temperature decreases **slowly enough**, then simulated annealing search will find a global optimum with probability approaching one.

However:

- This usually takes impractically long.

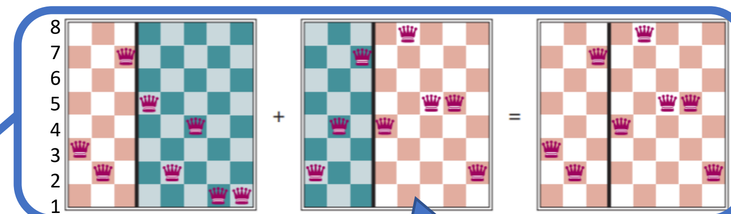


Evolutionary Algorithms

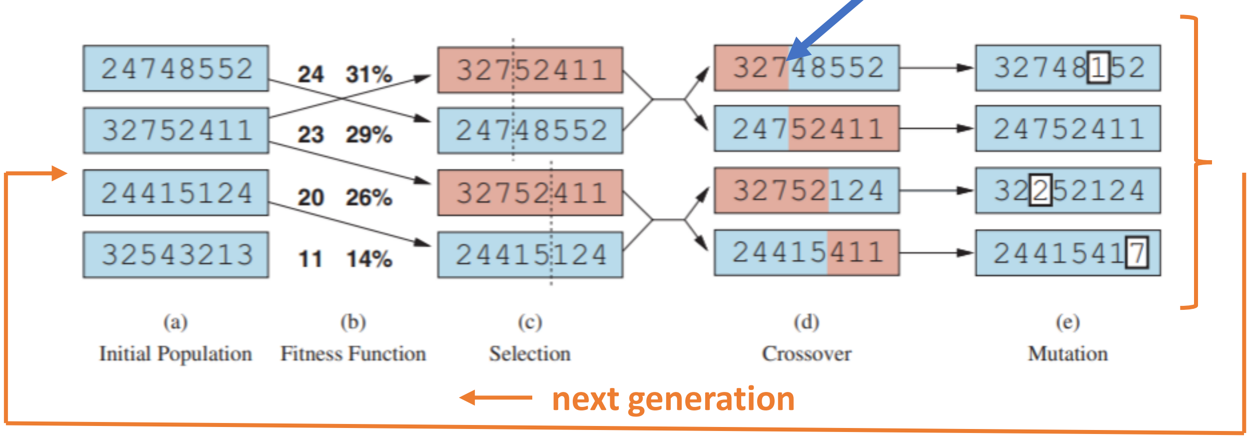
A Population-based Metaheuristics

Evolutionary Algorithms / Genetic Algorithms

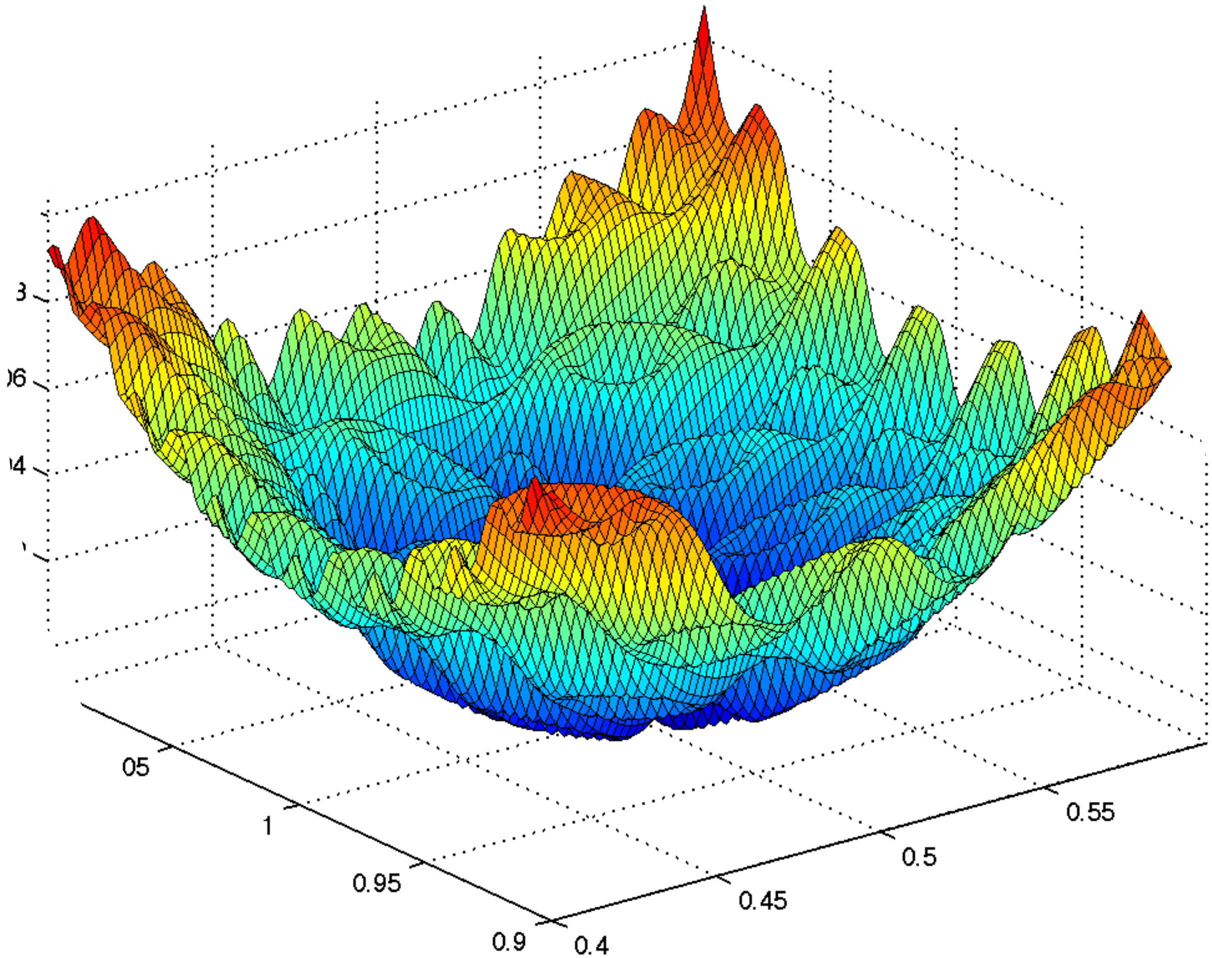
- A metaheuristic for **population-**based optimization.
- Uses mechanisms inspired by biological evolution (genetics):
  - Reproduction: Random selection with probability based on a **fitness** function.
  - Random recombination (crossover)
  - Random mutation
  - Repeated for many generations



- Example: 8-queens problem



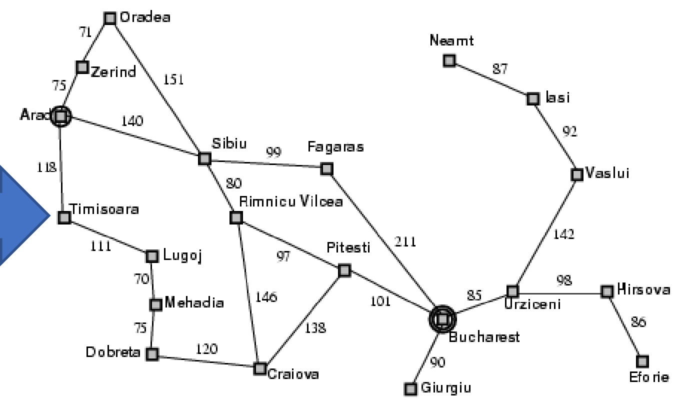
Individual = state **representation** as a chromosome: row of the queen in each column



Search in Continuous Spaces

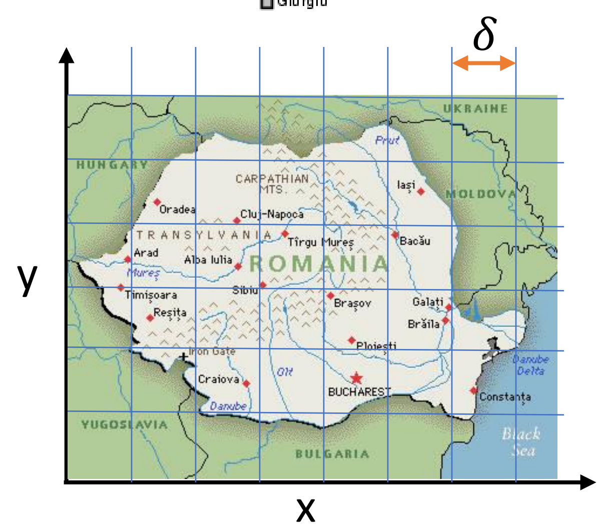
Discretization of Continuous Space

- Use atomic states and create a graph as the transition function.



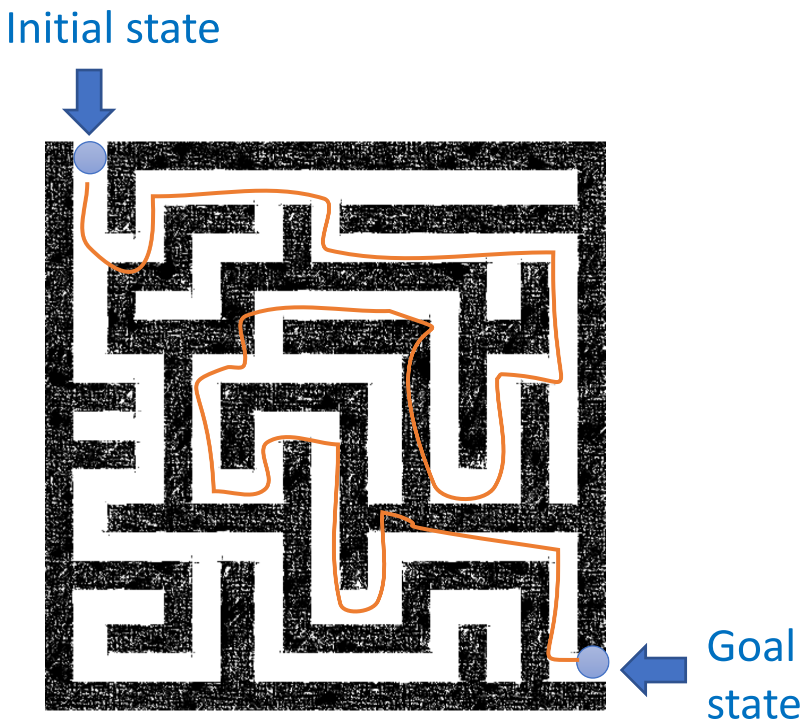
- Use a grid with spacing of size δ

  Note: You probably need a way finer grid!



Discretization of Continuous Space

How did we discretize this space?



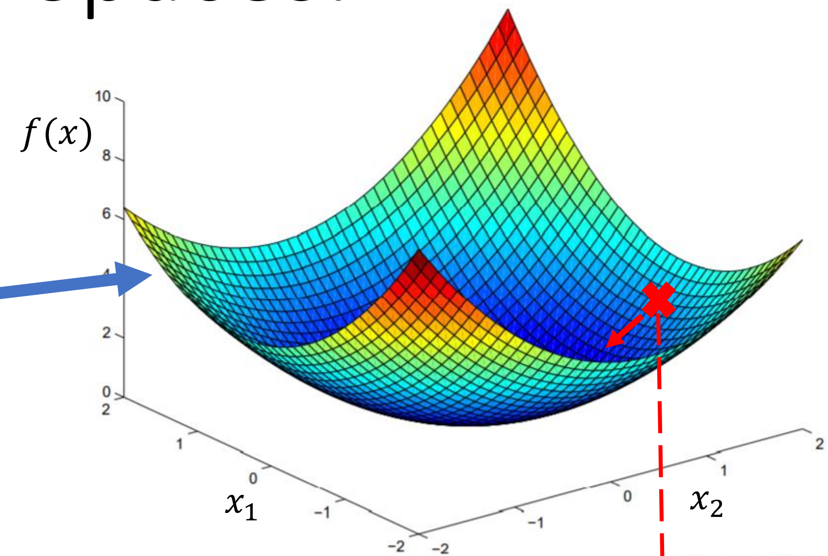
Search in Continuous Spaces: Gradient Descent

State space: infinite

State representation: $\boldsymbol{x} = (x_1, x_2, \dots, x_k)$

Objective function: min $f(\boldsymbol{x}) = f(x_1, x_2, \dots, x_k)$

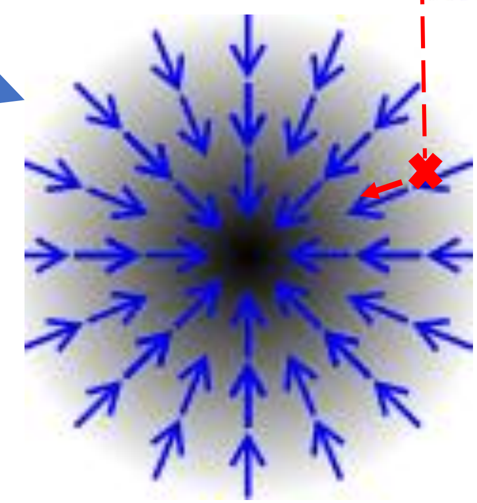
Local neighborhood: small changes in $x_1, x_2, \dots, x_k$



Gradient at point $\boldsymbol{x}$: $\nabla f(\boldsymbol{x}) = \left(\frac{\partial f(\boldsymbol{x})}{\partial x_1}, \frac{\partial f(\boldsymbol{x})}{\partial x_2}, \dots, \frac{\partial f(\boldsymbol{x})}{\partial x_k}\right)$

(=evaluation of the Jacobian matrix at $\boldsymbol{x}$)

Find optimum by solving: $\nabla f(\boldsymbol{x}) = 0$



- **Gradient descent (= Steepest-ascend hill climbing for minimization)** with step size α

  Repeat: $\boldsymbol{x} \leftarrow \boldsymbol{x} - \alpha \nabla f(\boldsymbol{x})$

- **Newton-Raphson method** uses the inverse of the Hessian matrix (second-order partial derivative of $f(\boldsymbol{x})$) $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ for the step size α

  Repeat: $\boldsymbol{x} \leftarrow \boldsymbol{x} - \boldsymbol{H}_f^{-1}(\boldsymbol{x}) \nabla f(\boldsymbol{x})$

Note: May get stuck in a local optima if the search space is non-convex! Use simulated annealing, momentum or other methods to escape local optima.

Search in Continuous Spaces: Empirical Gradient Methods

- What if the mathematical formulation of the objective function is not known?
- We may have objective values at fixed points, called the **training data**.
- In this case, we can estimate the gradient using the data and use **empirical gradient search.**

→ We will talk more about search in continuous spaces with loss functions using gradient descend when we talk about **parameter learning for machine learning.**