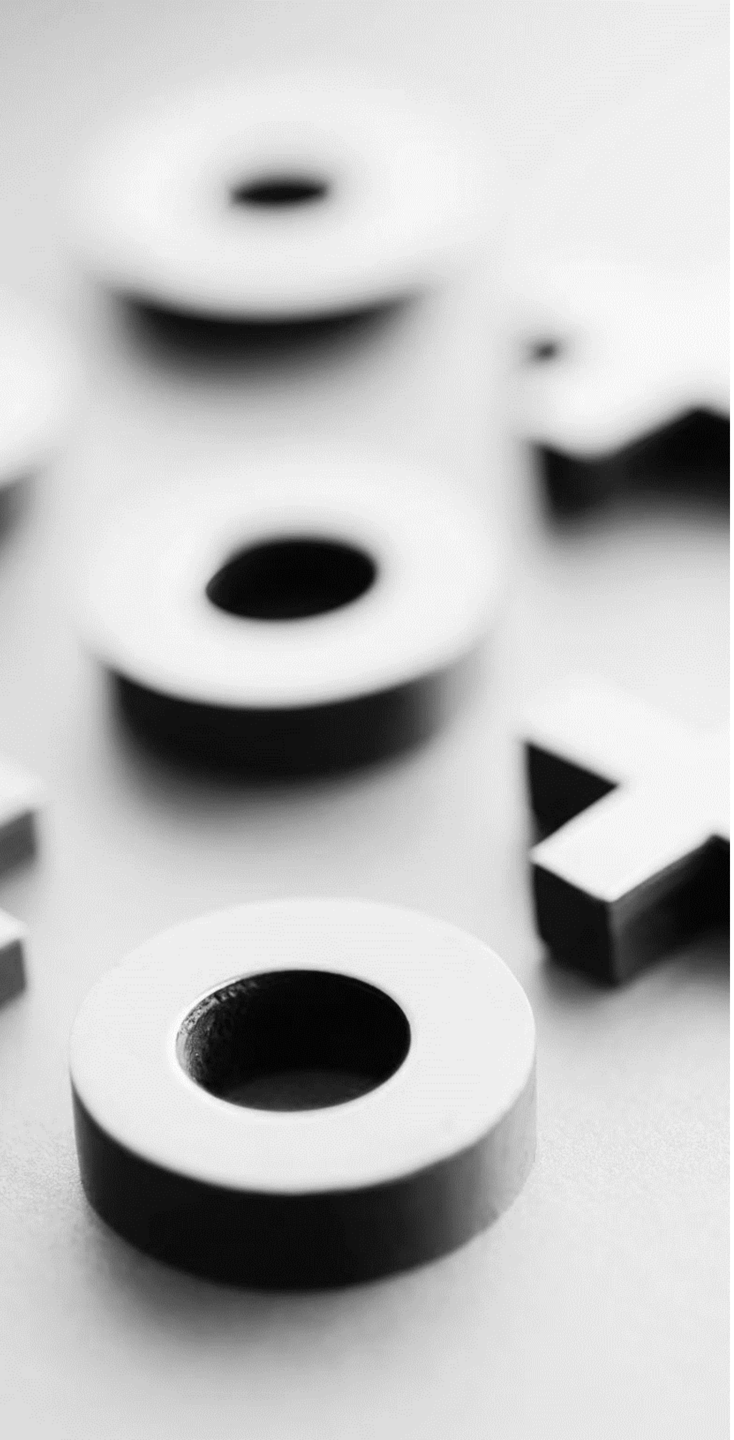
CS 5/7320
Artificial Intelligence

# Adversarial Search and Games

AIMA Chapter 5

Slides by Michael Hahsler
with figures from the AIMA textbook

"Reflected Chess pieces" by Adrian Askew



# Games

- Games typically confront the agent with a competitive (adversarial) environment affected by an opponent (strategic environment).

- We will focus on planning for
  - two-player zero-sum games with
  - deterministic game mechanics and
  - perfect information (i.e., fully observable environment).

- We call the two players:
  1) **Max** tries to maximize his utility.
  2) **Min** tries to minimize Max's utility since it is a zero-sum game.

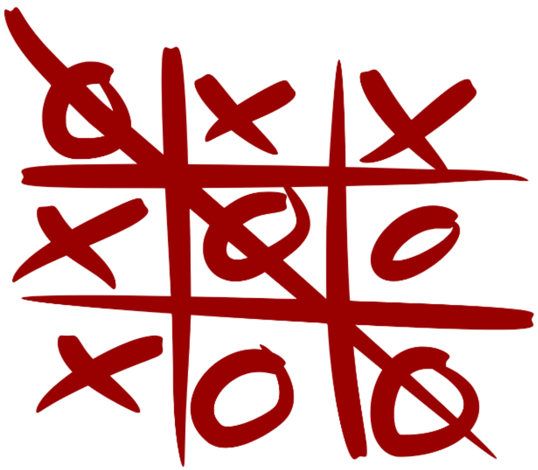
# Definition of a Game

- **Definition:**

| | |
|---|---|
| $s_0$ | The initial state (position, board). |
| $Actions(s)$ | Legal moves in state $s$. |
| $Result(s, a)$ | Transition model. |
| $Terminal(s)$ | Test for terminal states. |
| $Utility(s)$ | Utility for player Max. |

- **State space**: a graph defined by the initial state and the transition function containing all reachable states (e.g., chess positions).
- **Game tree**: a search tree superimposed on the state space. A complete game tree follows every sequence from the current state to the terminal state (the game ends).

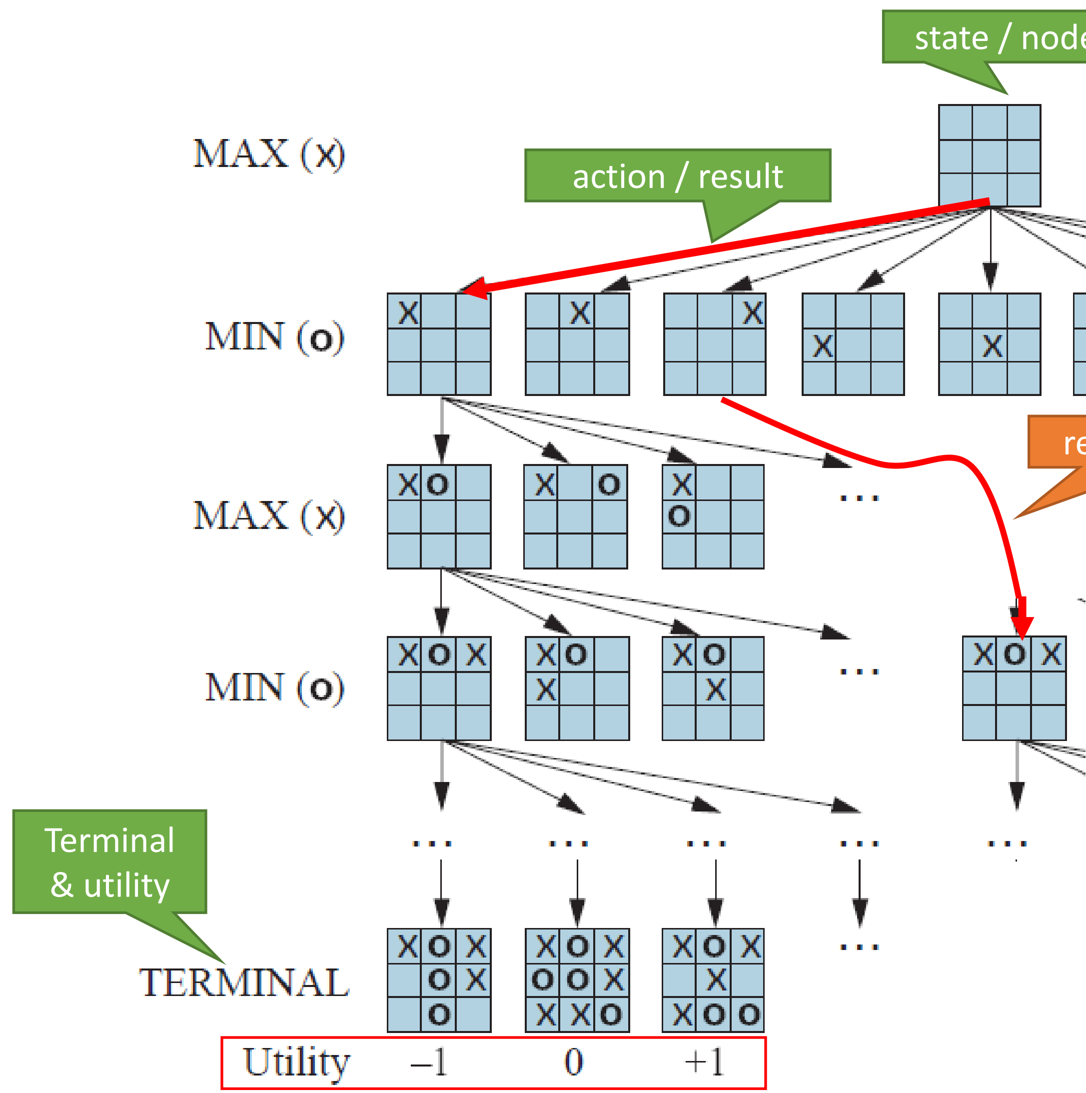
# Example: Tic-tac-toe



| | |
|---|---|
| $s_0$ | Empty board. |
| $Actions(s)$ | Empty squares. |
| $Result(s, a)$ | Place symbol (x/o) on empty square. |
| $Terminal(s)$ | Did a player win or is the game a draw? |
| $Utility(s)$ | +1 if x wins, -1 if o wins and 0 for a draw. |
| | Utility is only defined for terminal states. |

Here player x is Max
and player o is Min.

# Tic-tac-toe: Partial Game Tree




# of nodes

1

9

$9 \times 8$

The **state space** size (number of possible boards) is much smaller than:

$$3^9 = 19{,}683 \text{ states.}$$

However, the complete **game tree** is much larger because the same state (board) can be reached in different subtrees (**redundant paths**). The game tree here is a little smaller than:

$$1 + (9 \times 8) + (9 \times 8 \times 7) + \cdots (9!)$$
$$= 986{,}409 \text{ nodes}$$

# Methods for Adversarial Games

## Exact Methods

- **Model as nondeterministic actions**: The opponent is seen as part of an environment with nondeterministic actions. Non-determinism is the result of the unknown moves by the opponent. We **consider all possible moves** by the opponent.
- **Find optimal decisions**: Minimax search and Alpha-Beta pruning where **each player plays optimal** to the end of the game.

## Heuristic Methods

(game tree is too large)

- **Heuristic Alpha-Beta Tree Search**:
  a. Cut off game tree and use heuristic for utility.
  b. Forward Pruning: ignore poor moves.
- **Monte Carlo Tree search**: Estimate utility of a state by simulating complete games and average the utility.

# Nondeterministic Actions

Recall AND-OR Search from AIMA Chapter 4

# Methods for Adversarial Games

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# Recall: Nondeterministic Actions

For **planning**, we do not know what the opponents moves will be. We have already modeled this issue using nondeterministic actions.

Outcome of actions in the environment is nondeterministic = **transition model need to describe uncertainty about the opponent's behavior.**

Each action consists of the move by the player and all possible (i.e., nondeterministic) responses by the opponent.

Example transition:

$$Results(s_1, a) = \{s_2, s_4, s_5\}$$

i.e., action $a$ in $s_1$ can lead to one of several states (which is called a belief state of the agent).

# Recall: AND-OR DFS Search Algorithm

= nested If-then-else statements

**function** AND-OR-SEARCH(*problem*) **returns** a conditional plan, or *failure*
  **return** OR-SEARCH(*problem*, *problem*.INITIAL, [ ])

**function** OR-SEARCH(*problem*, *state*, *path*) **returns** *a conditional plan, or failure*
  **if** *problem*.IS-GOAL(*state*) **then return** the empty plan
  **if** IS-CYCLE(*path*) **then return** *failure* // don't follow loops
  **for each** *action* **in** *problem*.ACTIONS(*state*) **do** // check all possible actions
    *plan* ← AND-SEARCH(*problem*, RESULTS(*state*, *action*), [*state*] + *path*])
    **if** *plan* ≠ *failure* **then return** [*action*] + *plan*]
  **return** *failure*

my moves

all states that can result from opponent's moves

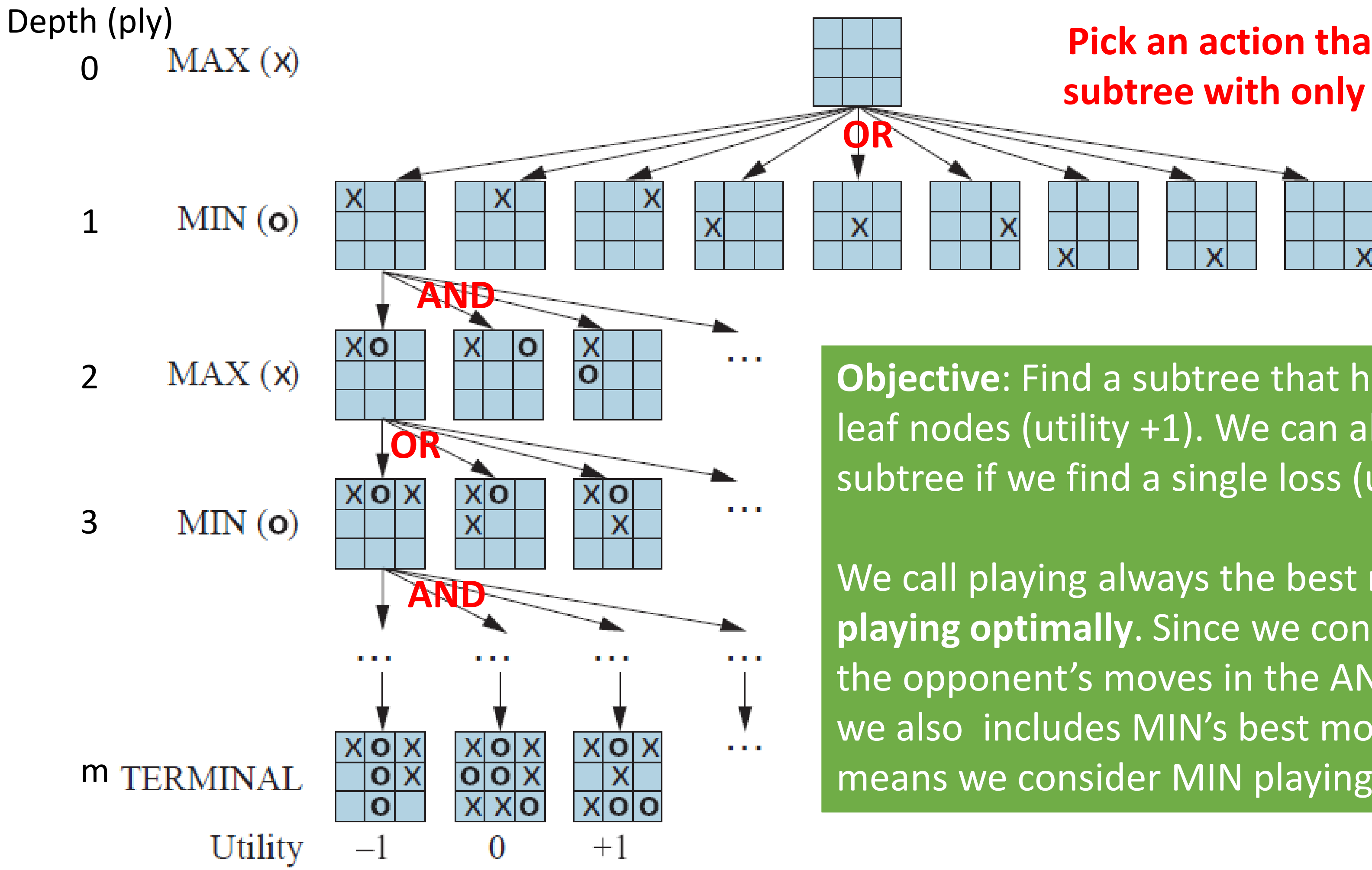
**function** AND-SEARCH(*problem*, *states*, *path*) **returns** *a conditional plan, or failure*
  **for each** $s_i$ **in** *states* **do** // check all possible current states
    $plan_i$ ← OR-SEARCH(*problem*, $s_i$, *path*)
    **if** $plan_i$ = *failure* **then return** *failure*
  **return** [**if** $s_1$ **then** $plan_1$ **else if** $s_2$ **then** $plan_2$ **else** ... **if** $s_{n-1}$ **then** $plan_{n-1}$ **else** $plan_n$]

abandon subtree if a loss is found

Go through opponent moves

# Tic-tac-toe: AND-OR Search

We play MAX and decide on our actions (OR).
MIN's actions introduce non-determinism (AND).




**Pick an action that leads to a subtree with only win leaves.**

**Objective**: Find a subtree that has only win leaf nodes (utility +1). We can abandon a subtree if we find a single loss (utility -1).

We call playing always the best move **playing optimally**. Since we consider all the opponent's moves in the AND stage, we also includes MIN's best move. This means we consider MIN playing optimally.

# Optimal Decisions

## Minimax Search and Alpha-Beta Pruning

# Methods for Adversarial Games

**Exact Methods**

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**Heuristic Methods**
(game tree is too large)

- **Heuristic Alpha-Beta Tree Search**:
    a. Cut off game tree and use heuristic for utility.
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- **Monte Carlo Tree search**: Estimate utility of a state by simulating complete games and average the utility.

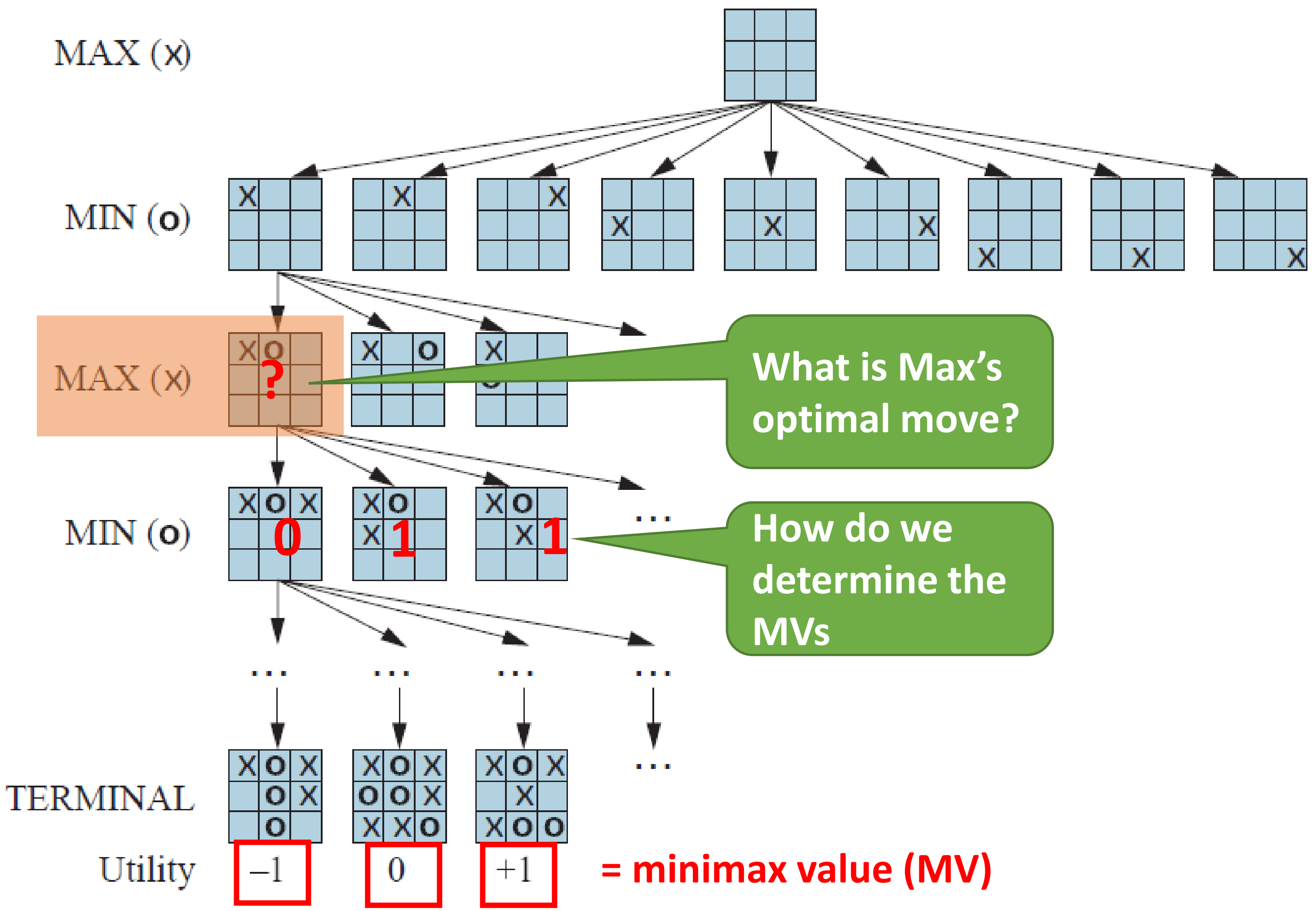
# Idea: Minimax Decision

- Assign each state a **minimax value** that reflects how much Max prefers the state (= Min dislikes the state).

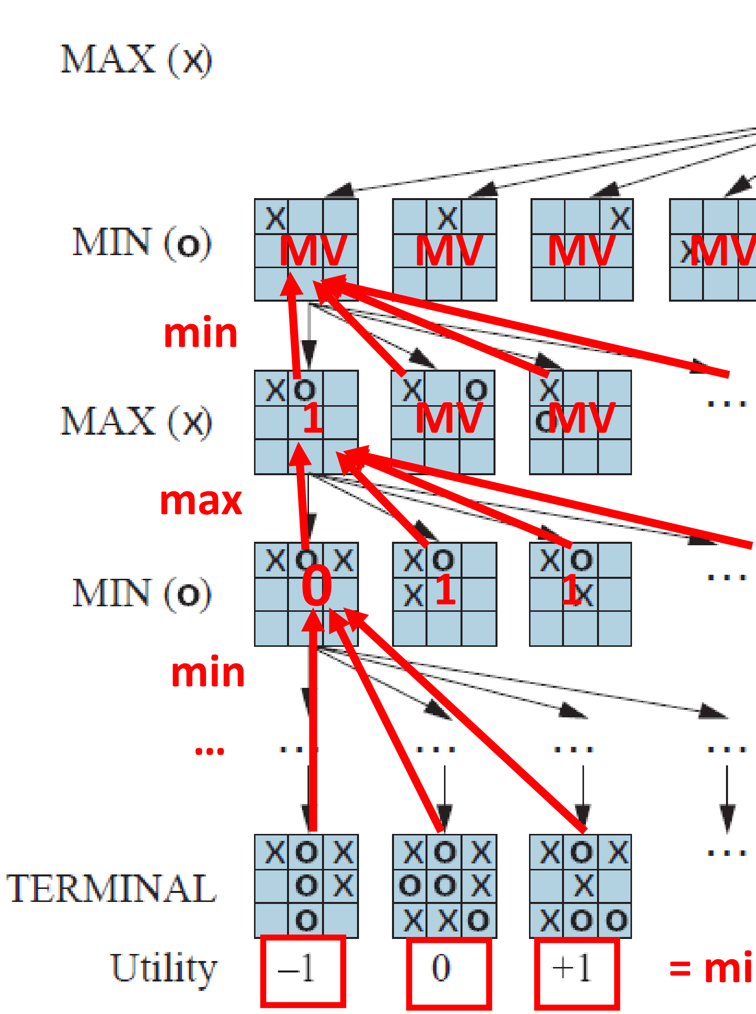
$$Minimax(s) = \begin{cases} Utility(s) & \text{if } terminal(s) \\ \max_{a \in Actions(s)} Minimax(Result(s,a)) & \text{if } move = Max \\ \min_{a \in Actions(s)} Minimax(Result(s,a)) & \text{if } move = Min \end{cases}$$

- The minimax value is the utility for Max in state $s$ assuming that **both players play optimally** from $s$ to the end of the game written as a recursion.
- The **optimal decision** for Max is the action that leads to the state with the largest minimax value.

# Minimax Search: Determining MV Values



# Minimax Search: Back-up Minimax Values

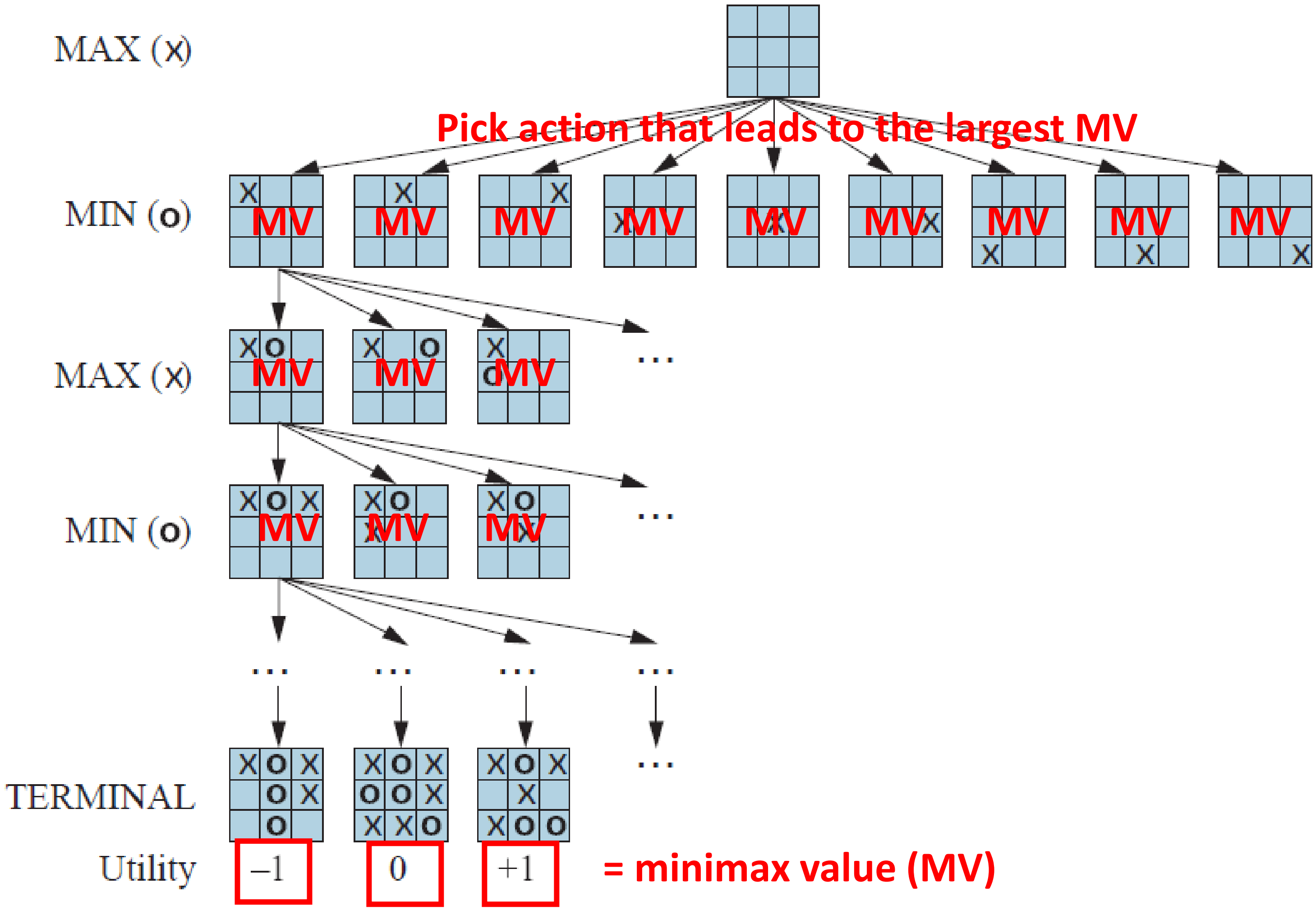



= minimax value (MV)

Determine MVs using a bottom-up strategy

- **Max** always picks the action that has the largest value.
- **Min** always picks the action that has the smallest value.

# Minimax Search: Decision



**Approach**: Follow tree to each terminal node and back up minimax value.

**Note**: This is just a generalization of the AND-OR Tree Search and returns the first action of the conditional plan.

```
function MINIMAX-SEARCH(game, state) returns an action
  player ← game.TO-MOVE(state)
  value, move ← MAX-VALUE(game, state)
  return move

function MAX-VALUE(game, state) returns a (utility, move) pair
  if game.IS-TERMINAL(state) then return game.UTILITY(state, player), null
  v ← −∞
  for each a in game.ACTIONS(state) do
    v2, a2 ← MIN-VALUE(game, game.RESULT(state, a))
    if v2 > v then
      v, move ← v2, a
  return v, move

function MIN-VALUE(game, state) returns a (utility, move) pair
  if game.IS-TERMINAL(state) then return game.UTILITY(state, player), null
  v ← +∞
  for each a in game.ACTIONS(state) do
    v2, a2 ← MAX-VALUE(game, game.RESULT(state, a))
    if v2 < v then
      v, move ← v2, a
  return v, move
```

Found a better action?

Found a better action?

Represents OR Search

Represents AND Search

b: max branching factor
m: max depth of tree

# Issue: Game Tree Size

- **Minimax search traverses the complete game tree using DFS!**

Space complexity: $O(bm)$
Time complexity: $O(b^m)$

- Fast solution is only feasible for very simple games with small branching factor!

- Example: Tic-tac-toe

$$b = 9, m = 9 \rightarrow O(9^9) = O(387{,}420{,}489)$$

$b$ decreases from 9 to 8, 7, … the actual size is smaller than:

$$1(9)(9 \times 8)(9 \times 8 \times 7) \ldots (9!) = 986{,}409 \text{ nodes}$$

- We need to reduce the search space! → **Game tree pruning**

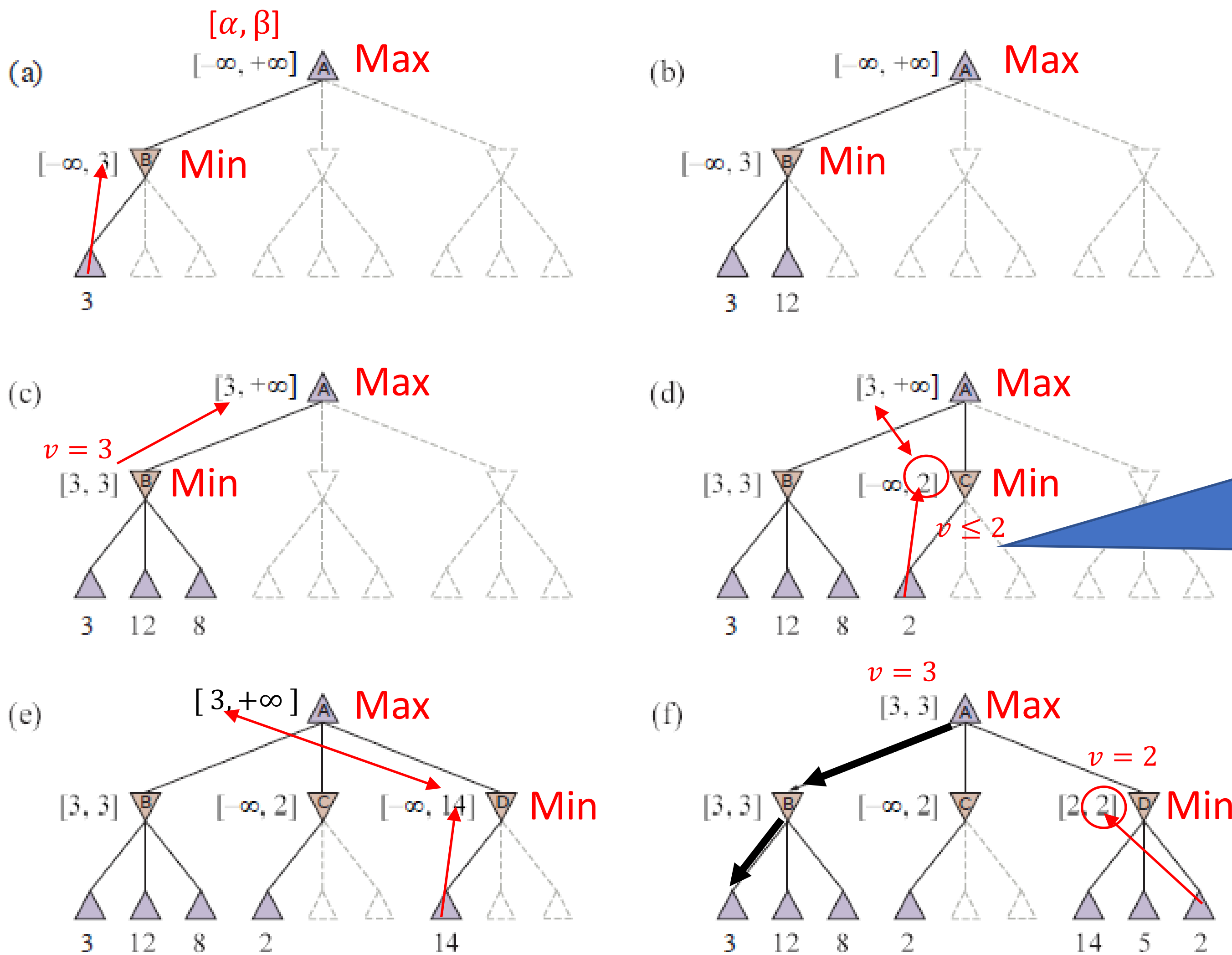
# Alpha-Beta Pruning

- **Idea**: Do not search parts of the tree if they do not make a difference to the outcome.

- **Observations**:
  - $\min(3, x, y)$ can never be more than 3
  - $\max(5, \min(3, x, y, \dots))$ does not depend on the values of $x$ or $y$.
  - Minimax search applies alternating min and max.

- **Approach**: maintain bounds for the minimax value $[\alpha, \beta]$ and prune subtrees (i.e., don't follow actions) that do not affect the current minimax value bound.
  - Alpha is used by Max and means "$Minimax(s)$ is at least $\alpha$."
  - Beta is used by Min and means "$Minimax(s)$ is at most $\beta$."

# Example: Alpha-Beta Search




Max updates α (utility is at least)

Min updates $\beta$ (utility is at most)

Utility cannot be more than 2 in the subtree, but we already can get 3 from the first subtree. Prune the rest.

Once a subtree is fully evaluated, the interval has a length of 0 ($\alpha = \beta$).

= minimax search + pruning

```
function ALPHA-BETA-SEARCH(game, state) returns an action
  player ← game.TO-MOVE(state)
  value, move ← MAX-VALUE(game, state, −∞, +∞)
  return move

function MAX-VALUE(game, state, α, β) returns a (utility, move) pair
  if game.IS-TERMINAL(state) then return game.UTILITY(state, player), null
  v ← −∞      // v is the minimax value
  for each a in game.ACTIONS(state) do
    v2, a2 ← MIN-VALUE(game, game.RESULT(state, a), α, β)
    if v2 > v then
      v, move ← v2, a
      α ← MAX(α, v)
    if v ≥ β then return v, move
  return v, move

function MIN-VALUE(game, state, α, β) returns a (utility, move) pair
  if game.IS-TERMINAL(state) then return game.UTILITY(state, player), null
  v ← +∞
  for each a in game.ACTIONS(state) do
    v2, a2 ← MAX-VALUE(game, game.RESULT(state, a), α, β)
    if v2 < v then
      v, move ← v2, a
      β ← MIN(β, v)
    if v ≤ α then return v, move
  return v, move
```

Found a better action?

Abandon subtree if Max finds an actions that has more value than the best known move Min has in another subtree.

Found a better action?

Abandon subtree if Min finds an actions that has less value than the best known move Max has in another subtree.

# Move Ordering for Alpha-Beta Search

- **Idea:** Pruning is more effective if good alpha-beta bounds can be found in the first few checked subtrees.
- **Move ordering for DFS** = Check good moves for Min and Max first.
- We need expert knowledge or some heuristic to determine what a good move is.
- **Issue:** Optimal decision algorithms still scale poorly even when using alpha-beta pruning with move ordering.

# Heuristic Alpha-Beta Tree Search

# Methods for Adversarial Games

**Exact Methods**

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**Heuristic Methods**
(game tree is too large or search takes too long)

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# Cutting off search

Reduce the search cost by restricting the search depth:

1. Stop search at a non-terminal node.
2. Use a heuristic evaluation function $Eval(s)$ to approximate the utility for that node/state.

Needed properties of the evaluation function:

- Fast to compute.
- $Eval(s) \in [Utility(loss), Utility(win)]$
- Correlated with the actual chance of winning (e.g., using features of the state).

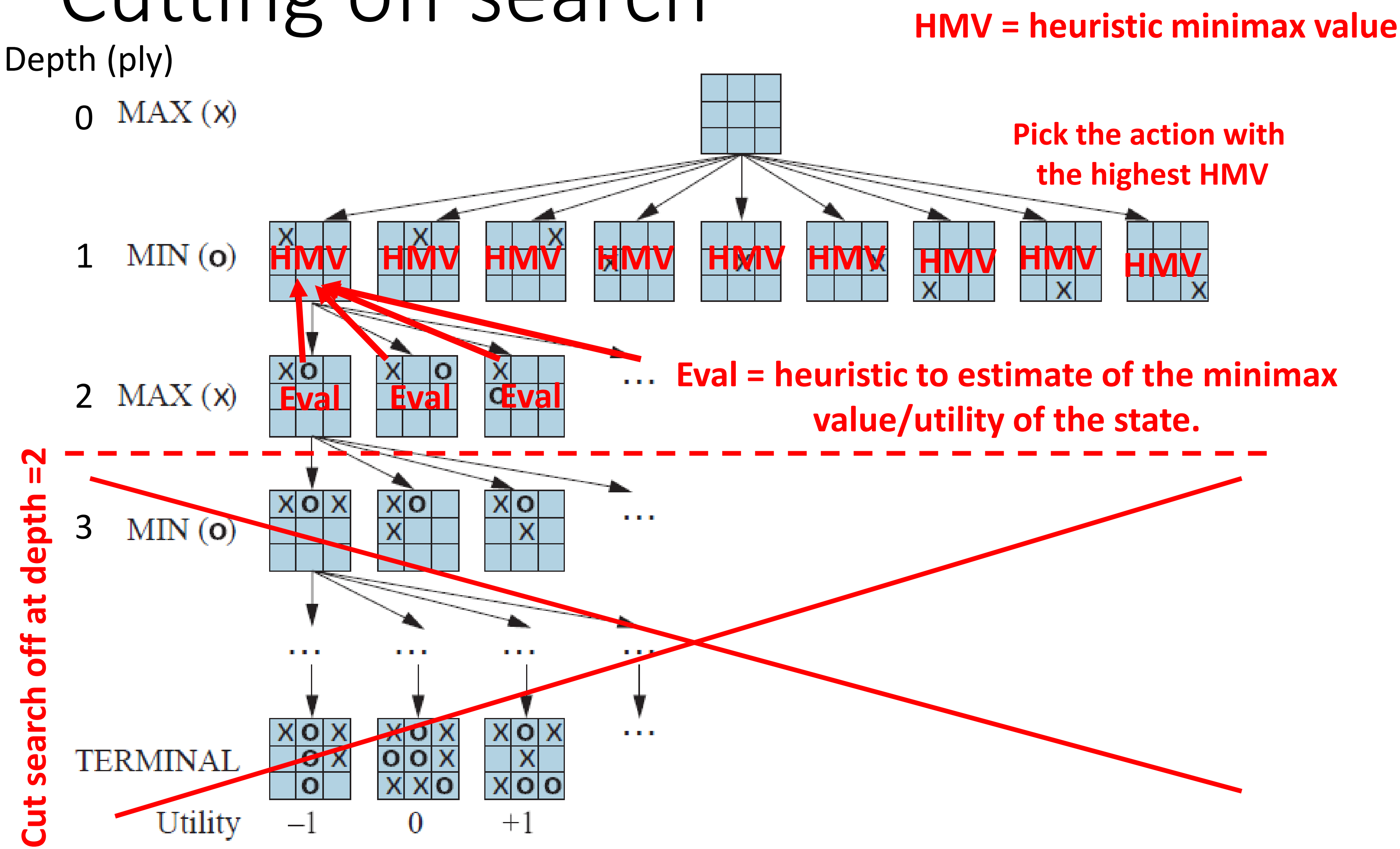
**Examples**:

1. A weighted linear function

   $$Eval(s) = w_1 f_1(s) + w_2 f_2(s) + \cdots + w_n f_n(s)$$

   where $f_i$ is a feature of the state (e.g., # of pieces captured in chess).
2. A deep neural network trained on complete games.

# Heuristic Alpha-Beta Tree Search: Cutting off search



# Forward pruning

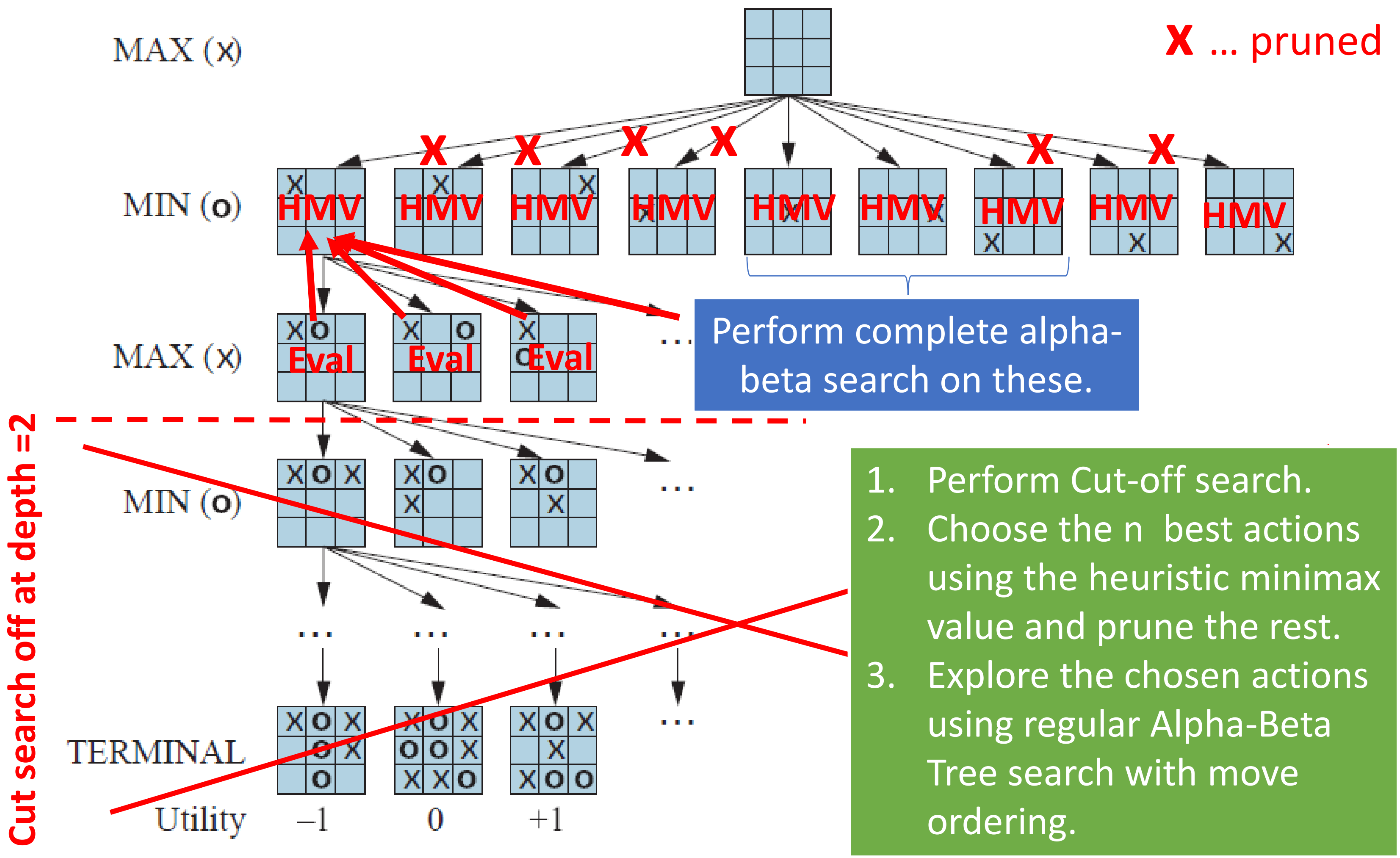
To save time, we can prune moves that appear bad.

There are many ways move quality can be evaluated:

- Low heuristic value.
- Low evaluation value after shallow search (cut-off search).
- Past experience.

**Issue**: May prune important moves.

# Heuristic Alpha-Beta Tree Search: Example for Forward Pruning



# Monte Carlo Tree Search (MCTS)

# Methods for Adversarial Games

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# Idea

- **Approximate $\boldsymbol{Eval(s)}$** as the average utility of several simulation runs to the terminal state (called playouts).

- **Playout policy**: How to choose moves during the simulation runs? Example policies:
    - Random.
    - Heuristics for good moves developed by experts.
    - Learn good moves from self-play (e.g., with deep neural networks). We will talk about this when we talk about "Learning from Examples."

- Typically used for problems with
    - High branching factor (many possible moves).
    - Unknown or hard to define good evaluation functions.

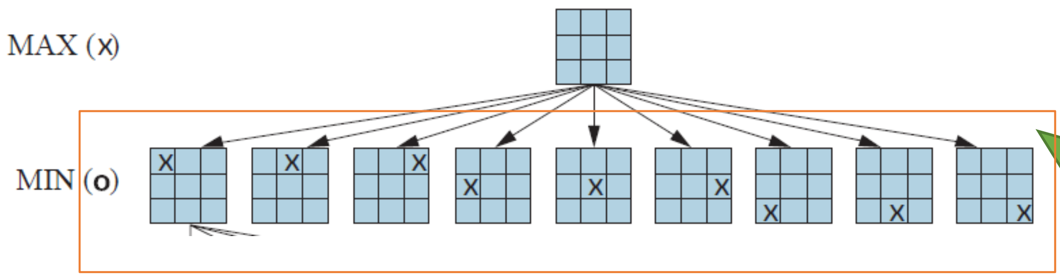
# Pure Monte Carlo Search

Find the next best move.

- Method
  1. Simulate $N$ playouts from the **current state**.
  2. Select the move that leads the highest win percentage.

- **Guarantee**: Converges to optimal play for stochastic games as $N$ increases.
- **Do as many playouts as you can** given the available time.

# Playout Selection Strategy




Max can start each playout at any of these states. Which one should it choose?

**Issue**: Pure Monte Carlo Search spends a lot of time to create playouts for bad move.

**Better:** Select the starting state for playouts to focus on important parts of the game tree.

This presents the following tradeoff between:

**Exploration**: perform more playouts from states that currently have no or few playouts.

**Exploitation**: more playouts for states that have done well to get more accurate estimates.

# Selection using Upper Confidence Bounds (UCB1)

Tradeoff constant $\approx \sqrt{2}$
can be optimizes using experiments

$$UCB1(n) = \frac{U(n)}{N(n)} + C\sqrt{\frac{\log N(Parent(n))}{N(n)}}$$

Average utility (=**exploitation**)

High for nodes with few playouts relative to the parent node (=**exploration**). Goes to 0 for large $N(n)$

$n$ ... node in the game tree
$U(n)$ ... total utility of all playouts going through node n
$N(n)$ ... number of playouts through n

**Selection strategy**: Select node with highest UCB1 score.

# Monte Carlo Tree Search

We do not need to always start playouts from the current node, we can build a **partial game tree** and simulate from any node in that tree.

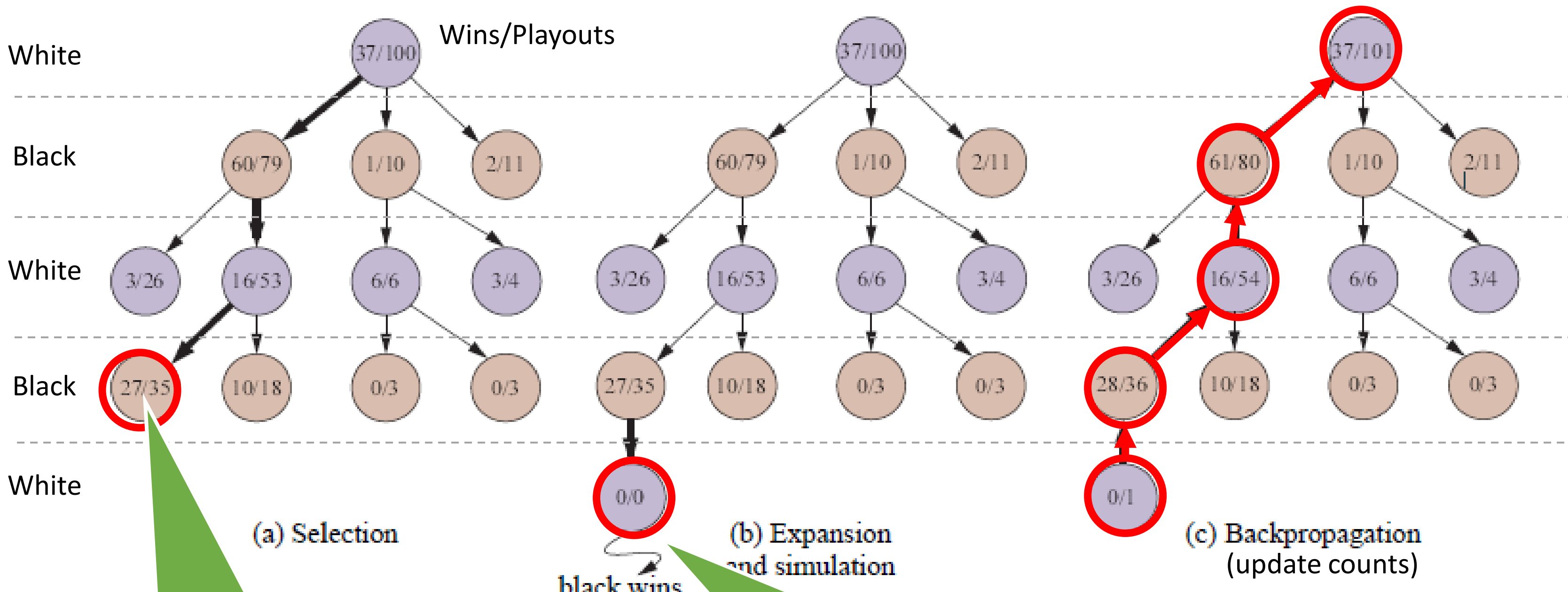
Important considerations:

- We can use UCB1 as the **selection strategy** to decide what part of the tree we should focus on for the next playout. This balances exploration and exploitation.
- We can only store a small **part of the game tree**, so we do not store the complete playout runs.

```
function MONTE-CARLO-TREE-SEARCH(state) returns an action
  tree ← NODE(state)
  while IS-TIME-REMAINING() do
    leaf ← SELECT(tree)
    child ← EXPAND(leaf)
    result ← SIMULATE(child)
    BACK-PROPAGATE(result, child)
  return the move in ACTIONS(state) whose node has highest number of playouts
```

Highest UCB1 score

UCB1 selection favors win percentage more and more.




(a) Selection

(b) Expansion and simulation

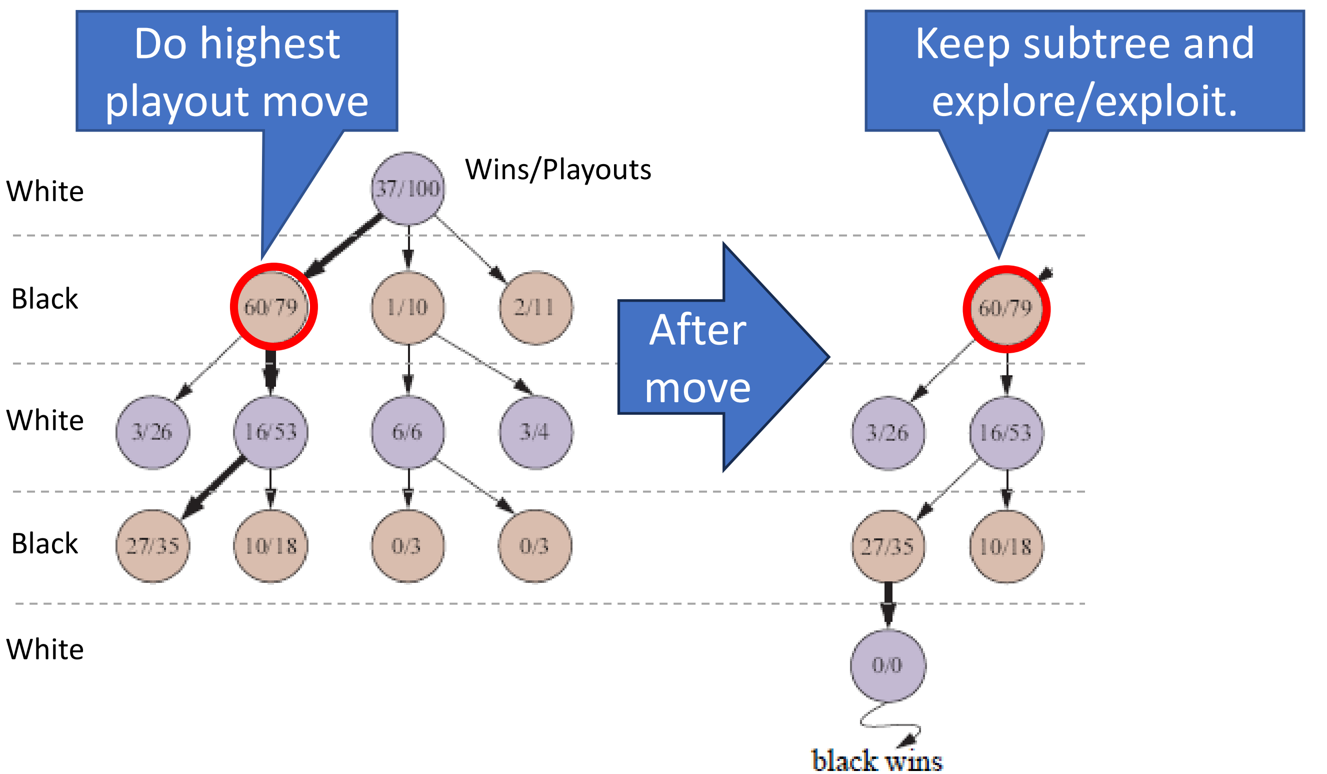
(c) Backpropagation (update counts)

Select leaf with highest UCB1 score

Note: the simulation path is not recorded to preserve memory!

# Online Play Using MCTS

- Search and update partial tree to use up the time budget for the move.
- Keep the relevant subtree from move to move and expand from there.

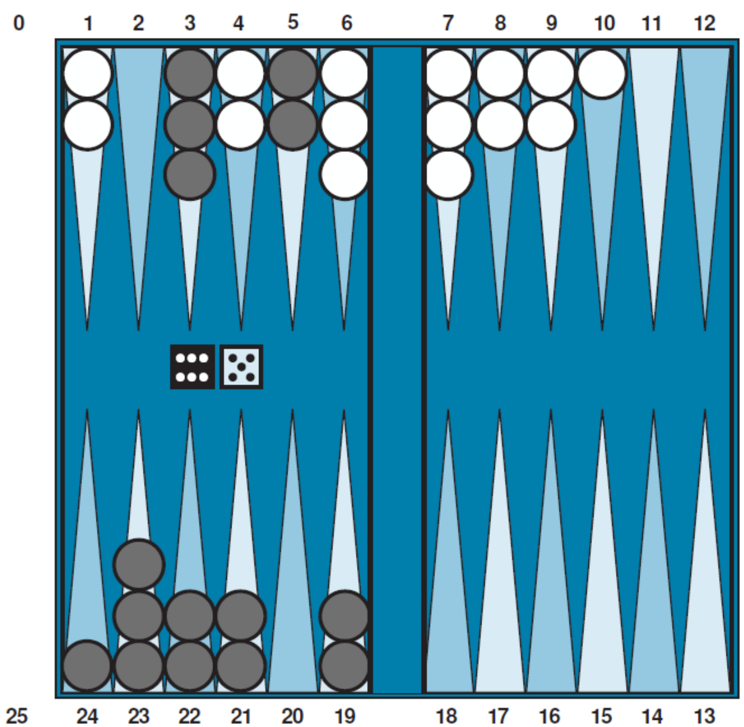


# Stochastic Games

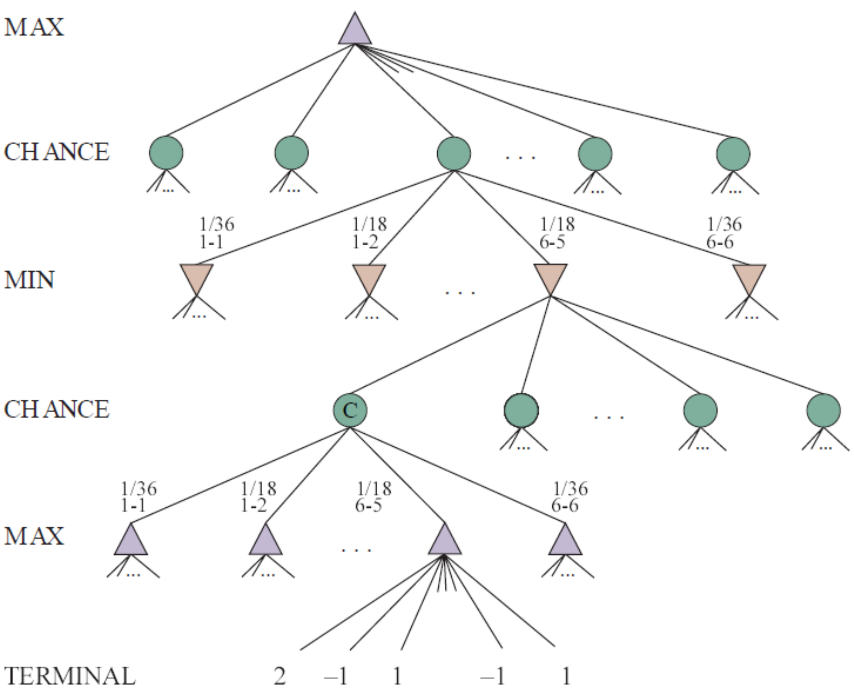
Games With Random Events

# Stochastic Games

- Game includes a “random action” $r$ (e.g., dice, dealt cards)
- Add **chance nodes** that calculate the expected value.




Backgammon



# Expectiminimax

- Game includes a “random action” $r$ (e.g., dice, dealt cards).
- For **chance nodes** we calculate the expected minimax value.

$$Expectiminimax(s) =$$

$$\begin{cases} Utility(s) & \text{if } terminal(s) \\ \max_{a \in Actions(s)} Expectiminimax(Result(s,a)) & \text{if } move = Max \\ \min_{a \in Actions(s)} Expectiminimax(Result(s,a)) & \text{if } move = Min \\ \boxed{\sum_r P(r) Expectiminimax(Result(s,r)) \qquad \text{if } move = Chance} \end{cases}$$

- Options:
  - Use Minimax algorithm. Issue: Search tree size explodes if the number of “random actions” is large. Think of drawing cards for poker!
  - Cut-off search and approximate Expectiminimax with an evaluation function.
  - Perform Monte Carlo Tree Search.

# Conclusion

**Nondeterministic actions**:

- The opponent is seen as part of an environment with nondeterministic actions. Non-determinism is the result of the unknown moves by the opponent. *All possible moves are considered*.

**Optimal decisions**:

- Minimax search and Alpha-Beta pruning where *each player plays optimal* to the end of the game.
- Choice nodes and Expectiminimax for stochastic games.

Scale only for tiny problems!

**Heuristic Alpha-Beta Tree Search**:

- Cut off game tree and use *heuristic evaluation function* for utility (based on state features).
- Forward Pruning: ignore poor moves.
- Learn heuristic from data using MCTS

**Monte Carlo Tree search**:

- Simulate complete games and calculate proportion of wins.
- Use modified UCB1 scores to expand the partial game tree.
- Learn playout policy using self-play and deep learning.

State of the Art