CS 5/7320
Artificial Intelligence

Solving problems by searching

AIMA Chapter 3

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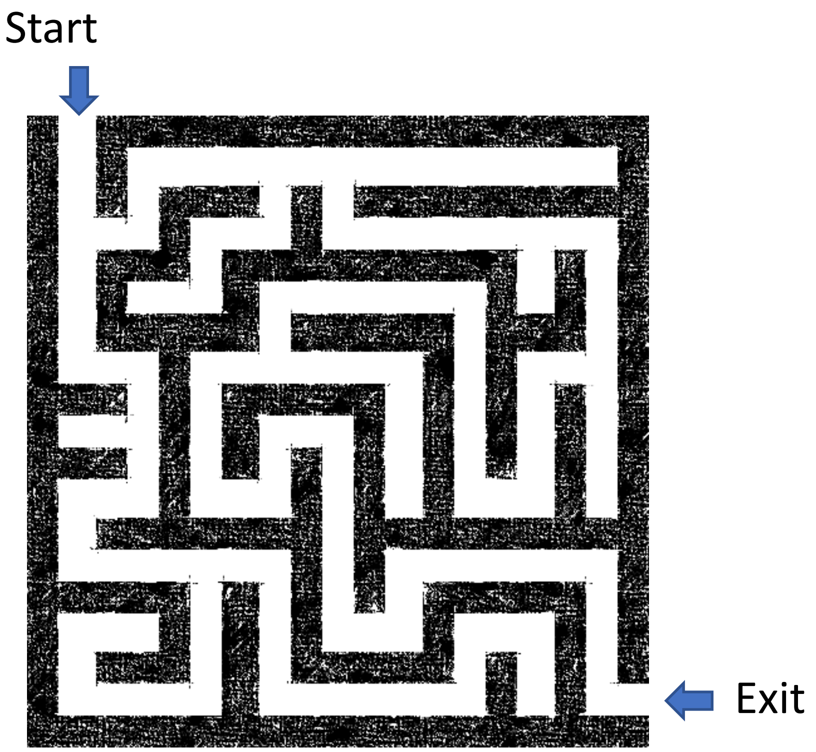
based on slides by Svetlana Lazepnik
with figures from the AIMA textbook.

Contents

- What are search problems?
- State space
- Tree search
- Uninformed search
- Informed search

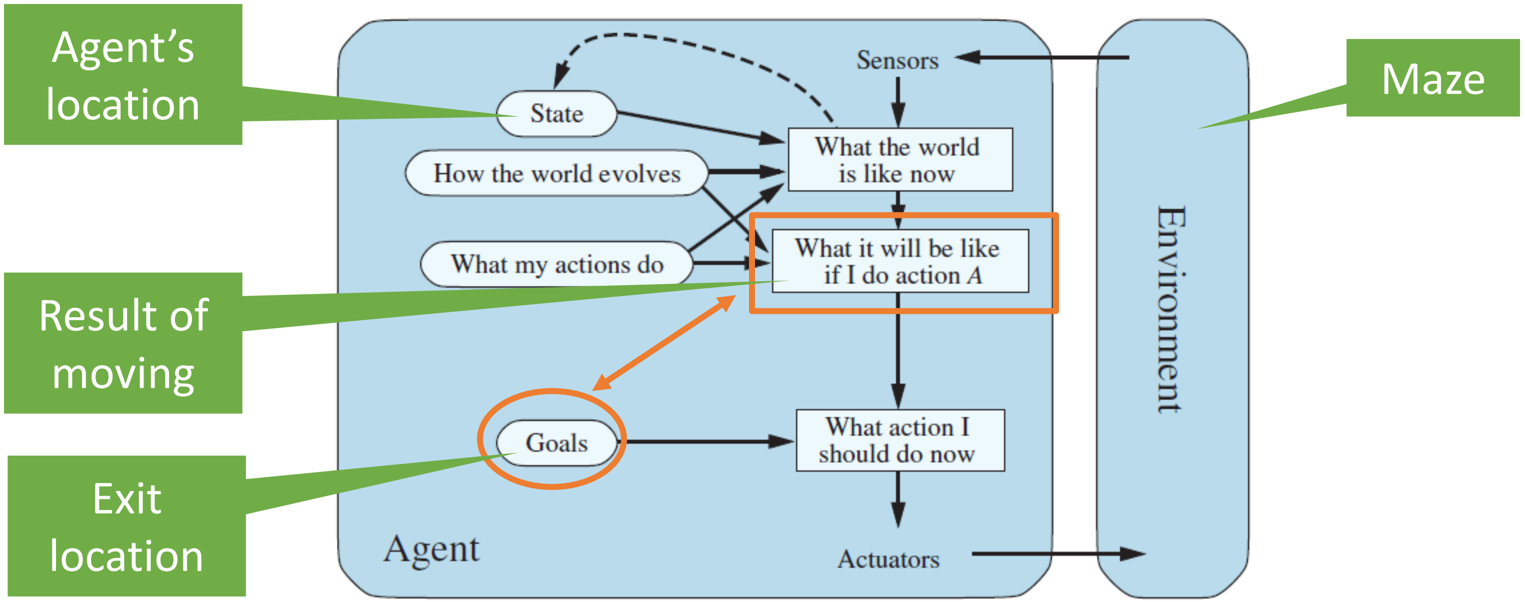
What are search problems?

- We will consider the problem of designing **goal-based agents** in **known, fully observable, and deterministic** environments.
- Example environment:



Remember: Goal-based Agent

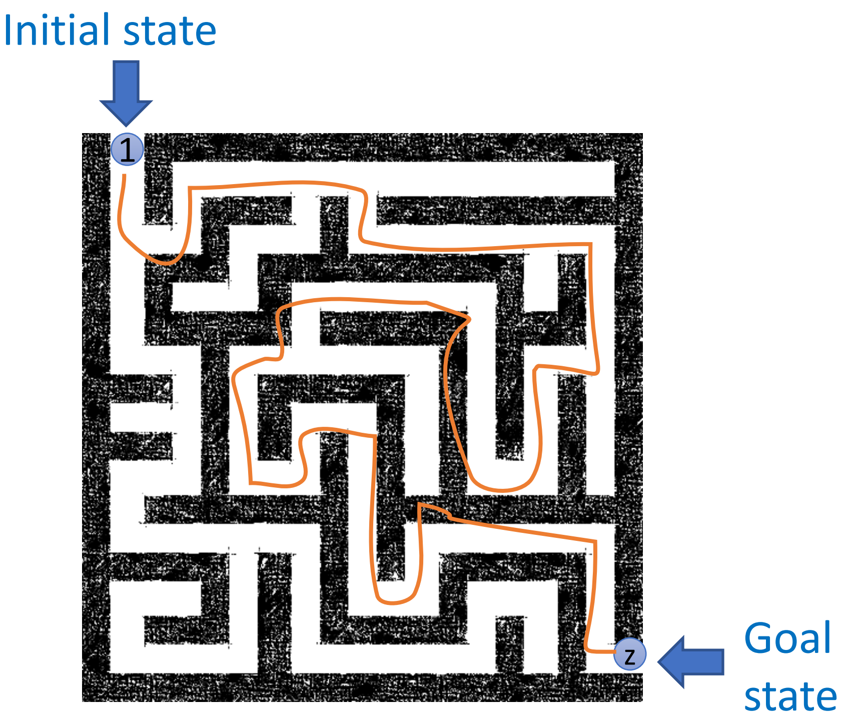
- The agent has the task to reach a defined **goal state**.
- The agent needs to move towards the goal. It can use **search algorithms** to plan actions that lead to the goal.
- The performance measure is typically the cost to reach the goal.



$$a_s = \mathrm{argmin}_{a \in A}[cost(s, s_1, s_2, \dots, s_n | s_n \in S^{goal})]$$

What are search problems?

- We will consider the problem of designing **goal-based agents** in**, known, fully observable, deterministic** environments.
- For now, we consider only a discrete environment using an **atomic state representation** (states are just labeled 1, 2, 3, …).
- The **state space** is the set of all possible states of the environment and some states are marked as **goal states**.

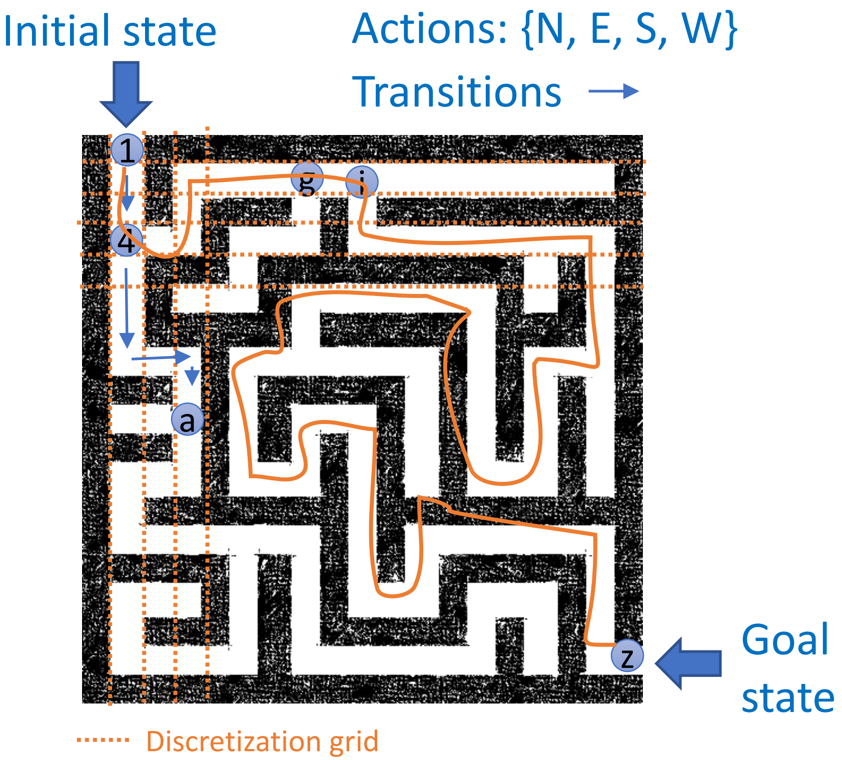


Phases:

1) **Search/Planning**: the process of looking for the **sequence of actions** that reaches a goal state. Requires that the agent knows what happens when it moves!
2) **Execution**: Once the agent begins executing the search solution in a deterministic, known environment, it can ignore its percepts (**open-loop system**).

Search problem components

- **Initial state:** state description
- **Actions:** set of possible actions A
- **Transition model:** a function that defines the new state resulting from performing an action in the current state
- **Goal state:** state description
- **Path cost:** the sum of *step costs*

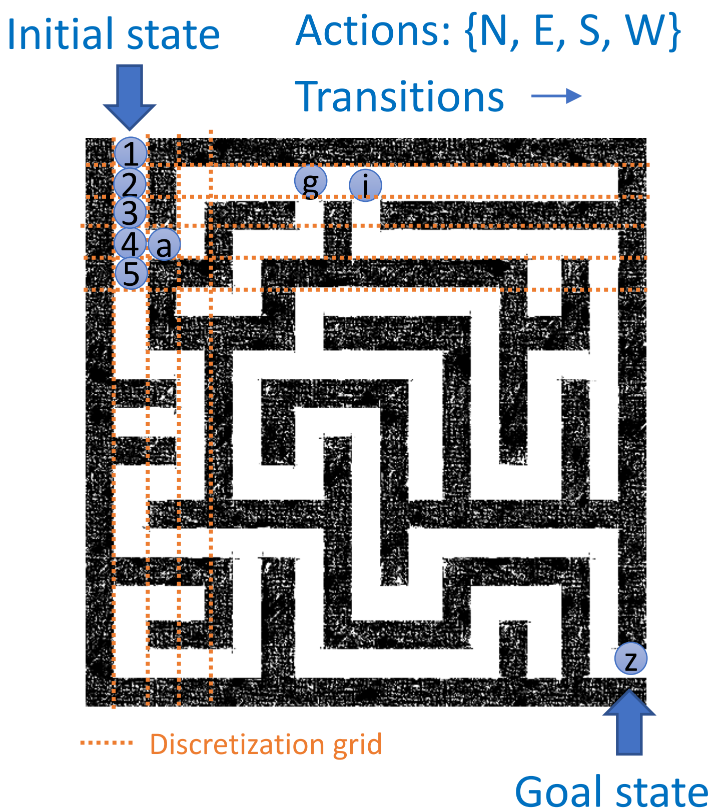


Notes:

- The **state space** is typically too large to be enumerated or it is continuous. Therefore, the problem is defined by initial state, actions and the transition model and not the set of all possible states.
- The **optimal solution** is the sequence of actions (or equivalently a sequence of states) that gives the lowest path cost for reaching the goal.

Transition function and available actions

Original Description



- As an action schema:
  $Action(go(dir))$
  PRECOND: no wall in direction dir
  EFFECT: change the agent's location according to dir
- As a function:
  $f: S \times A \rightarrow S$ or $s' = result(a, s)$

Function implemented as a table representing the state space as a graph.

s	a	s'
1	S	2
2	N	1
2	S	3
…	…	…
4	E	a
4	S	5
4	N	3
…	…	…

- Available actions in a state come from the transition function. E.g.,
  $actions(4) = \{E, S, N\}$

Example: Romania Vacation

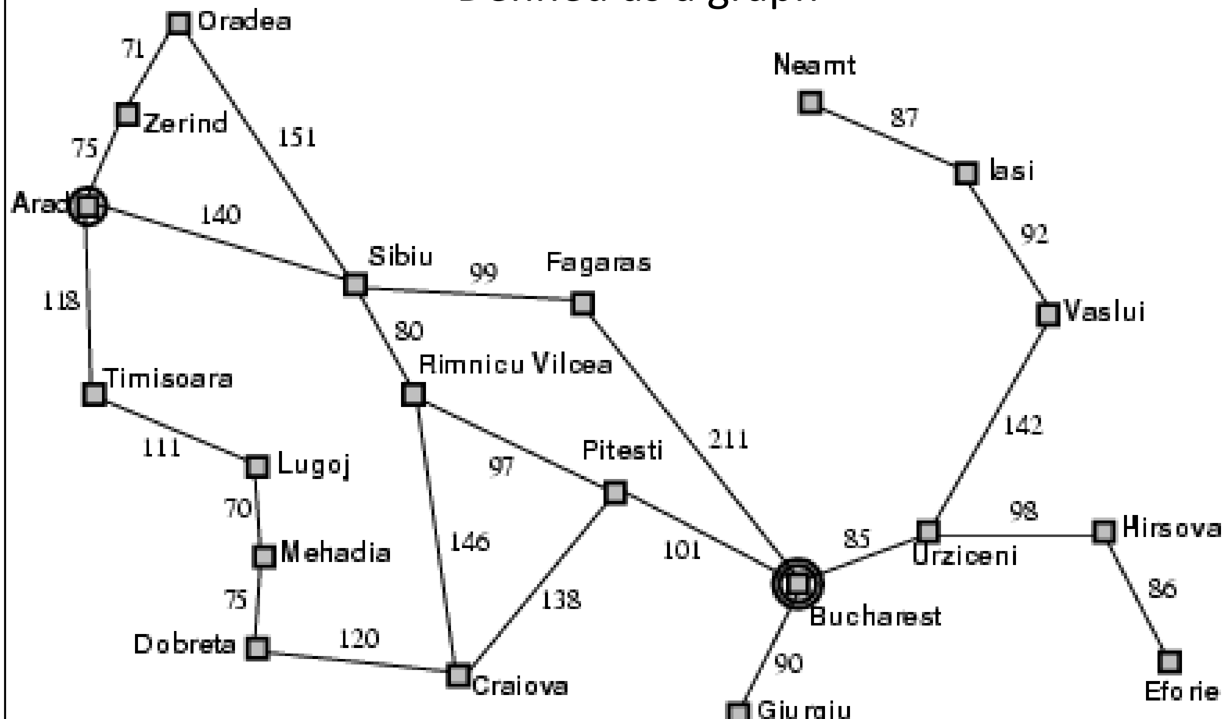
- On vacation in Romania; currently in Arad
- Flight leaves tomorrow from Bucharest

Original Description

- **Initial state:** Arad
- **Actions:** Drive from one city to another.
- **Transition model and states:** If you go from city A to city B, you end up in city B.
- **Goal state:** Bucharest
- **Path cost:** Sum of edge costs.

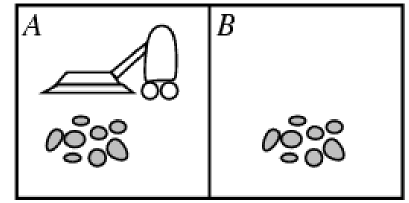
State Space/Transition model

Defined as a graph

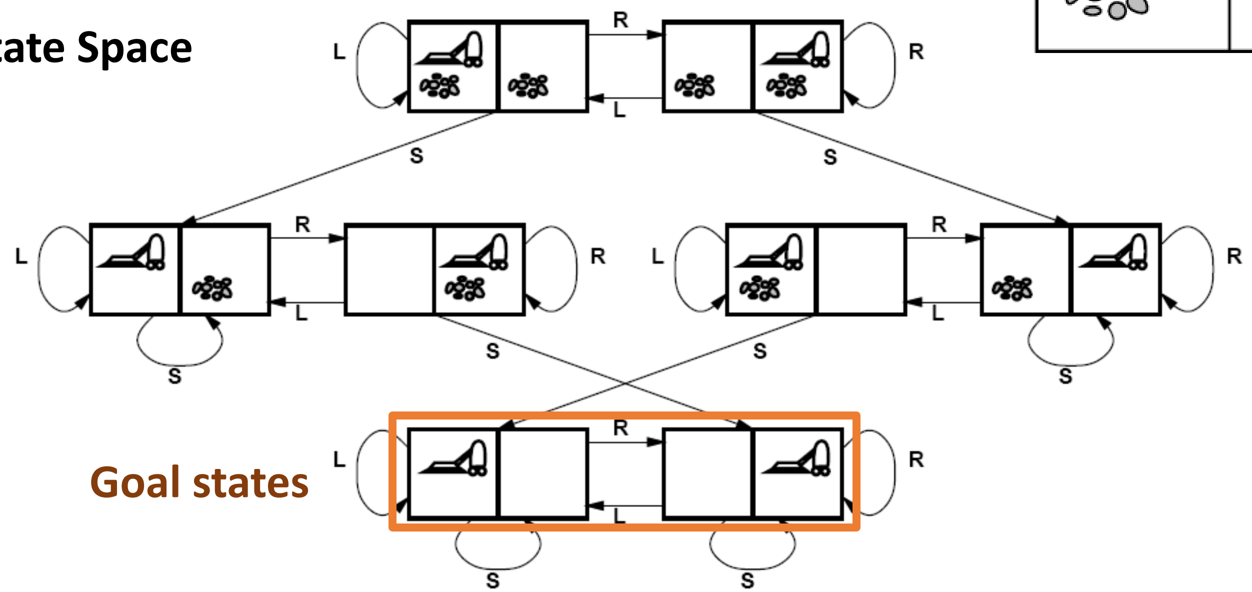


Distance in miles

Example: Vacuum world



State Space

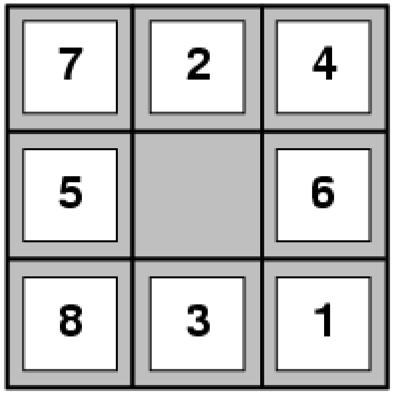


- **Initial State:** Defined by agent location and dirt location.
- **Actions:** Left, right, suck
- **Transition model:** Clean a location or move.
- **Goal state:** All locations are clean.
- **Path cost:** E.g., number if actions

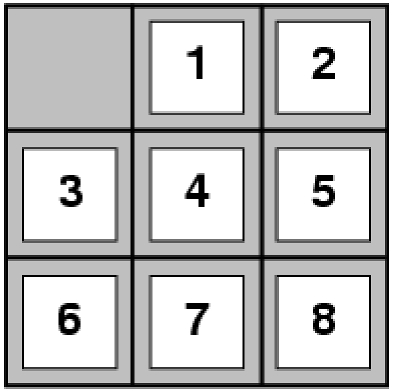
There are 8 possible atomic states of the system.
Why is the number of states for *n* possible locations $n(2^n)$?

Example: Sliding-tile puzzle

- **Initial State:** A given configuration.
- **Actions:** Move blank left, right, up, down
- **Transition model:** Move a tile
- **Goal state:** Tiles are arranged empty and 1-8 in order
- **Path cost:** 1 per tile move.



Start State



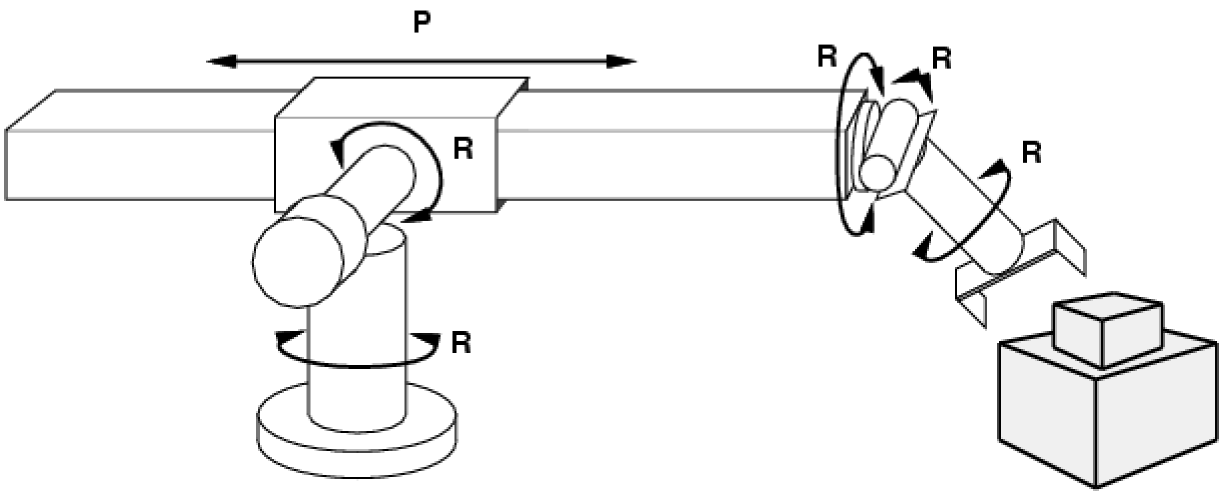
Goal State

State space size

Each state describes the location of each tile (including the empty one). ½ of the permutations are unreachable.

- 8-puzzle: $9!/2 = 181{,}440$ states
- 15-puzzle: $16!/2 \approx 10^{13}$ states
- 24-puzzle: $25!/2 \approx 10^{25}$ states

Example: Robot motion planning



- **Initial State:** Current arm position.
- **States:** Real-valued coordinates of robot joint angles.
- **Actions:** **Continuous** motions of robot joints.
- **Goal state:** Desired final configuration (e.g., object is grasped).
- **Path cost:** Time to execute, smoothness of path, etc.

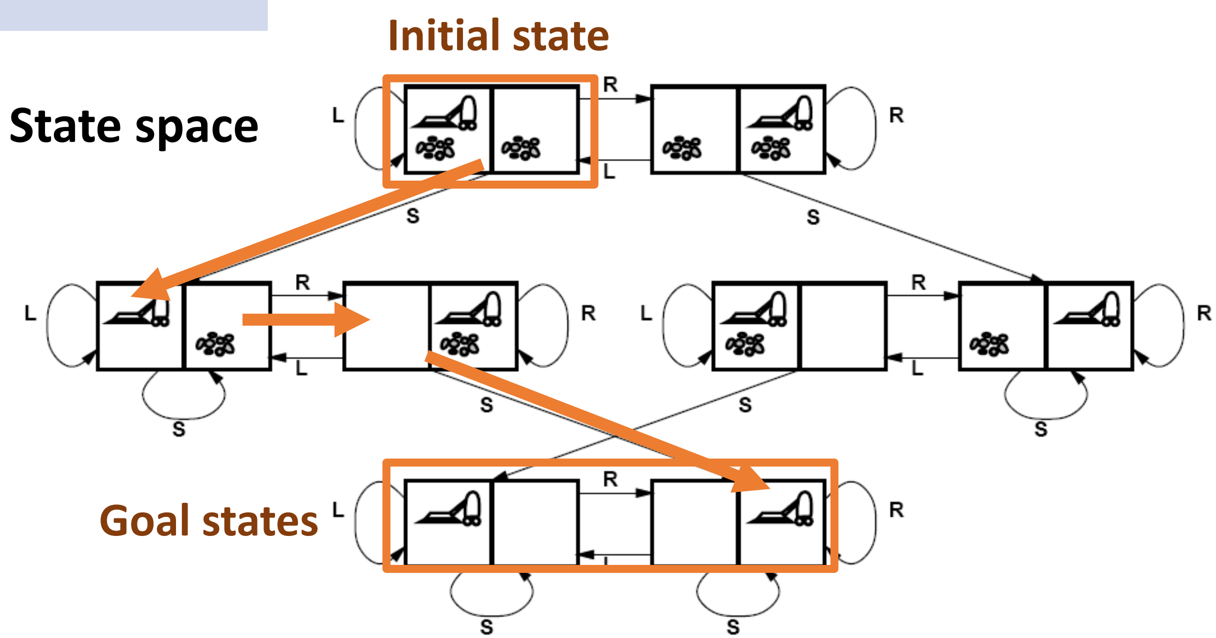
Solving search problems

Given a search problem definition

- Initial state
- Actions
- Transition model
- Goal state
- Path cost

How do we find the optimal solution (sequence of actions/states)?

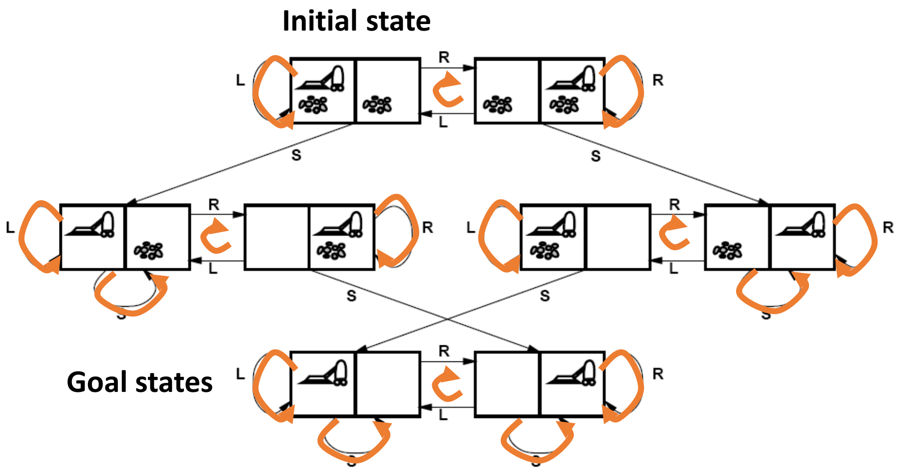
Construct a search tree for the state space graph!



Issue: Transition model is not a tree!
It has cycles vs. redundant paths

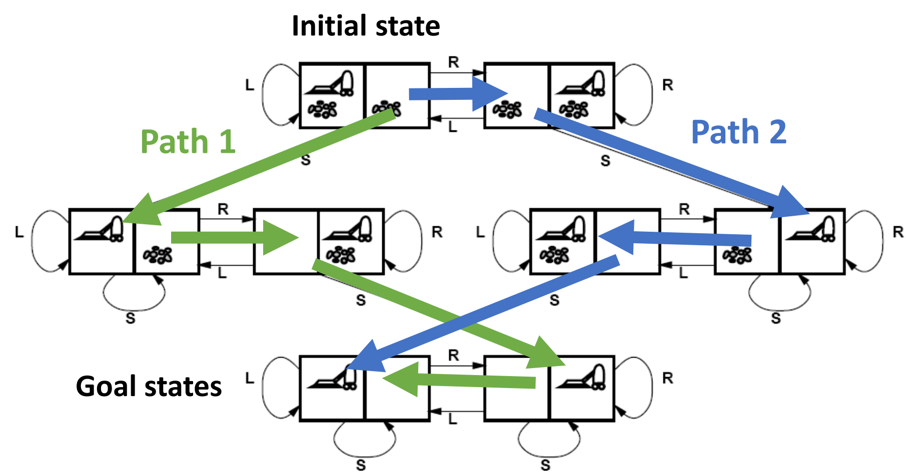
Cycles

Return to the same state. The search tree will create a new node!



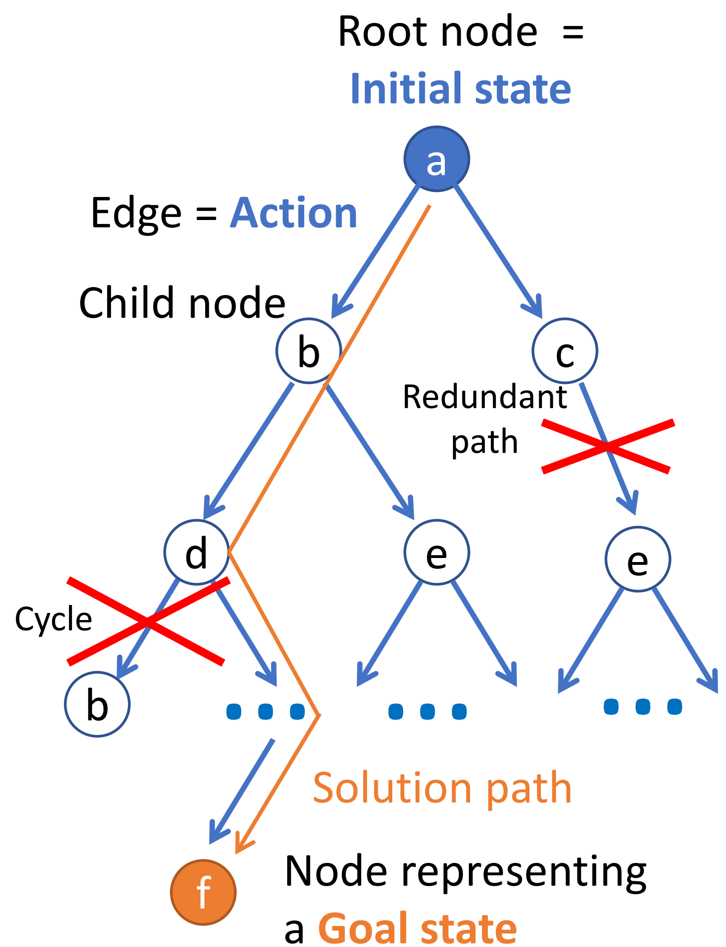
Redundant paths

Multiple paths to get to the same state



Search tree

- Superimpose a "what if" tree of possible actions and outcomes (states) on the state space graph.
- The **Root node** represents the initial stare.
- An action child node is reached by an **edge** representing an action. The corresponding state is defined by the transition model.
- Trees have no **cycles** or **redundant paths**. Cycles in the search space must be broken. To prevent infinite loops. Removing redundant paths improves search efficiency.
- A **path** through the tree corresponds to a sequence of actions (states).
- A **solution** is a path ending in a node representing a goal state.
- **Nodes vs. states**: Each tree node represents a state of the system. If redundant path cannot be prevented then state can be represented by multiple nodes.



Differences between typical Tree search and AI search

Typical tree search

- Assumes a given tree that fits in memory.

- Trees have by construction no cycles or redundant paths.

AI tree/graph search

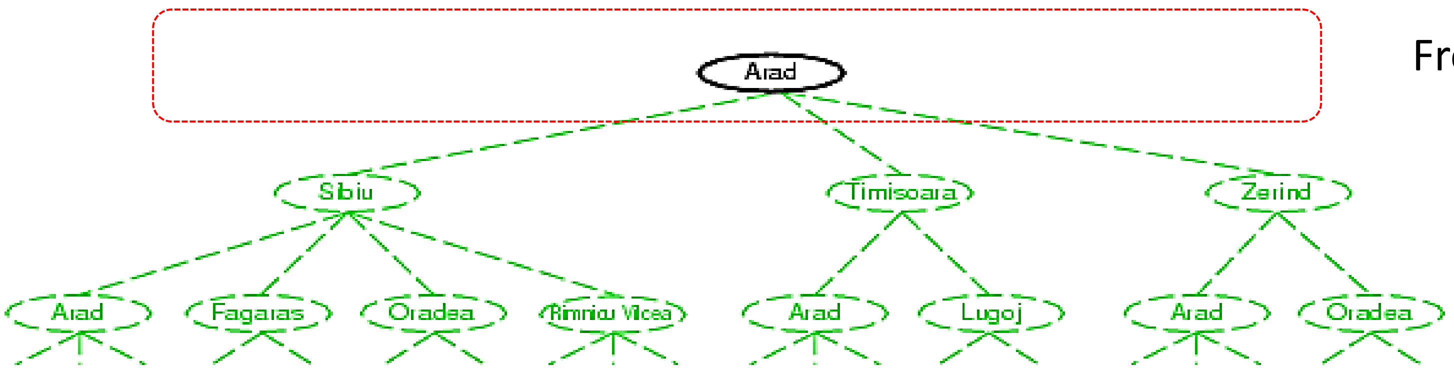
- The search tree is too large to fit into **memory**.
  a. **Builds parts of the tree** from the initial state using the transition function representing the graph.
  b. **Memory management** is very important.

- The search space is typically a very large and complicated graph. Memory-efficient **cycle checking** is very important to avoid infinite loops or minimize searching parts of the search space multiple times.
- Checking redundant paths often requires too much memory and we accept searching the same part multiple times.

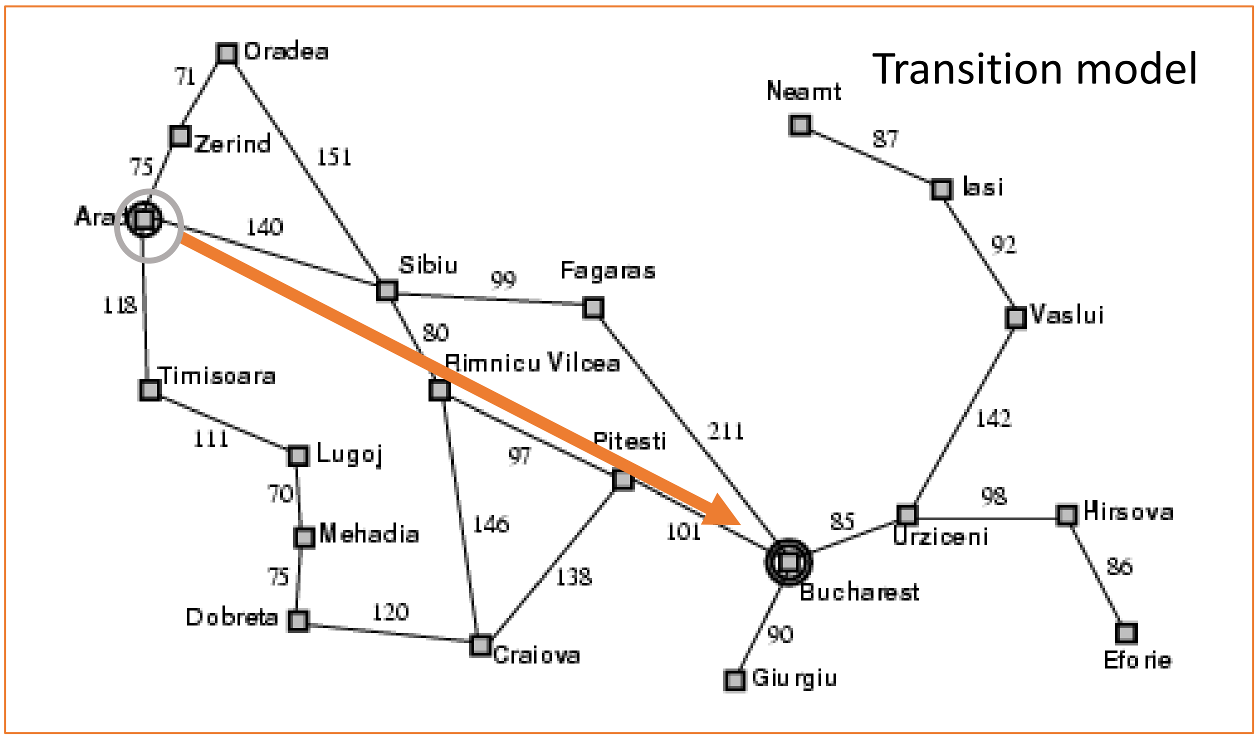
Tree Search Algorithm Outline

1. Initialize the **frontier** (set of unexplored know nodes) using the **starting state/root node.**
2. While the frontier is not empty:
   a) Choose next frontier node to expand according to **search strategy.**
   b) If the node represents a **goal state,** return it as the solution.
   c) Else **expand** the node (i.e., apply all possible actions to the transition model) and add its children nodes representing the newly reached states to the frontier.

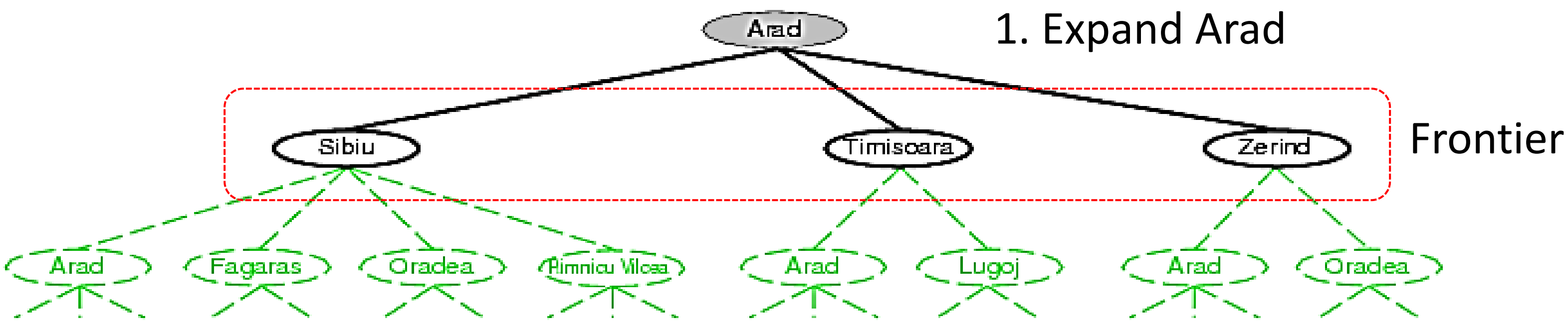
Tree search example

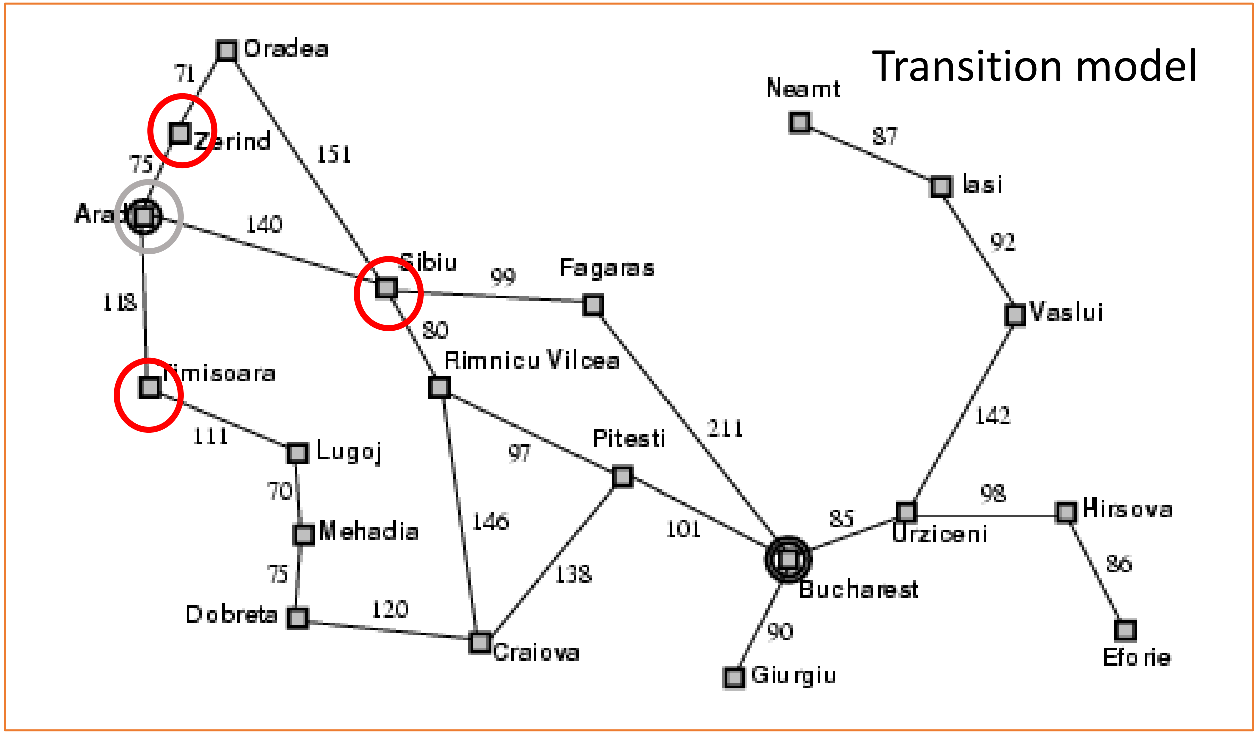


Frontier

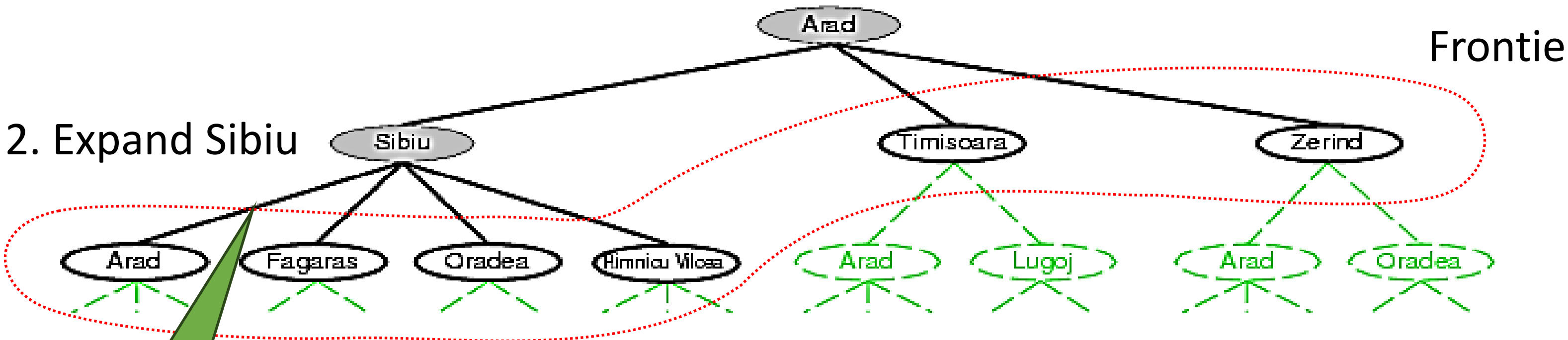


Tree search example

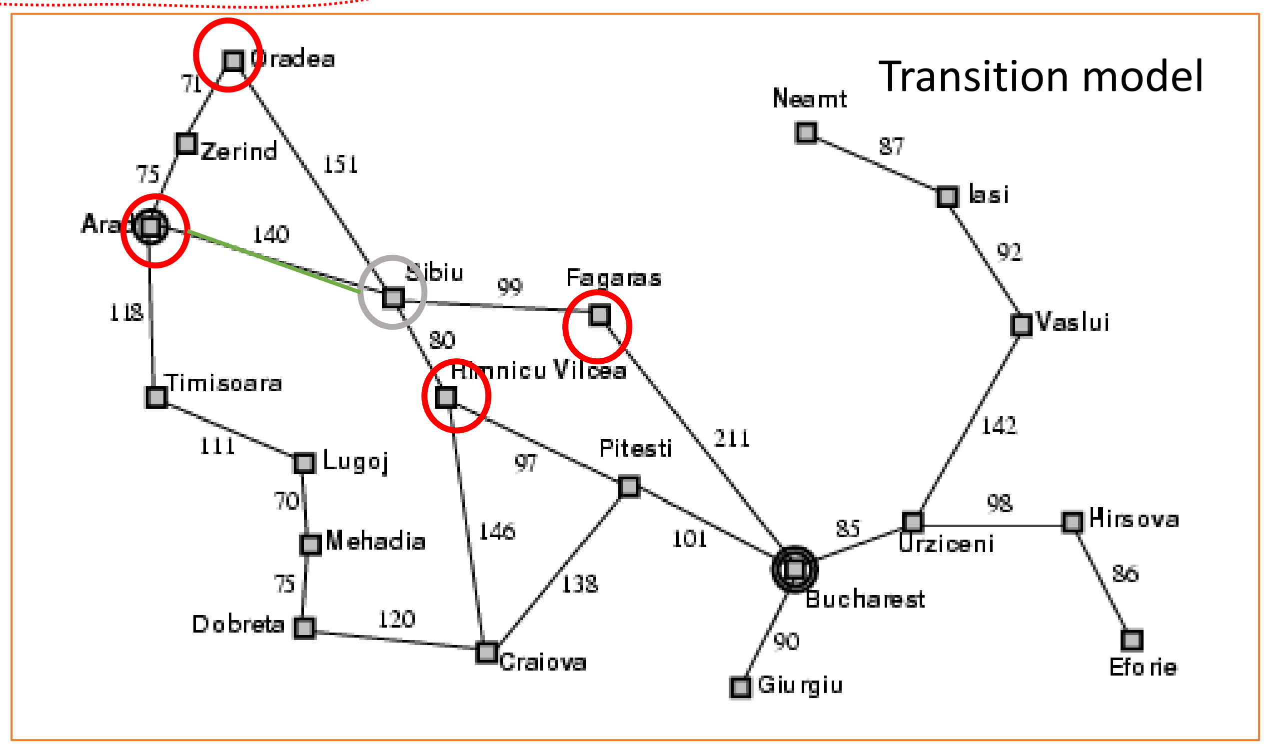




Tree search example



Example of a cycle



We could have also expanded Timisoara or Zerind!

Search strategies

- A **search strategy** is defined by picking the **order of node expansion**.

- Strategies are evaluated along the following dimensions:
  - **Completeness:** does it always find a solution if one exists?
  - **Optimality:** does it always find a least-cost solution?
  - **Time complexity:** how long does it take?
  - **Space complexity:** how much memory does it need?

- Worst case time and space complexity are measured in terms of the **size of the state space *n*** (= number of nodes in the search tree).

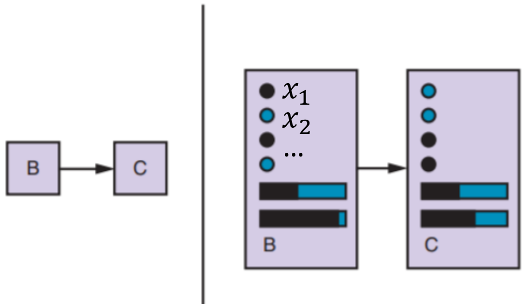
  Metrics used if the state space is only implicitly defined by initial state, actions and a transition function are:
  - *d:* depth of the optimal solution (= number of actions needed)
  - *m:* the number of actions in any path (may be infinite with loops)
  - *b:* maximum branching factor of the search tree (number of successor nodes for a parent)

State Space for Search

State Space

- Number of different states the agent and environment can be in.
- **Reachable states** are defined by the initial state and the transition model. Not all states may be reachable from the initial state.
- **Search tree** spans the state space. Note that a single state can be represented by several search tree nodes if we have redundant paths.
- State space size is an indication of problem size.

State representation



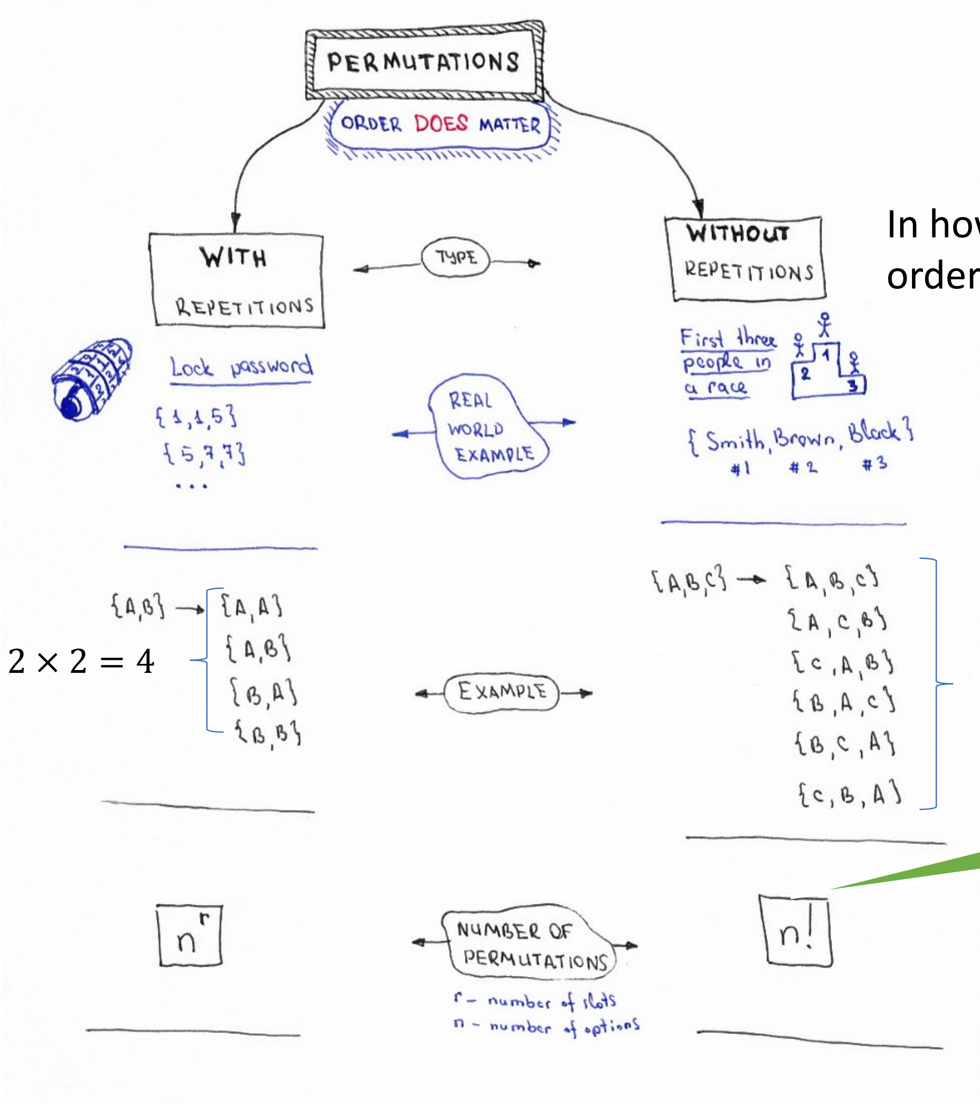
(a) Atomic (b) Factored

State Space Size Estimation

- Even if the used algorithm represents the state space using atomic states, we may know that internally they have a factored representation that can be used to estimate the problem size.
- The basic rule to calculate (estimate) the state space size for factored state representation with n variables is:

$$|x_1| \times |x_2| \times \cdots \times |x_n|$$

where $|\cdot|$ is the number of possible values.



In how many ways can we order/arrange n objects?

$2 \times 2 = 4$

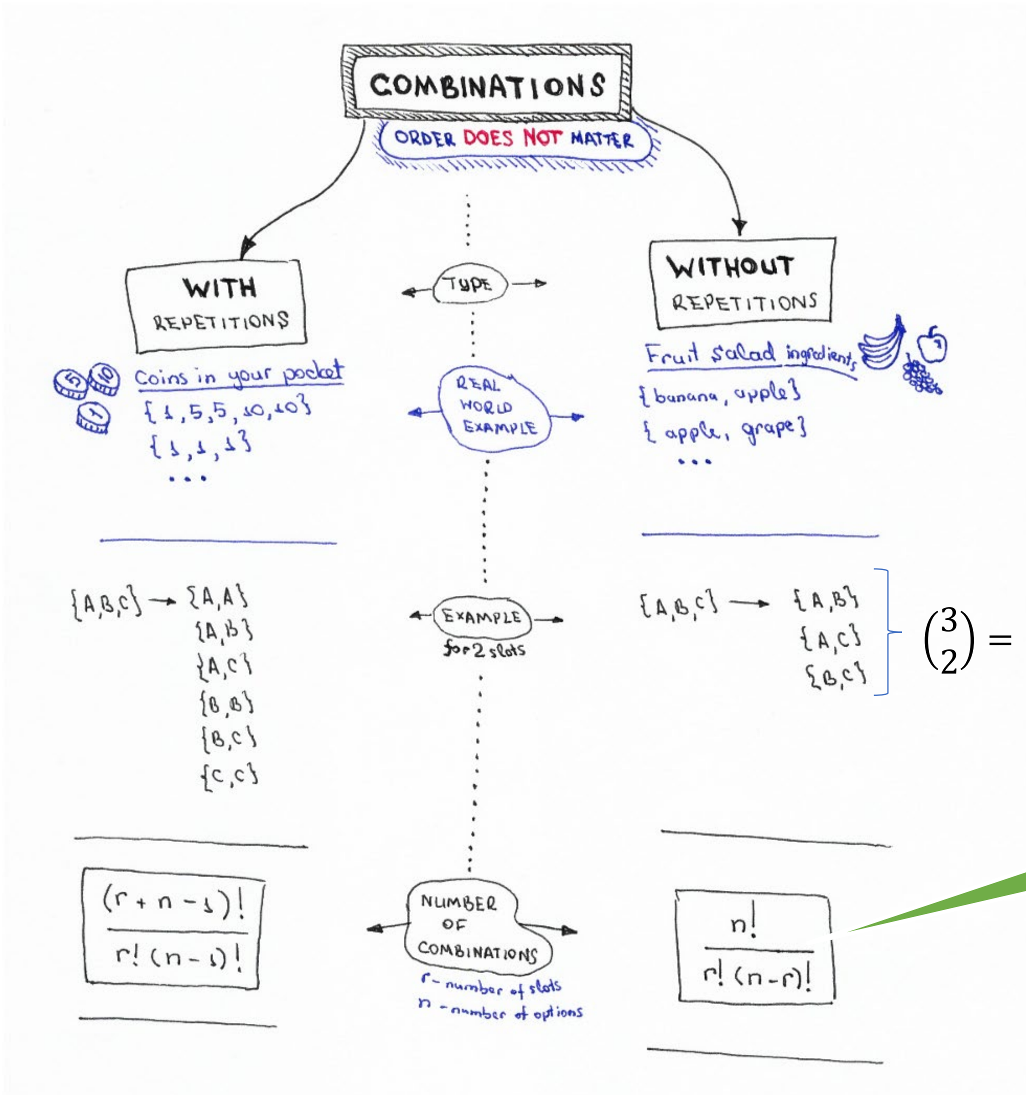
$3 \times 2 \times 1 = 6$

Factorial: $n! = n \times (n-1) \times \cdots \times 2 \times 1$

#Python
import math

print (math.factorial(23))

Source: Permutations/Combinations Cheat Sheets by Oleksii Trekhleb
https://itnext.io/permutations-combinations-algorithms-cheat-sheet-68c14879aba5



$$\binom{3}{2} = 3$$

Binomial Coefficient: $\binom{n}{r} = C(n,r) = {}_nC_r$

Read as "n choose r" because it is the number of ways can we choose r out of n objects?

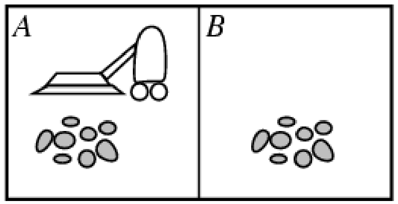
Special case for $r = 2$: $\binom{n}{2} = \frac{n(n-1)}{2}$

```
#Python
import scipy.special

# the two give the same results
scipy.special.binom(10, 5)
scipy.special.comb(10, 5)
```

Source: Permutations/Combinations Cheat Sheets by Oleksii Trekhleb
https://itnext.io/permutations-combinations-algorithms-cheat-sheet-68c14879aba5

Examples: What is the state space size?



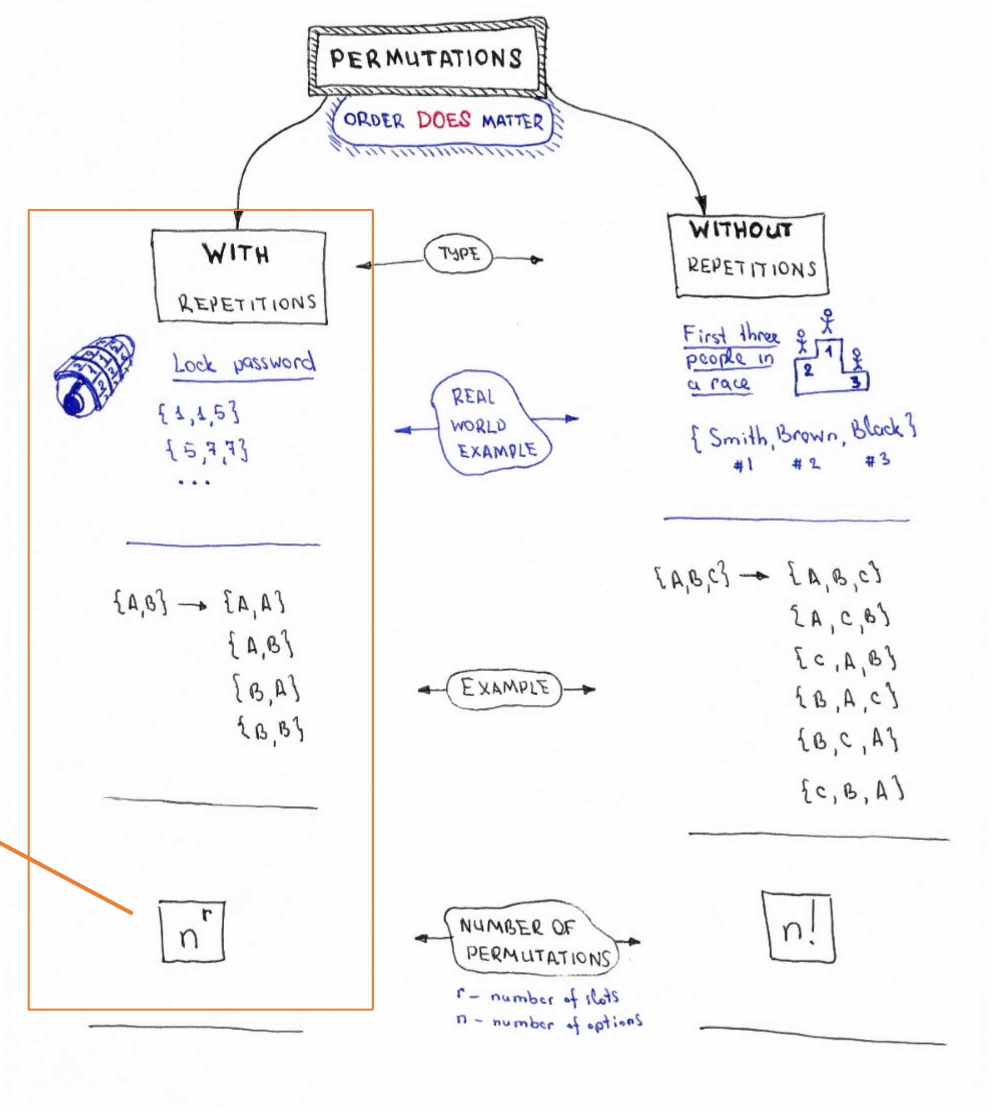
Dirt
- Order of A and B does not matter!
- Repetition: Dirt can be in both places
- There are 2 options (clean/dirty)

$\rightarrow 2^2$

Robot
- Can be in 1 out of 2 squares.

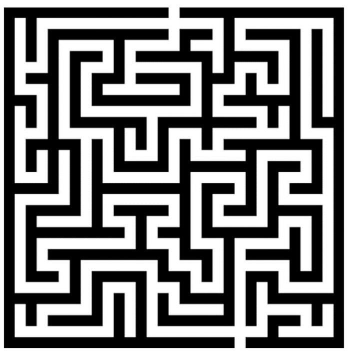
$\rightarrow 2$

Total: $2 \times 2^2 = 2^3$

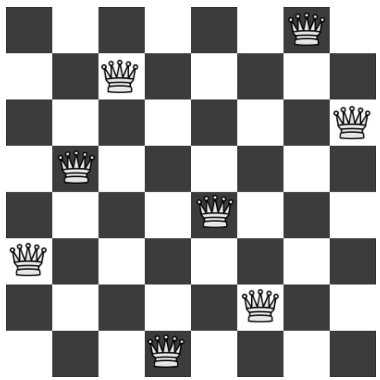


Examples: What is the state space size?

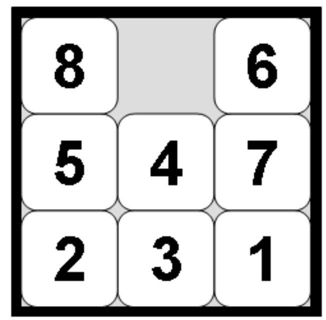
Often a rough upper limit is sufficient to determine how hard the search problem is.



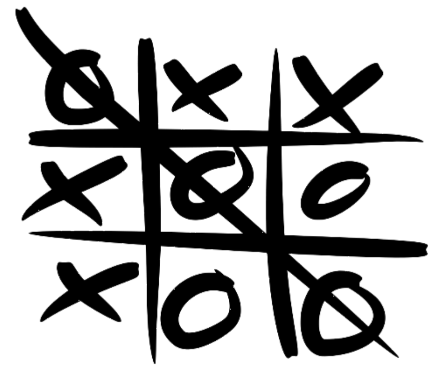
Maze



8-queens problem



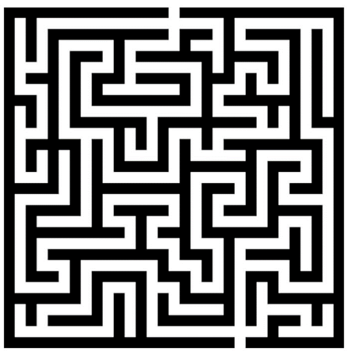
8-puzzle problem



Tic-tac-toe

Examples: What is the state space size?

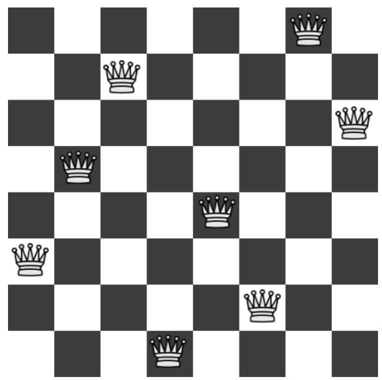
Often a rough upper limit is sufficient to determine how hard the search problem is.



Maze

Positions the agent can be in.

n = Number of white squares.



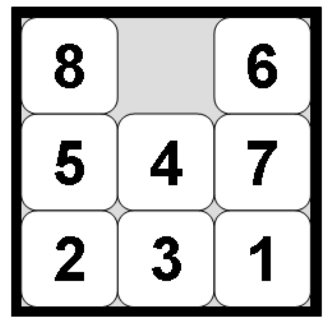
8-queens problem

All arrangements with 8 queens on the board.

$$n < 2^{64} \approx 1.8 \times 10^{19}$$

We can only have 8 queens:

$$n = \binom{64}{8} \approx 4.4 \times 10^9$$



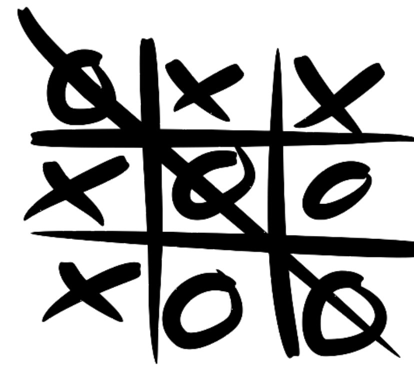
8-puzzle problem

All arrangements of 9 elements.

$$n \leq 9!$$

Half is unreachable:

$$n = \frac{9!}{2} = 181{,}440$$



Tic-tac-toe

All possible boards.

$$n < 3^9 = 19{,}683$$

Many boards are not legal (e.g., all x's)

Uninformed Search

Uninformed search strategies

The search algorithm/agent is **not** provided information about how close a state is to the goal state.

It blindly searches following a simple strategy until it finds the goal state by chance.

Search strategies we will discuss:

- Breadth-first search
- Uniform-cost search
- Depth-first search
- Iterative deepening search

Breadth-first search (BFS)

Expansion rule: Expand shallowest unexpanded node in the frontier (=**FIFO**).

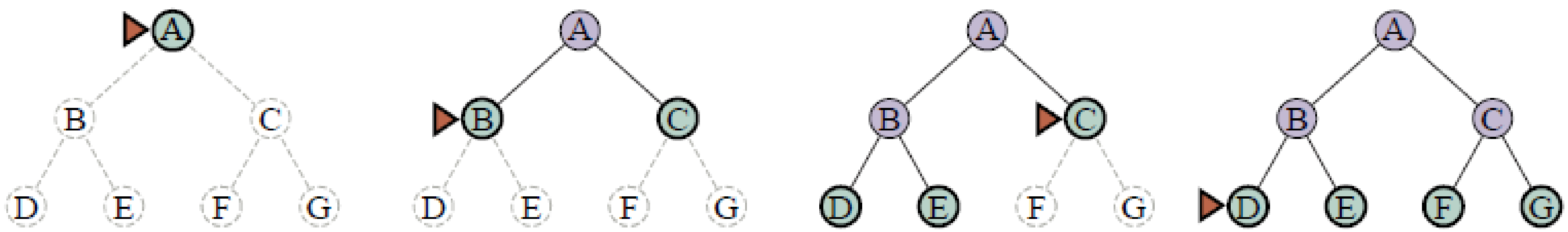


Figure 3.8 Breadth-first search on a simple binary tree. At each stage, the node to be expanded next is indicated by the triangular marker.

Data Structures

- **Frontier** data structure: holds references to the green nodes (green) and is implemented as a FIFO **queue**.
- **Reached** data structure: holds references to all visited nodes (gray and green) and is used to prevent visiting nodes more than once (cycle checking).
- Builds a **tree** with links from parent to child.

Implementation: BFS

```
function BREADTH-FIRST-SEARCH(problem) returns a solution node or failure
  node ← NODE(problem.INITIAL)
  if problem.IS-GOAL(node.STATE) then return node
  frontier ← a FIFO queue, with node as an element
  reached ← {problem.INITIAL}
  while not IS-EMPTY(frontier) do
    node ← POP(frontier)
    for each child in EXPAND(problem, node) do
      s ← child.STATE
      if problem.IS-GOAL(s) then return child
      if s is not in reached then
        add s to reached
        add child to frontier
  return failure
```

Expand adds the next level below node to the search tree.

reached makes sure we do not visit nodes twice (e.g., in a cycle or a redundant path). Fast lookup is important.

Implementation: Expanding the search tree

- AI tree search creates the search tree while searching.
- The EXPAND function uses the current search tree node (i.e., current state) and the problem description to create new nodes for all reachable states.
- It tries all actions in the current state by checking the transition function (RESULTS) and then returns a list of new nodes for the frontier.

function EXPAND(*problem*, *node*) **yields** nodes
  $s \leftarrow node.$STATE
  for each *action* **in** $problem.$ACTIONS(s) **do**
    $s' \leftarrow problem.$RESULT(s, *action*)
    $cost \leftarrow node.$PATH-COST + $problem.$ACTION-COST(s, *action*, s')
    yield NODE(STATE=s', PARENT=*node*, ACTION=*action*, PATH-COST=*cost*)

Transition function

Node structure for the search tree. Yield can also be implemented by returning a list of Nodes.

Properties of Breadth-first search

d: depth of the optimal solution
m: max. depth of tree
b: maximum branching factor

- **Complete?**
  Yes

- **Optimal?**
  Yes – if cost is the same per step (action). Otherwise: Use uniform-cost search.

- **Time?**
  Sum of the number of nodes created in at each level in a *b*-ary tree of depth *d*:
  $$1 + b + b^2 + \cdots + b^d = O(b^d)$$

- **Space?**
  Stored nodes: $O(b^d)$

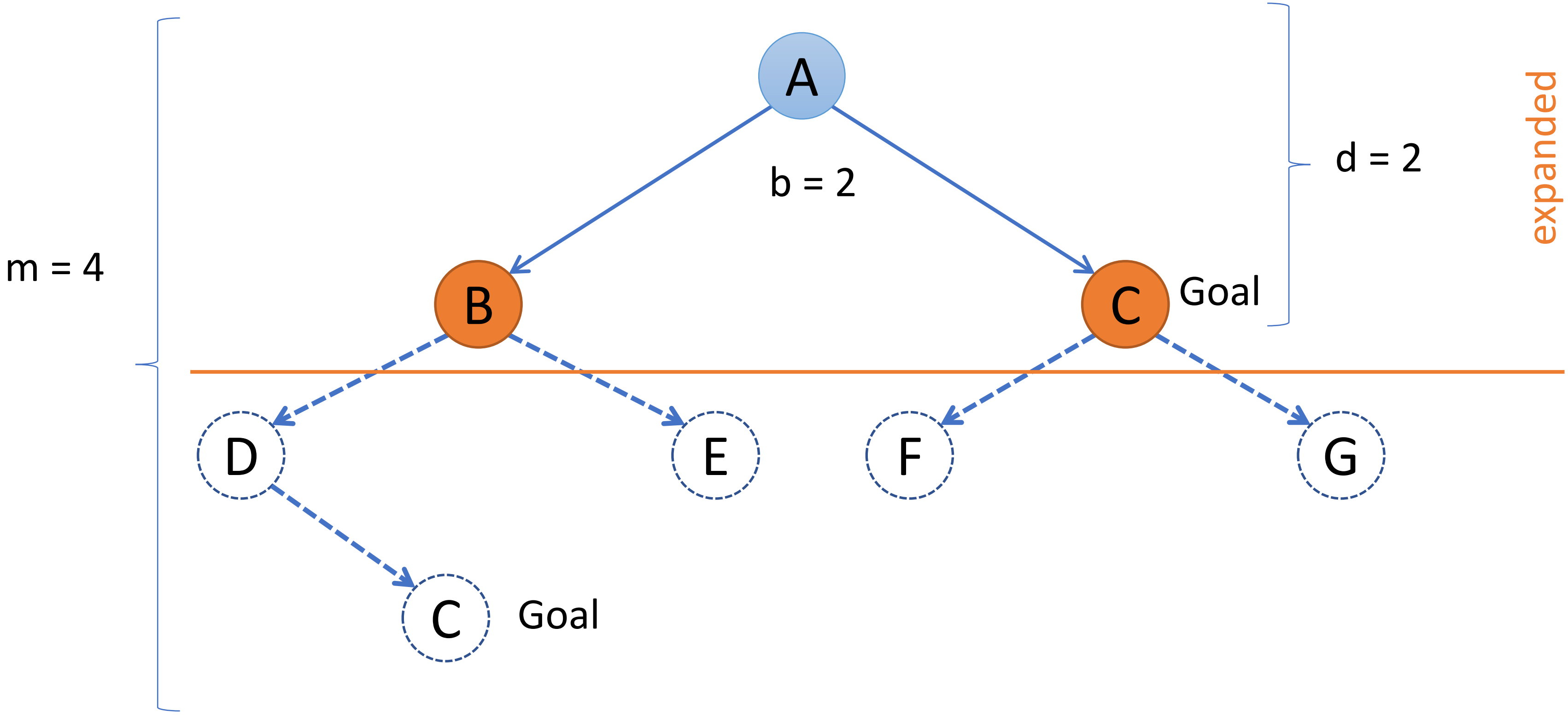
Note:

- The large space complexity is usually a bigger problem than time!

Breadth-first search

d: depth of the optimal solution
m: max. depth of tree
b: maximum branching factor

- Time and Space: $O(b^d)$ - all paths to the depth of the goal are expanded



Uniform-cost search
(= Dijkstra's shortest path algorithm)

- **Expansion rule**: Expand node in the frontier with **the least path cost** from the initial state.
- Implementation: **best-first search** where the frontier is a **priority queue** ordered by lower $f(n)$ = **path cost** (cost of all actions starting from the initial state).
- Breadth-first search is a special case when all step costs being equal, i.e., each action costs the same!

d: depth of the optimal solution
m: max. depth of tree
b: maximum branching factor

- **Complete?**
  Yes, if all step cost is greater than some small positive constant $\varepsilon > 0$
- **Optimal?**
  Yes – nodes expanded in increasing order of path cost
- **Time?**
  Number of nodes with path cost ≤ cost of optimal solution (C^*) is $O(b^{1+C^*/\varepsilon})$.
  This can be greater than $O(b^d)$: the search can explore long paths consisting of small steps before exploring shorter paths consisting of larger steps
- **Space?**
  $O(b^{1+C^*/\varepsilon})$

See [Dijkstra's algorithm on Wikipedia]

Implementation: Best-First Search Strategy

```
function UNIFORM-COST-SEARCH(problem) returns a solution node, or failure
  return BEST-FIRST-SEARCH(problem, PATH-COST)
```

```
function BEST-FIRST-SEARCH(problem, f) returns a solution node or failure
  node ← NODE(STATE=problem.INITIAL)
  frontier ← a priority queue ordered by f, with node as an element
  reached ← a lookup table, with one entry with key problem.INITIAL and value node
  while not IS-EMPTY(frontier) do
    node ← POP(frontier)
    if problem.IS-GOAL(node.STATE) then return node
    for each child in EXPAND(problem, node) do
      s ← child.STATE
      if s is not in reached or child.PATH-COST < reached[s].PATH-COST then
        reached[s] ← child
        add child to frontier
  return failure
```

The order for expanding the frontier is determined by *f(n)* = path cost from the initial state to node *n*.

This check is the difference to BFS! It visits a node again if it can be reached by a better (cheaper) path.

See BFS for function EXPAND.

Depth-first search (DFS)

- **Expansion rule**: Expand deepest unexpanded node in the frontier (last added).
- **Frontier**: **stack** (LIFO)
- **No reached data structure!**

  Cycle checking checks only the current path.

  Redundant paths can not be identified and lead to replicated work.

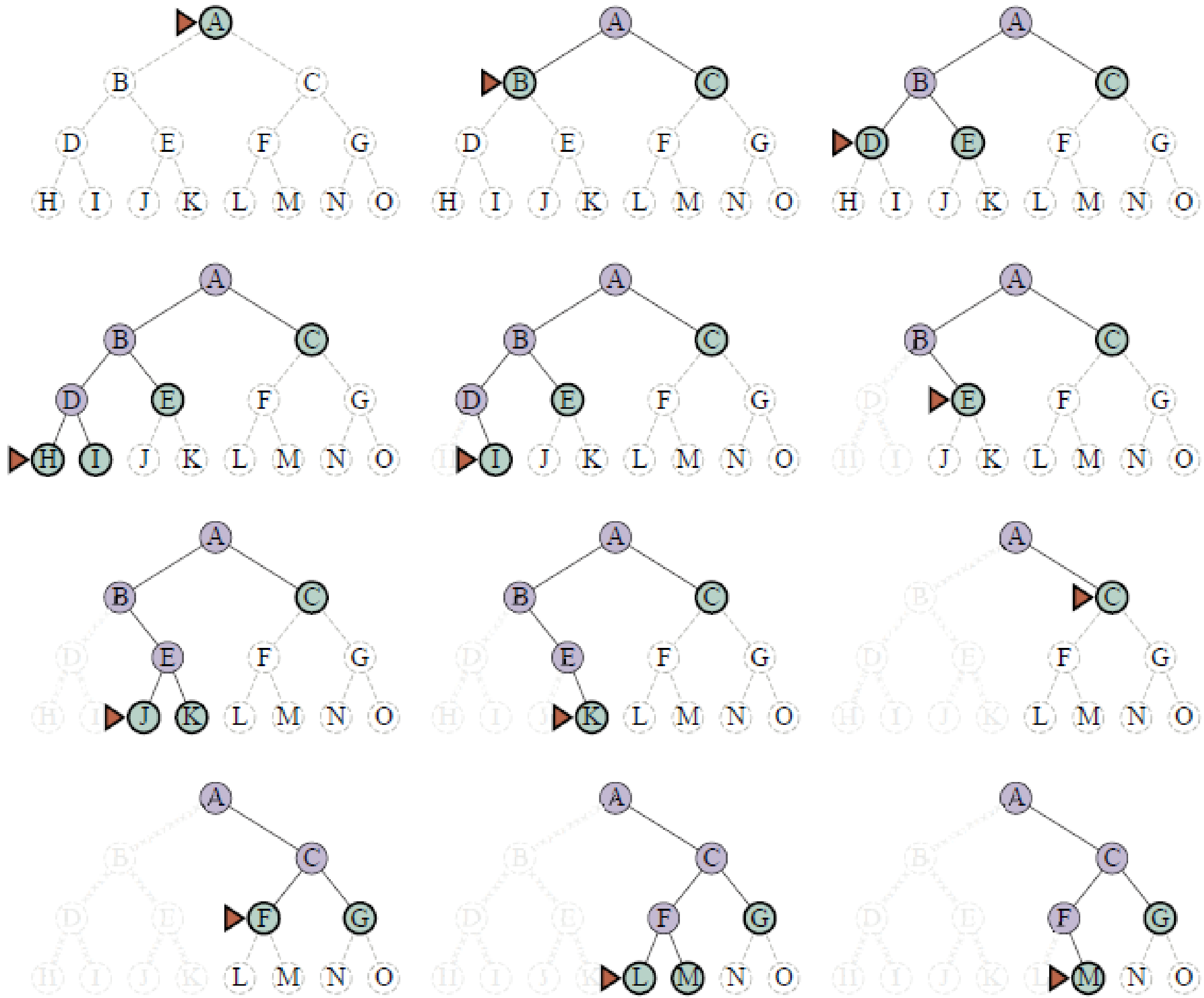


Figure 3.11 A dozen steps (left to right, top to bottom) in the progress of a depth-first search on a binary tree from start state A to goal M. The frontier is in green, with a triangle marking the node to be expanded next. Previously expanded nodes are lavender, and potential future nodes have faint dashed lines. Expanded nodes with no descendants in the frontier (very faint lines) can be discarded.

Implementation: DFS

- DFS could be implemented like BFS/Best-first search and just taking the last element from the frontier (LIFO).
- However, to reduce the space complexity to $O(bm)$, the reached data structure needs to be removed! Options:
  - ~~Recursive implementation: cycle checking is a problem leading to infinite loops.~~
  - Iterative implementation: Build tree and abandoned branches are removed from memory. Cycle checking is only done against the current path. This is similar to Backtracking search.

DFS uses $\ell = \infty$

```
function DEPTH-LIMITED-SEARCH(problem, ℓ) returns a node or failure or cutoff
  frontier ← a LIFO queue (stack) with NODE(problem.INITIAL) as an element
  result ← failure
  while not IS-EMPTY(frontier) do
    node ← POP(frontier)
    if problem.IS-GOAL(node.STATE) then return node
    if DEPTH(node) > ℓ then
      result ← cutoff
    else if not IS-CYCLE(node) do
      for each child in EXPAND(problem, node) do
        add child to frontier
  return result
```

If we only keep the current path in memory, then we can only check against the path from the root to the current node and the frontier to prevent cycles.

See BFS for function EXPAND.

Properties of depth-first search

- **Complete?**
  - Only in finite search spaces. Some cycles can be avoided by checking for repeated states along the path.
  - **Incomplete in infinite search spaces** (e.g., with cycles).

- **Optimal?**

  No – returns the first solution it finds.

d: depth of the optimal solution
m: max. depth of tree
b: maximum branching factor

- **Time?**

  Could be the time to reach a solution at maximum depth m in the last path: $O(b^m)$
  Terrible if $m \gg d$, but if there are many shallow solutions, it can be much faster than BFS.

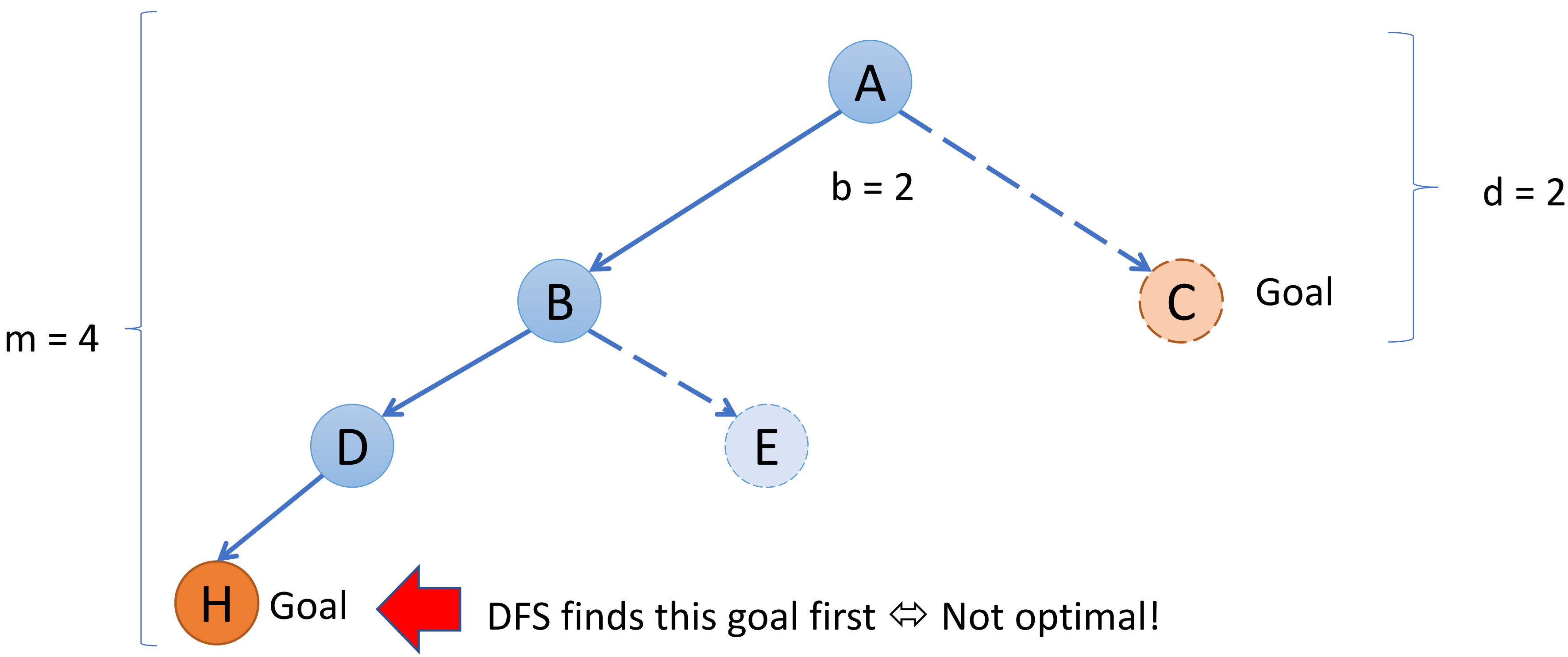
- **Space?**

  $O(bm)$ ⇔ **linear in max. tree depth** (only if **no reached data structure** is used!)

Depth-first search

d: depth of the optimal solution
m: max. depth of tree
b: maximum branching factor

- Time: $O(b^m)$ – worst case is expanding all paths.
- Space: $O(bm)$ - if it only stores the frontier nodes and the current path.



Note: The order in which we add new nodes to the frontier can change what goal we find!

Iterative deepening search (IDS)

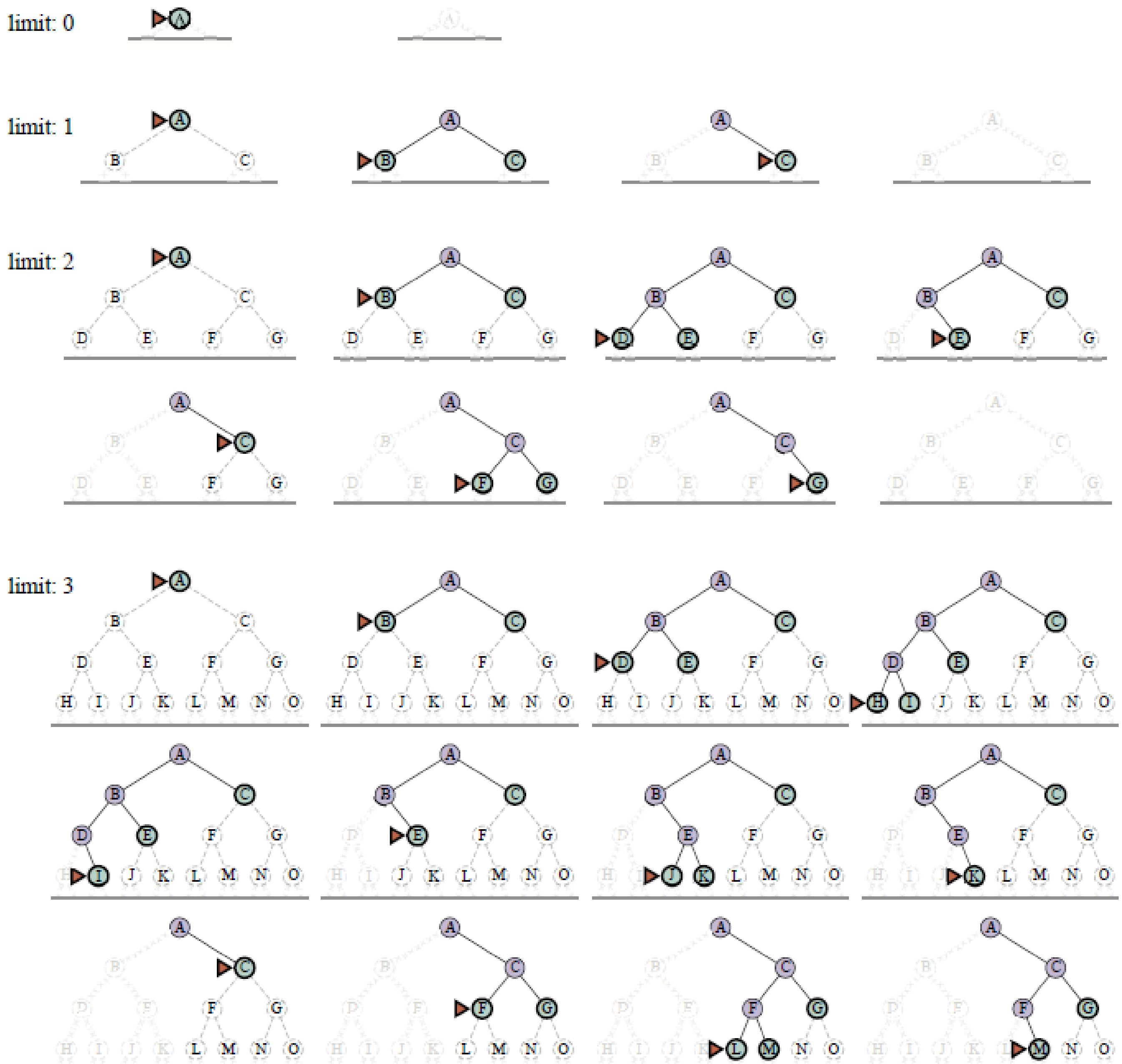
Can we

a) **get DFS's good memory footprint,**
b) **avoid infinite cycles, and**
c) **preserve BFS's optimality guaranty?**

Use depth-restricted DFS and gradually increase the depth.

1. Check if the root node is the goal.
2. Do a DFS searching for a path of length 1
3. If goal not found, do a DFS searching for a path of length 2
4. If goal not found, do a DFS searching for a path of length 3
5. …

Iterative deepening search (IDS)



Implementation: IDS

```
function ITERATIVE-DEEPENING-SEARCH(problem) returns a solution node or failure
  for depth = 0 to ∞ do
    result ← DEPTH-LIMITED-SEARCH(problem, depth)
    if result ≠ cutoff then return result
```

```
function DEPTH-LIMITED-SEARCH(problem, ℓ) returns a node or failure or cutoff
  frontier ← a LIFO queue (stack) with NODE(problem.INITIAL) as an element
  result ← failure
  while not IS-EMPTY(frontier) do
    node ← POP(frontier)
    if problem.IS-GOAL(node.STATE) then return node
    if DEPTH(node) > ℓ then
      result ← cutoff
    else if not IS-CYCLE(node) do
      for each child in EXPAND(problem, node) do
        add child to frontier
  return result
```

See BFS for function EXPAND.

Properties of iterative deepening search

- **Complete?**
  Yes

d: depth of the optimal solution
m: max. depth of tree
b: maximum branching factor

- **Optimal?**
  Yes, if step cost = 1

- **Time?**
  Consists of rebuilding trees up to d times
  $d\ b^1 + (d-1)b^2 + \ldots + 1b^d = O(b^d)$ ⇔ Slower than BFS, but the same complexity!

- **Space?**
  O(***bd***) ⇔ **linear space. Even less than DFS since m<=d.** Cycles need to be handled by the depth-limited DFS implementation.

Note: IDS produces the same result as BFS but trades better space complexity for worse run time.

This makes IDS/DFS into the workhorse of AI.

Informed Search

Informed search

- AI search problems are typically very large. We would like to improve efficiency by **expanding as few nodes as possible.**

- The agent can use **additional information** in the form of “hints” about how promising different states/nodes are to lead to the goal. These hints are derived from
  - information the agent has (e.g., a map) or
  - percepts coming from a sensor.

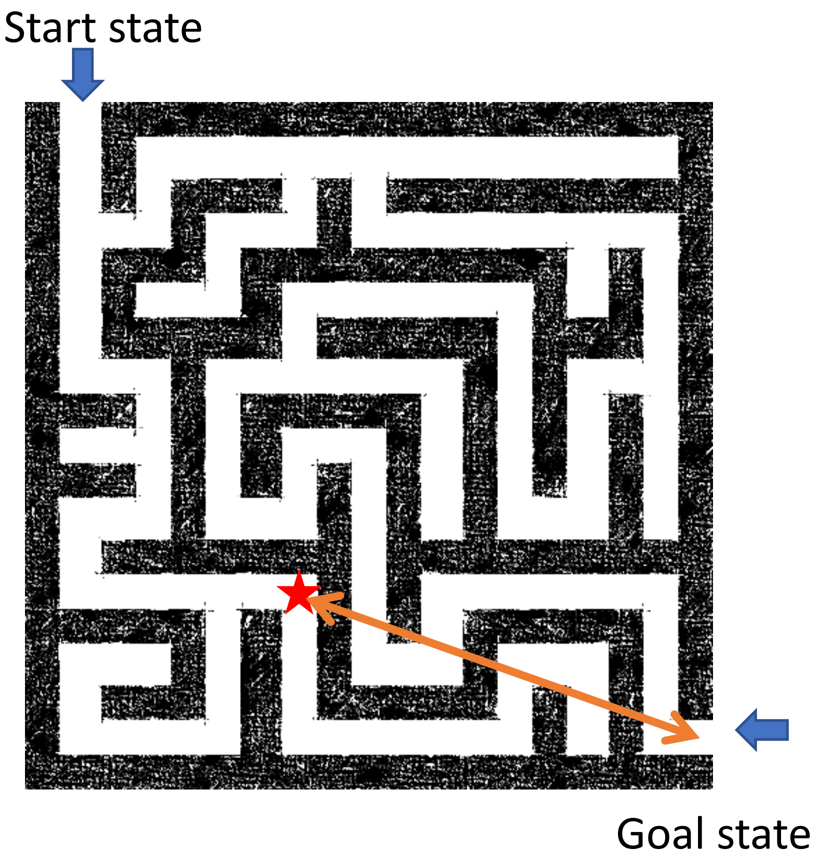
- The agent uses a **heuristic function** $h(n)$ to rank nodes in the frontier and select the most promising state in the frontier for expansion using a **best-first search** strategy.

- Algorithms:
  - Greedy best-first search
  - A* search

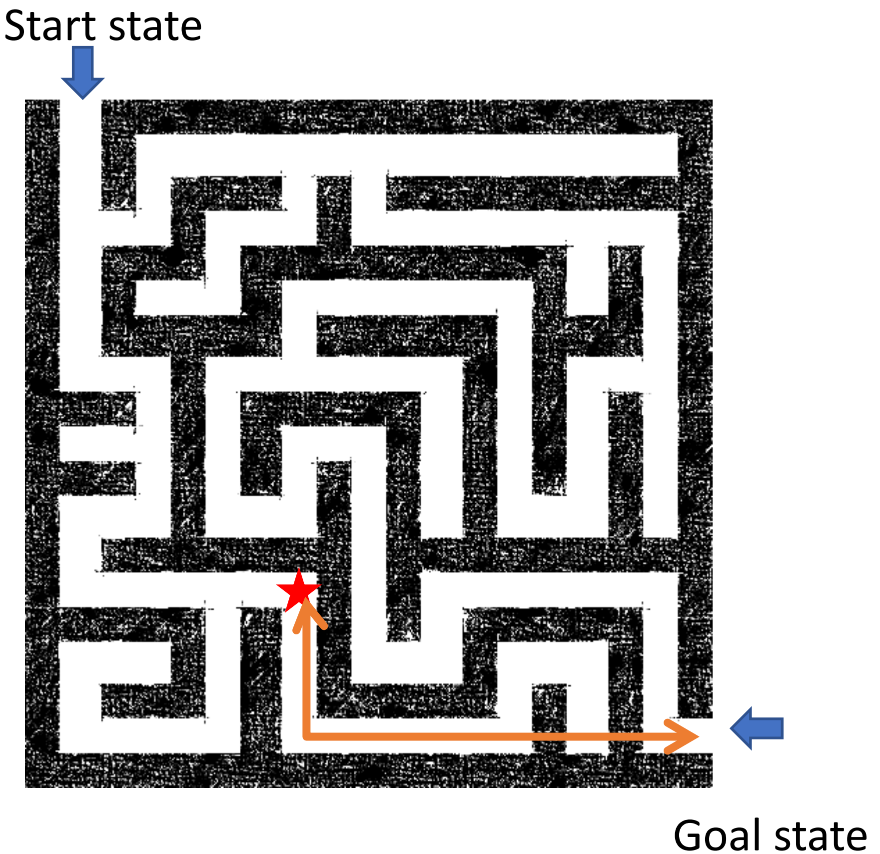
Heuristic function

- **Heuristic function** $h(n)$ estimates the cost of reaching a node representing the goal state from the current node n.
- Examples:

Euclidean distance

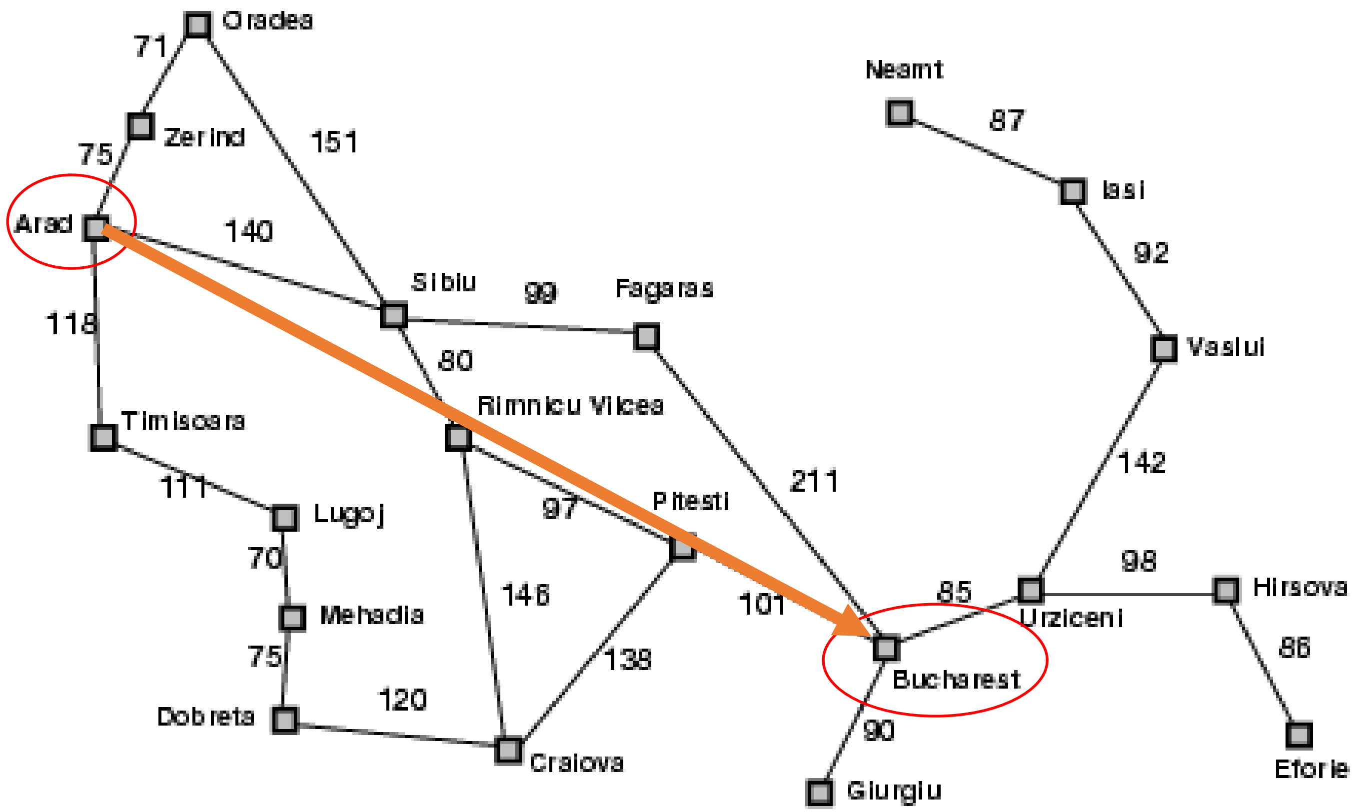


Manhattan distance



Heuristic for the Romania problem

Drive from Arad to Bucharest using a table with straight-line distances.

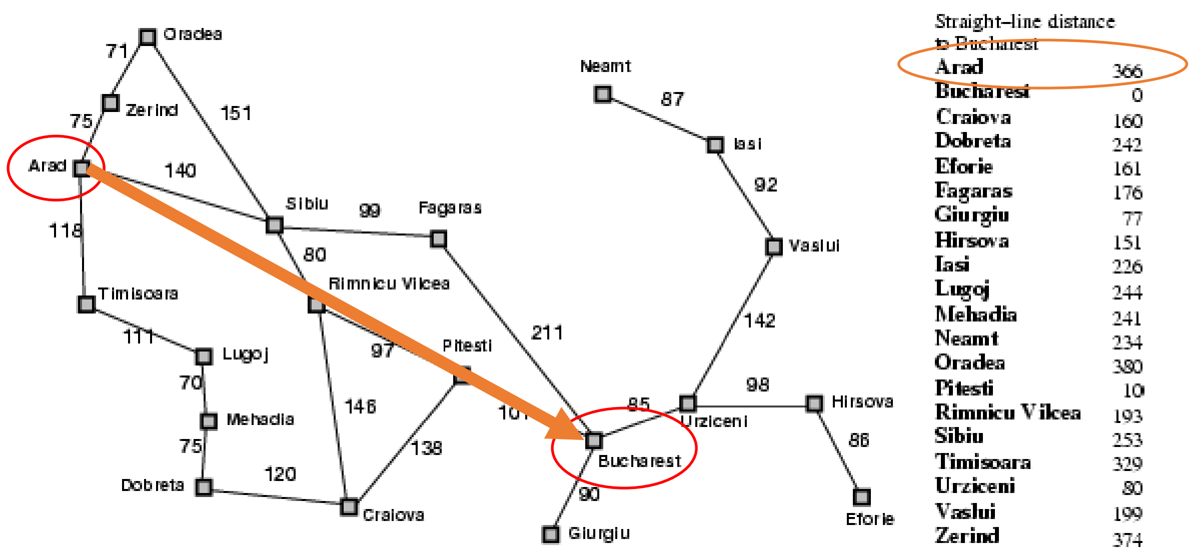


Straight–line distance to Bucharest	$h(n)$
Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	176
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
Oradea	380
Pitesti	10
Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

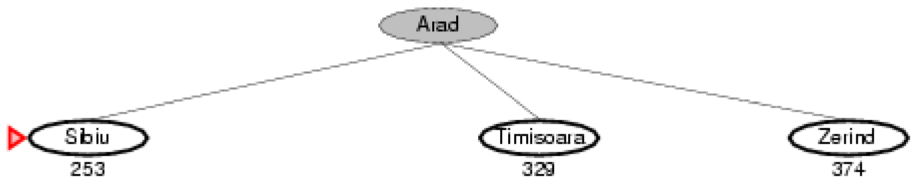
Greedy best-first search example

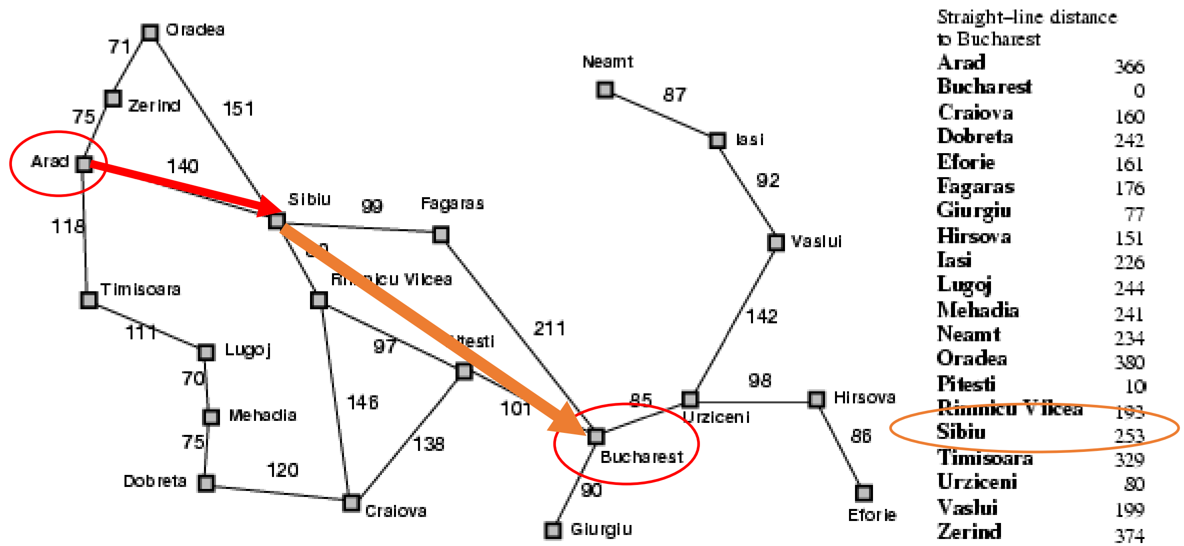
Expansion rule: Expand the node that has the lowest value of the heuristic function *h*(*n*)

$h(n)=$

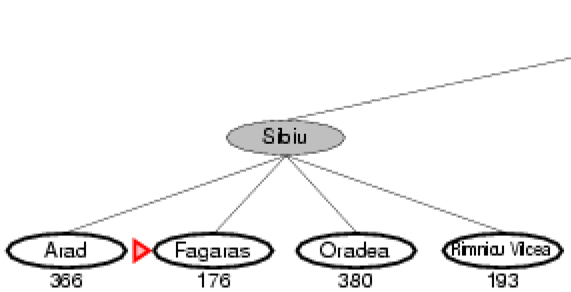


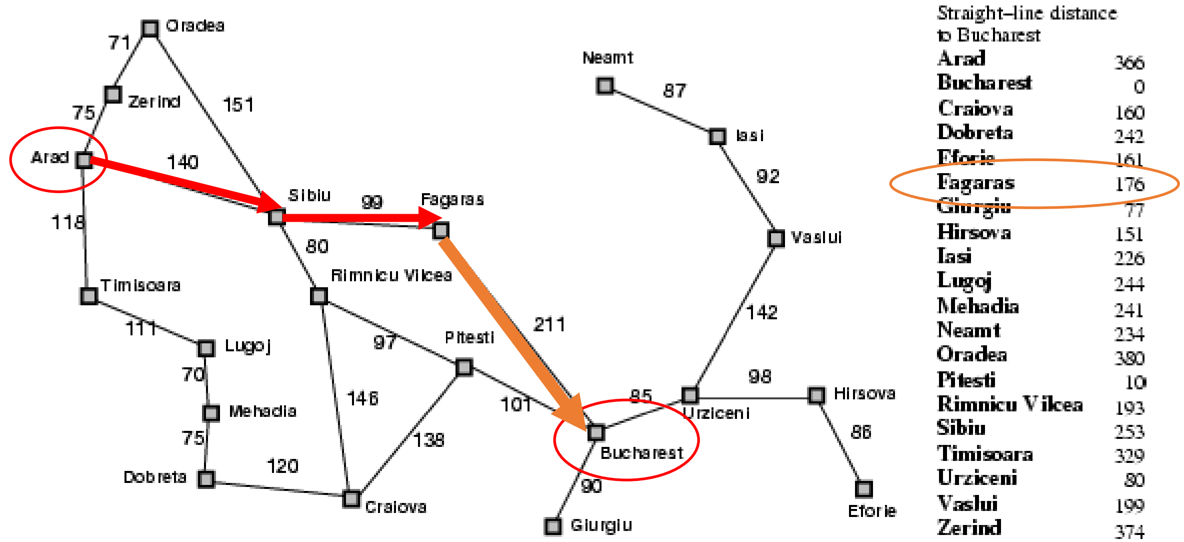
Greedy best-first search example



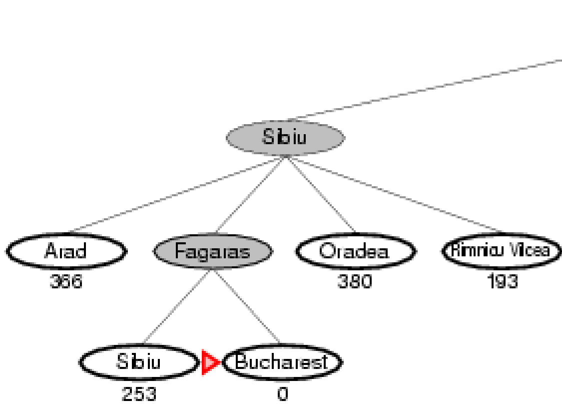


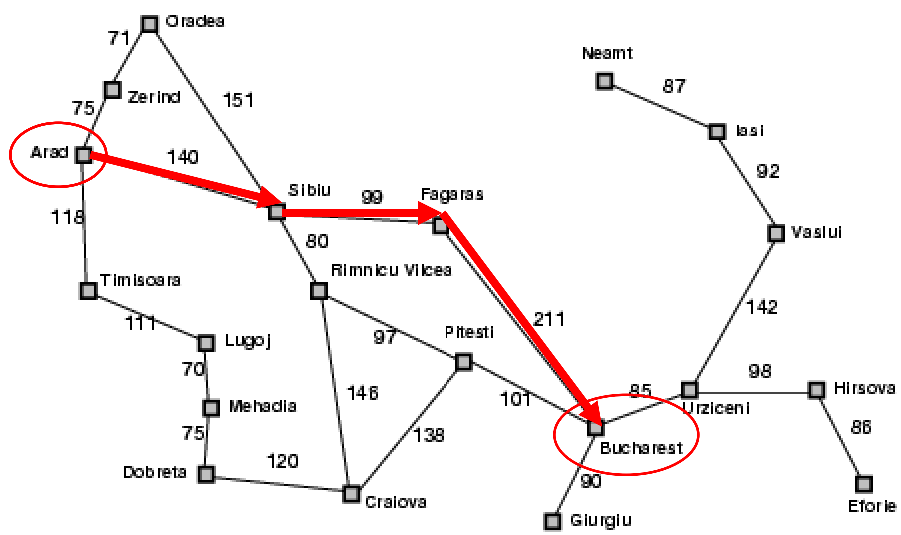
Greedy best-first search example





Greedy best-first search example





Straight–line distance to Bucharest

Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	176
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
Oradea	380
Pitesti	10
Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

Total:
140 + 99 + 211 = 450 miles

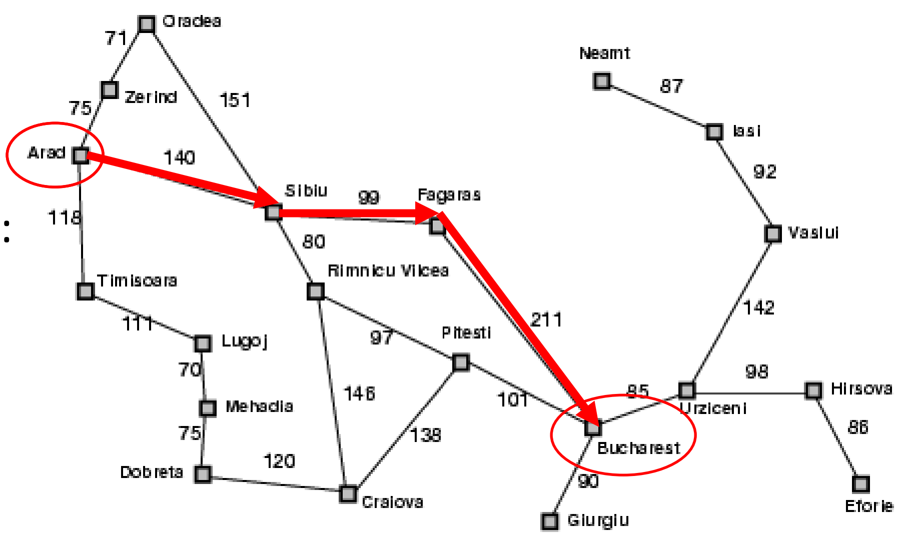
Properties of greedy best-first search

- **Complete?**
  Yes – Best-first search if complete in finite spaces.

- **Optimal?**
  No

Total:
140 + 99 + 211 = 450 miles

Alternative through Rimnicu Vilcea:
140 + 80 + 97 + 101 = 418 miles



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Sibiu	253
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Zerind	374

Implementation of greedy best-first search

Best-First Search

+

Expand the frontier using $f(n) = h(n)$

Implementation of greedy best-first search

Heuristic $h(n)$ so we expand the node with the lowest estimated cost

function BEST-FIRST-SEARCH(*problem*, f) **returns** a solution node or *failure*
  node ← NODE(STATE=*problem*.INITIAL)
  frontier ← a priority queue ordered by f, with *node* as an element
  reached ← a lookup table, with one entry with key *problem*.INITIAL and value *node*
  while not IS-EMPTY(*frontier*) **do**
    node ← POP(*frontier*)
    if *problem*.IS-GOAL(*node*.STATE) **then return** *node*
    for each *child* **in** EXPAND(*problem*, *node*) **do**
      s ← *child*.STATE
      if s is not in *reached* **or** *child*.PATH-COST < *reached*[s].PATH-COST **then**
        reached[s] ← *child*
        add *child* to *frontier*
  return *failure*

See BFS for function EXPAND.

The order for expanding the frontier is determined by *f(n)*

This check is the different to BFS! It visits a node again if it can be reached by a better (cheaper) path.

Properties of greedy best-first search

- **Complete?**
  Yes – Best-first search if complete in finite spaces.

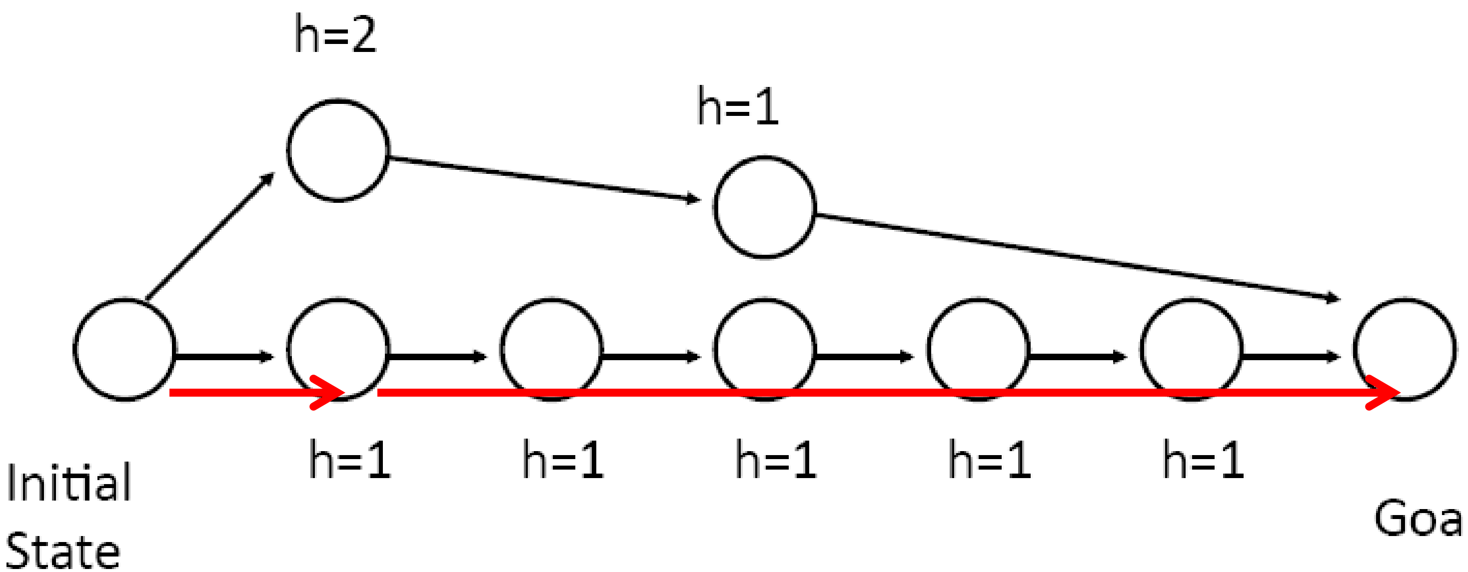
- **Optimal?**
  No

d: depth of the optimal solution
m: max. depth of tree
b: maximum branching factor

- **Time?**
  Worst case: $O(b^m)$ ⇔ like DFS
  Best case: $O(bm)$ – If $h(n)$ is 100% accurate

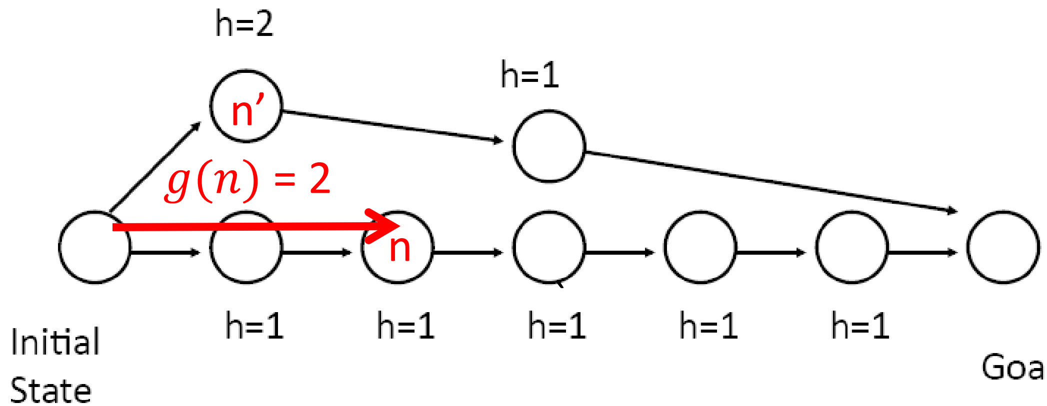
- **Space?**
  Same as time complexity.

How can we fix the optimality problem with greedy best-first search?



$h = 1$ is always better than $h = 2$.
Greedy best-first will go this way
and never reconsider!

A* search



- **Idea**: Take the cost of the path to n called $g(n)$ into account to avoid expanding paths that are already very expensive.
- The evaluation function $f(n)$ is the estimated total cost of the path through node n to the goal:

$$f(n) = g(n) + h(n)$$

$g(n)$: cost so far to reach *n* (path cost)

$h(n)$: estimated cost from *n* to goal (heuristic)

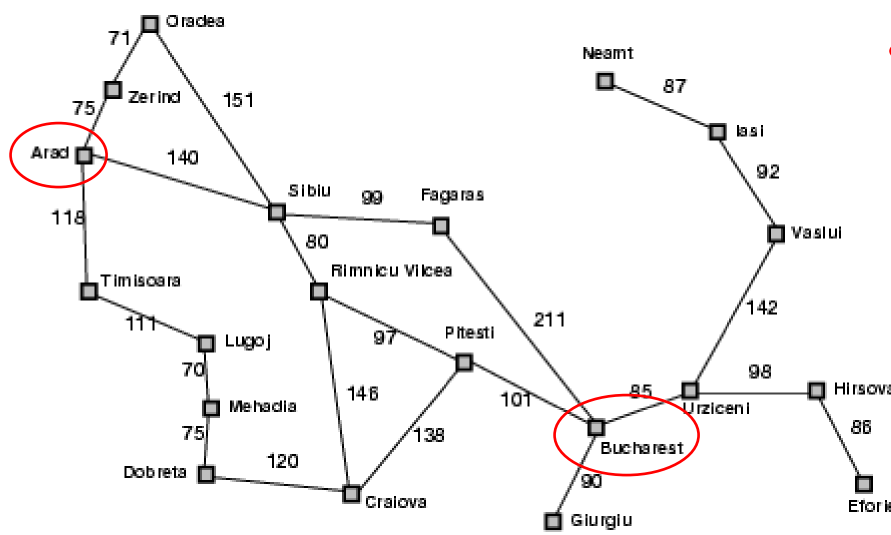
- The agent in the example above will stop at n with $f(n) = 3$ and chose the path up with a better $f(n') = 2$

Note: For greedy best-first search we just used $f(n) = h(n)$.

A* search example

Expansion rule:
Expand the node with the smallest f(n)

$$f(n) = g(n) + h(n) =$$



$h(n)$

Straight–line distance to Bucharest

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Bucharest	0
Craiova	160
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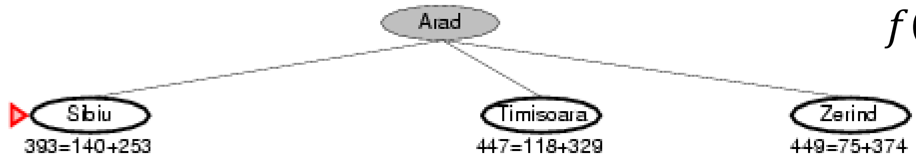
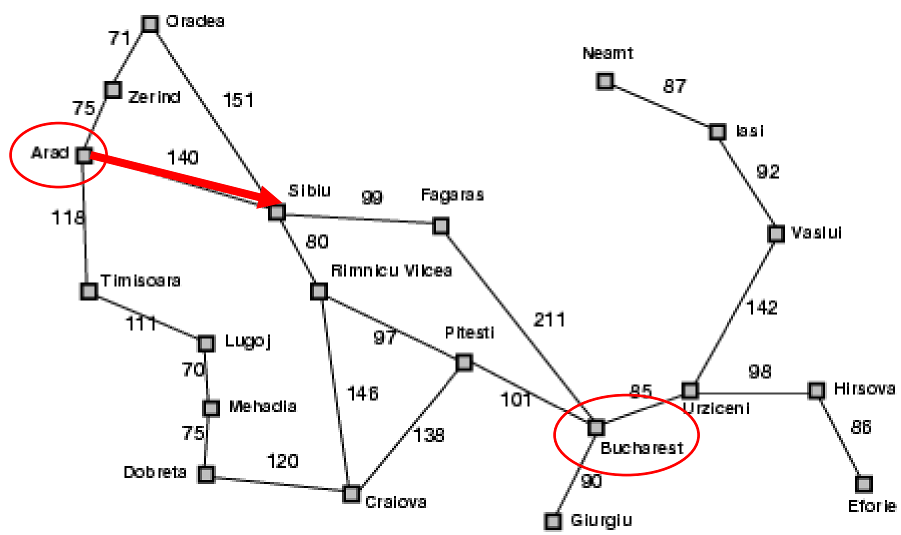
A* search example



$$f(n) = g(n) + h(n)$$



$h(n)$

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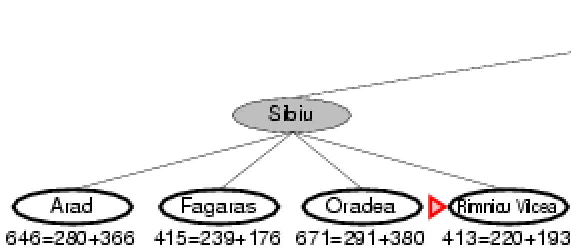
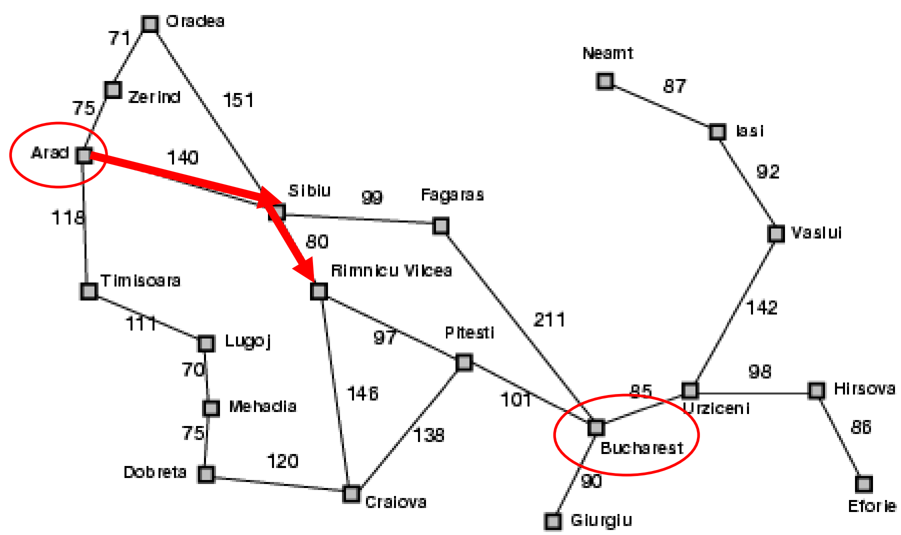
A* search example



$$f(n) = g(n) + h(n)$$



$h(n)$

Straight–line distance to Bucharest

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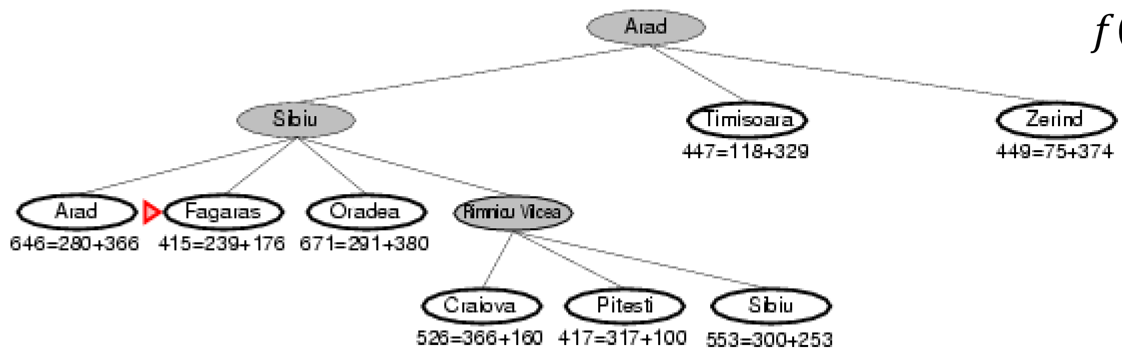
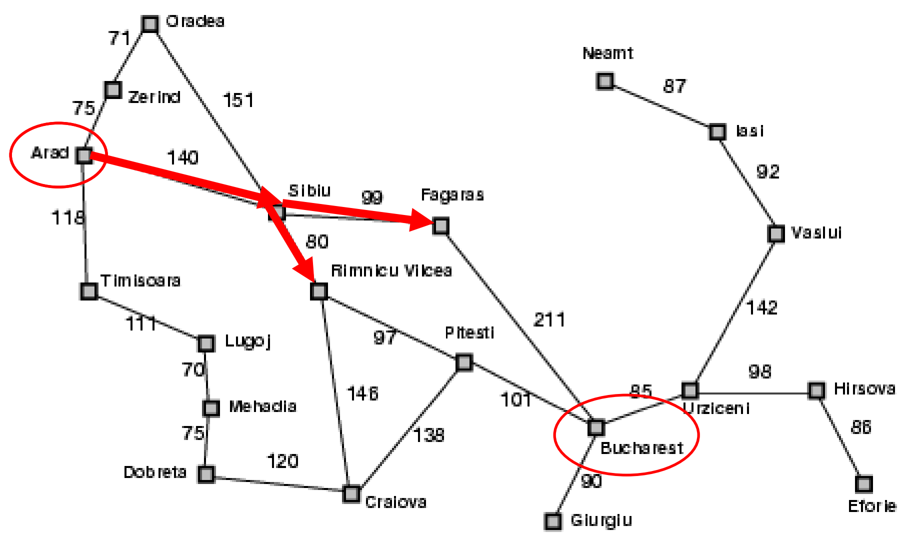
A* search example

$$f(n) = g(n) + h(n)$$





$h(n)$

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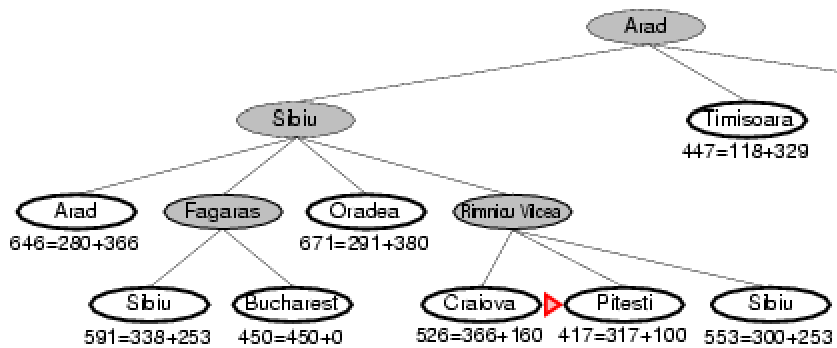
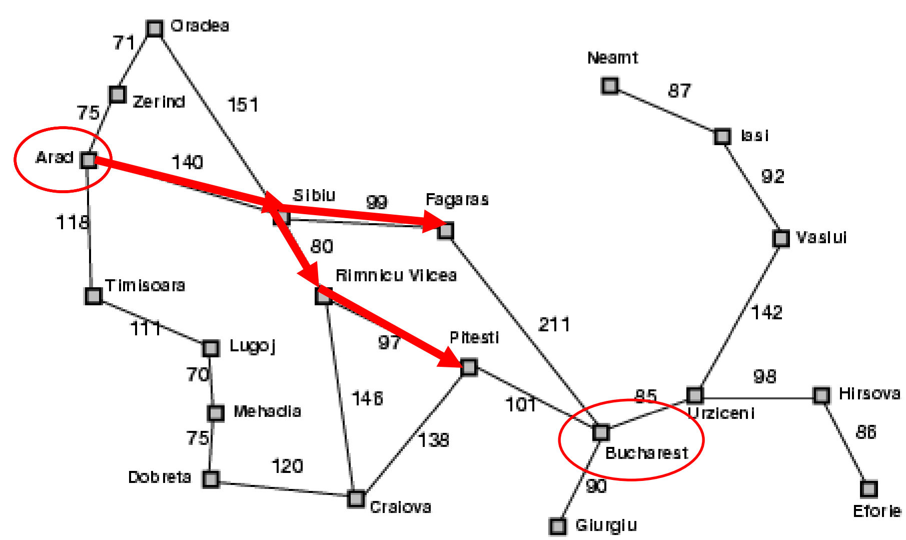
A* search example



$$f(n) = g(n) + h(n)$$



$h(n)$

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Sibiu	253
Timisoara	329
Urziceni	80
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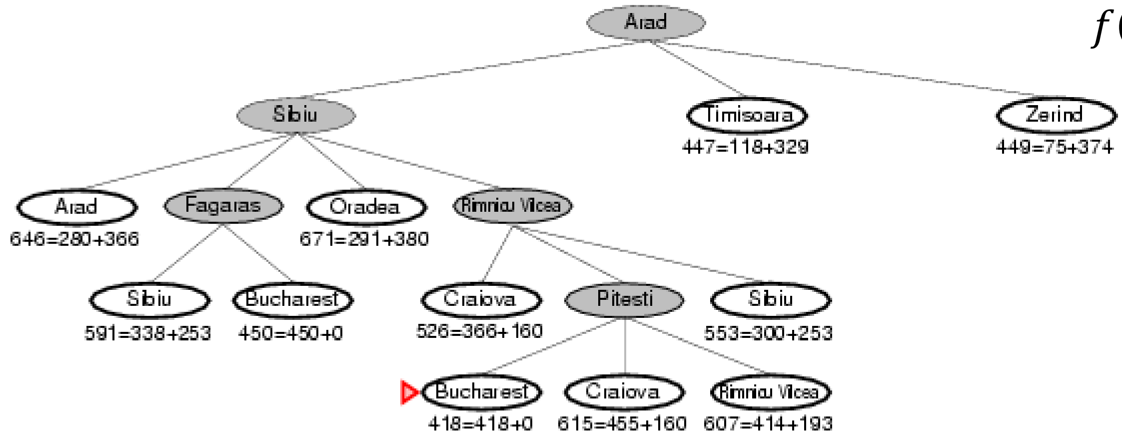
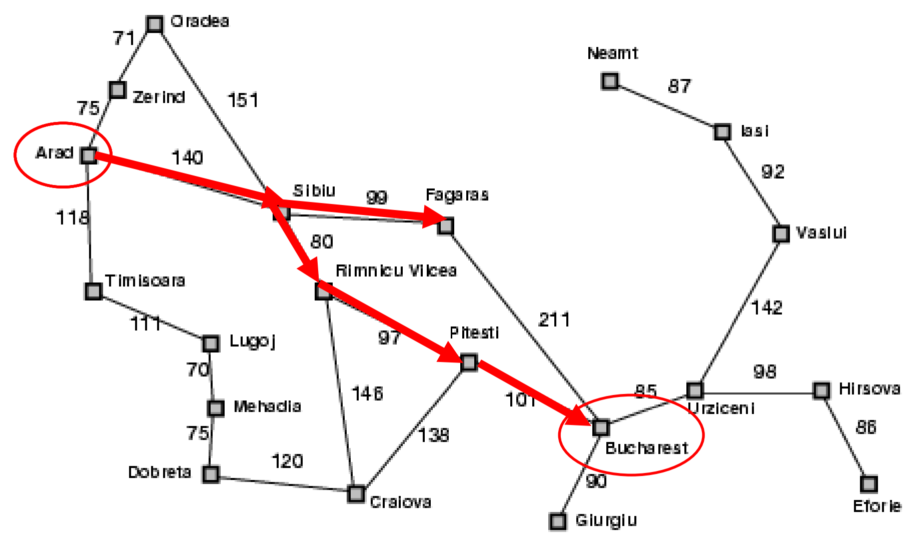
Zerind	374

A* search example

$$f(n) = g(n) + h(n)$$





$h(n)$

Straight–line distance to Bucharest	
Arad	366
Bucharest	0
Craiova	160
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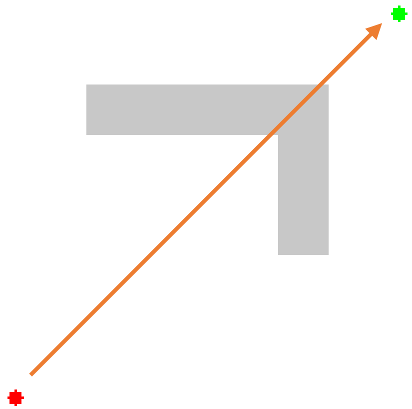
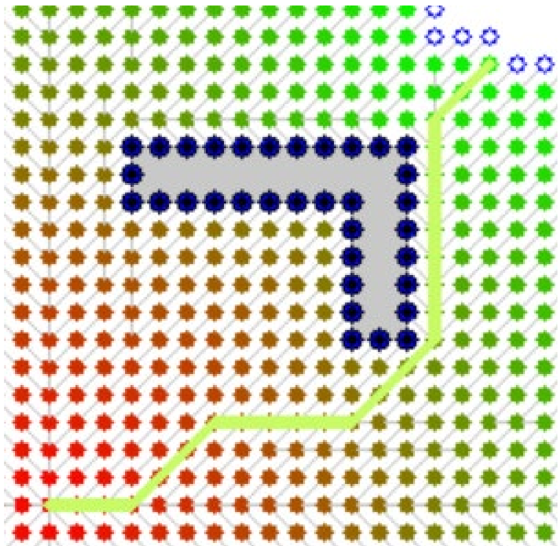
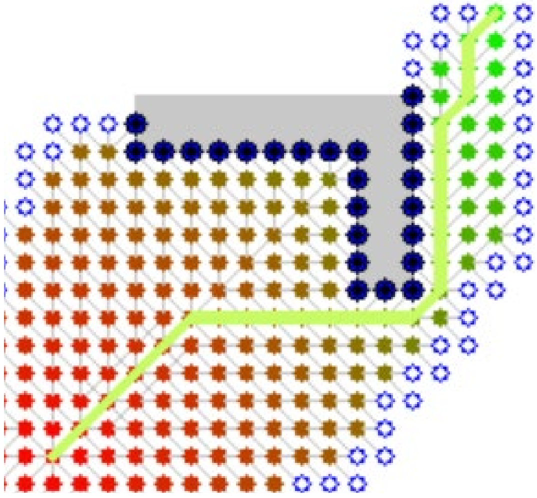
BFS vs. A* search



BFS



A*



Source: [Wikipedia]()

Implementation of A* Search

Best-First Search

+

Expand the frontier using $f(n) = g(n) + h(n)$

Implementation of A* Search

Path cost to n + heuristic from n to goal = estimate of the total cost

$g(n) + h(n)$

```
function BEST-FIRST-SEARCH(problem, f) returns a solution node or failure
  node ← NODE(STATE=problem.INITIAL)
  frontier ← a priority queue ordered by f, with node as an element
  reached ← a lookup table, with one entry with key problem.INITIAL and value node
  while not IS-EMPTY(frontier) do
    node ← POP(frontier)
    if problem.IS-GOAL(node.STATE) then return node
    for each child in EXPAND(problem, node) do
      s ← child.STATE
      if s is not in reached or child.PATH-COST < reached[s].PATH-COST then
        reached[s] ← child
        add child to frontier
  return failure
```

The order for expanding the frontier is determined by $f(n)$

This check is different to BFS! It visits a node again if it can be reached by a better (cheaper) redundant path.

See BFS for function EXPAND.

Optimality: Admissible heuristics

Definition: A heuristic h is **admissible** if for every node n, $h(n) \leq h^*(n)$, where $h^*(n)$ is the true cost to reach the goal state from n.

I.e., an admissible heuristic is a **lower bound** and never overestimates the true cost to reach the goal.

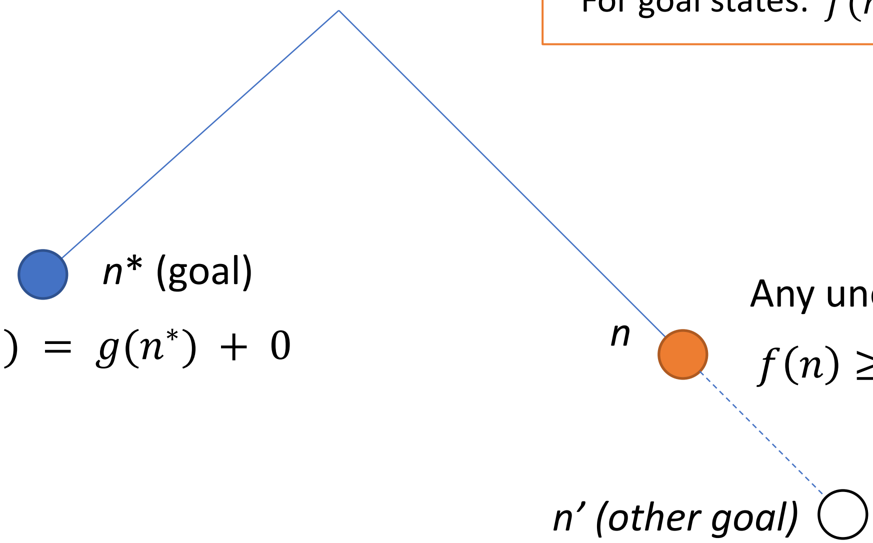
Example: straight line distance never overestimates the actual road distance.

Theorem: If h is admissible, A^* is optimal.

Proof of Optimality of A*

$$f(n) = g(n) + h(n)$$

For goal states: $f(n^*) = g(n) + 0$



$$C^* = f(n^*) = g(n^*) + 0$$

Any unexplored node n has:

$$f(n) \geq f(n^*)$$

$$g(n') \geq f(n) \Leftrightarrow g(n') \geq C^*$$

- Suppose A* terminates its search at goal n^* at a cost of $C^* = f(n^*)$.
- All unexplored nodes n have $f(n) \geq f(n^*)$ or they would have been explored before n^*.
- Since $f(n)$ is an *optimistic* estimate, it is impossible for n to have a successor goal state n' with $C' < C^*$.
- This proofs that n^* must be an optimal solution.

Guarantees of A*

A* is **optimally efficient**

a. No other tree-based search algorithm that uses the same heuristic can expand fewer nodes and still be guaranteed to find the optimal solution.

b. Any algorithm that does not expand all nodes with $f(n) < C^*$ (the lowest cost of going to a goal node) cannot be optimal. It risks missing the optimal solution.

Properties of A*

- **Complete?**
  Yes

- **Optimal?**
  Yes

- **Time?**
  Number of nodes for which $f(n) \leq C^*$ (exponential)

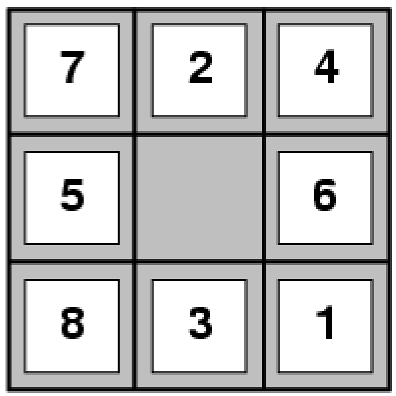
- **Space?**
  Same as time complexity.

Designing heuristic functions

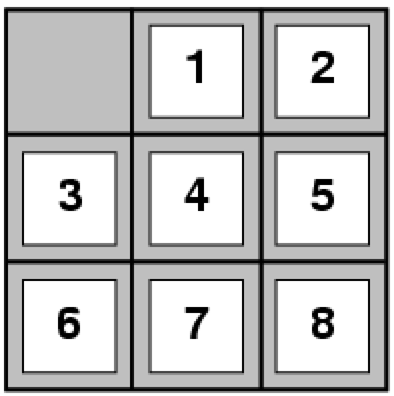
Heuristics for the 8-puzzle

$h_1(n)$ = number of misplaced tiles

$h_2(n)$ = total Manhattan distance (number of squares from desired location of each tile)



Start State



Goal State

$$h_1(start) \; = \; 8$$
$$h_2(start) \; = \; 3+1+2+2+2+3+3+2 \; = \; 18$$

1 needs to move 3 positions

Are h_1 and h_2 admissible?

Heuristics from relaxed problems

- A problem with fewer restrictions on the actions is called a relaxed problem.
- **The cost of an optimal solution to a relaxed problem is an admissible heuristic for the original problem. I.e., the true cost is never smaller.**
- h_1: If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then $h_1(n)$ gives the shortest solution.
- h_2: If the rules are relaxed so that a tile can move to any adjacent square, then $h_2(n)$ gives the shortest solution.

7	2	4
5		6
8	3	1

Start State

	1	2
3	4	5
6	7	8

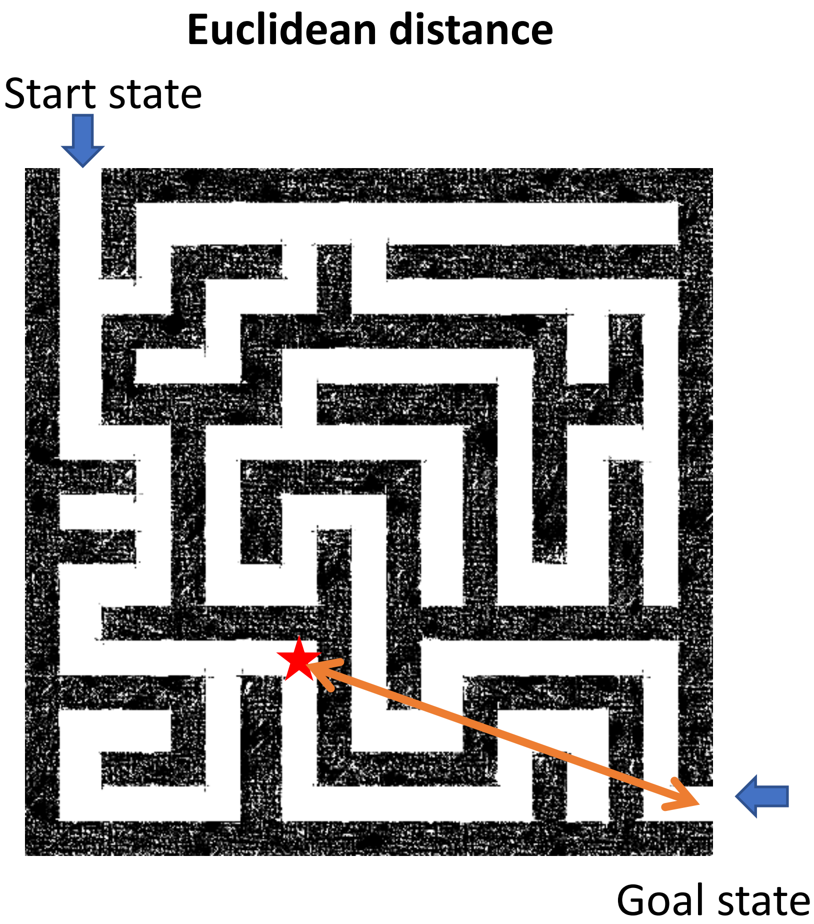
Goal State

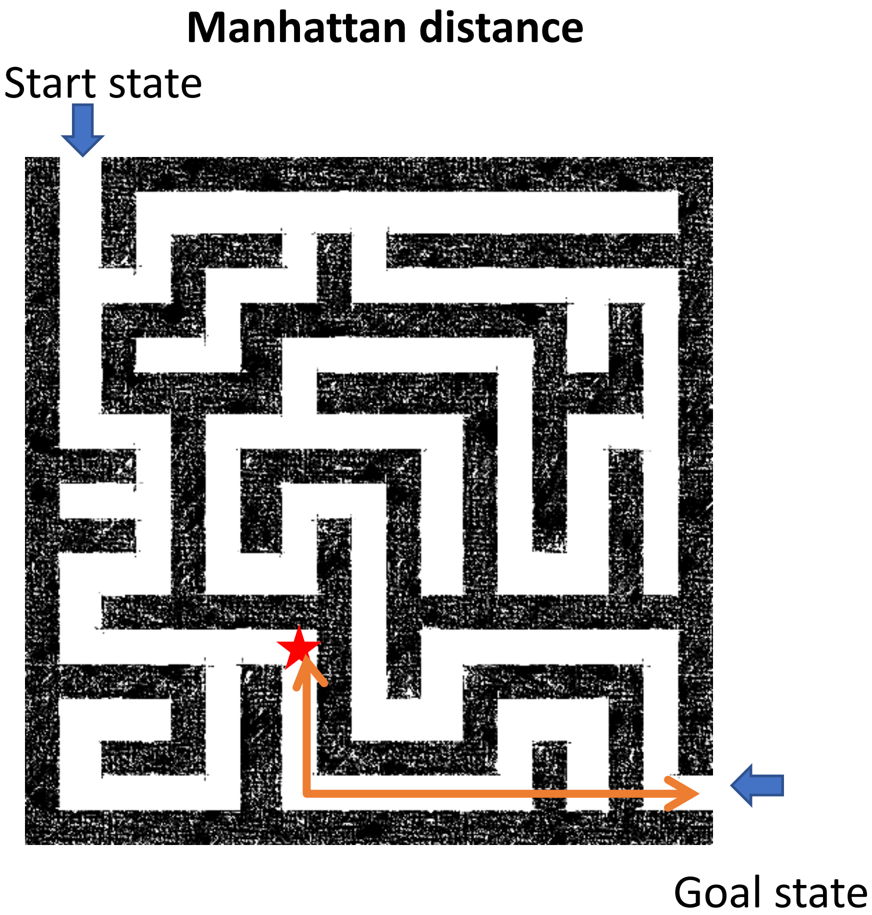
$h_1(start) = 8$

$h_2(start)$
$= 3 + 1 + 2 + 2 + 2 + 3 + 3 + 2$
$= 18$

Heuristics from relaxed problems

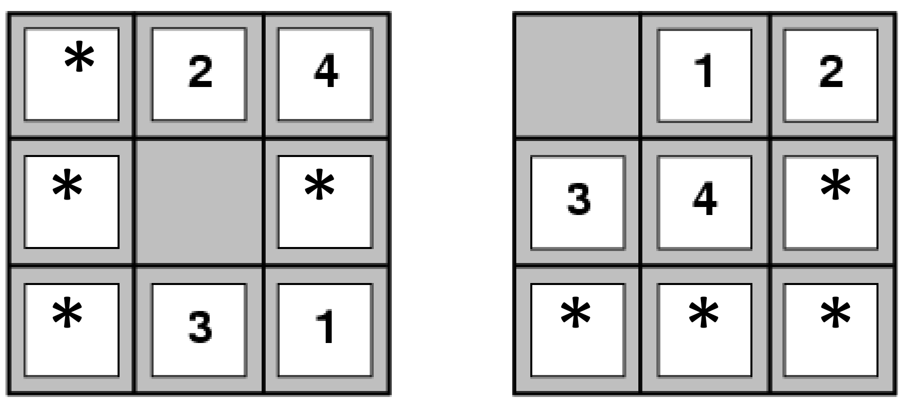
What relaxations are used in these two cases?





Heuristics from subproblems

- Let $h_3(n)$ be the cost of getting a subset of tiles (say, 1,2,3,4) into their correct positions. The final order of the * tiles does not matter.
- Small subproblems are often easy to solve.
- Can precompute and save the exact solution cost for every or many possible subproblem instances – ***pattern database***.



Start State Goal State

Dominance: What heuristic is better?

Definition: If h_1 and h_2 are both admissible heuristics and $h_2(n) \geq h_1(n)$ for all n, then h_2 dominates h_1

Is h_1 or h_2 better for A* search?

- A* search expands every node with $f(n) < C^* \Leftrightarrow h(n) < C^* - g(n)$
- h_2 is never smaller than h_1. A* search with h_2 will expand less nodes and is therefore better.

Dominance

- Typical search costs for the 8-puzzle (average number of nodes expanded for different solution depths d):

- $d = 12$ IDS = 3,644,035 nodes
  A*(h_1) = 227 nodes
  A*(h_2) = 73 nodes

- $d = 24$ IDS ≈ 54,000,000,000 nodes
  A*(h_1) = 39,135 nodes
  A*(h_2) = 1,641 nodes

Combining heuristics

- Suppose we have a collection of admissible heuristics $h_1, h_2, \dots, h_m$, but none of them dominates the others.
- Combining them is easy:

$$h(n) = \max\{h_1(n), h_2(n), \dots, h_m(n)\}$$

- That is, always pick for each node the heuristic that is closest to the real cost to the goal $h^*(n)$.

Satisficing Search: Weighted A* search

- Often it is sufficient to find a **"good enough" solution** if it can be found very quickly or with way less computational resources. I.e., expanding fewer nodes.

- We could use inadmissible heuristics in A* search (e.g., by multiplying $h(n)$ with a factor W) that sometimes overestimate the optimal cost to the goal slightly.
  1. It potentially reduces the number of expanded nodes significantly.
  2. **This will break the algorithm's optimality guaranty!**

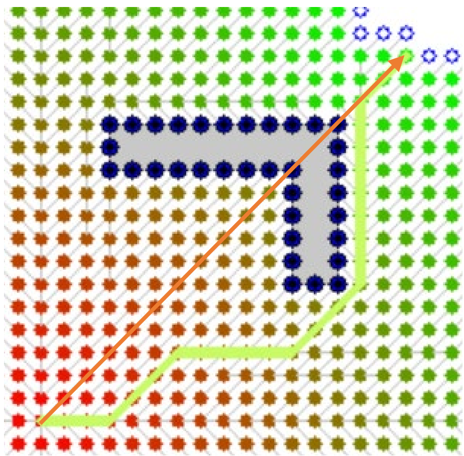
$$\mathrm{f}(n) = g(n) + W \times h(n)$$

Weighted A* search: $\boldsymbol{g(n) + W \times h(n)}$ $\boldsymbol{(1 < W < \infty)}$

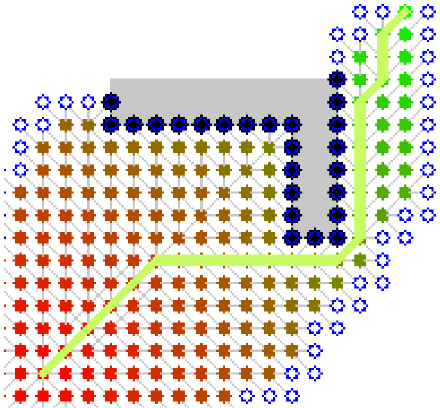
The presented algorithms are special cases:

A* search:	$g(n) + h(n)$	$(W = 1)$
Uniform cost search/BFS:	$g(n)$	$(W = 0)$
Greedy best-first search:	$h(n)$	$(W = \infty)$

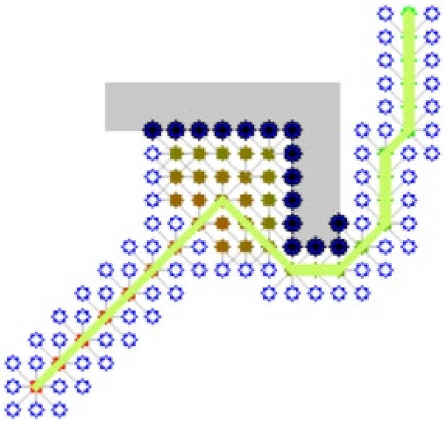
Example of weighted A* search



Breadth-first Search (BFS)
$f(n) = $ # actions to reach n



Exact A* Search
$f(n) = g(n) + h_{Eucl}(n)$



Weighted A* Search
$f(n) = g(n) + 5\, h_{Eucl}(n)$

Source and Animation: [Wikipedia]()

Memory-bounded search

- The memory usage of A* (search tree and frontier) can still be exorbitant.
- How can we make A* more memory-efficient while maintaining completeness and optimality?
  - Iterative deepening A* search.
  - Recursive best-first search, SMA*: Forget some subtrees but remember the best f-value in these subtrees and regenerate them later if necessary.

- **Problems**: memory-bounded strategies can be complicated to implement and suffer from “memory thrashing” (regenerating forgotten nodes like IDS).

Implementation as Best-first Search

- All discussed search strategies can be implemented using Best-first search.
- Best-first search expands always the **node with the minimum value of an evaluation function $\boldsymbol{f(n)}$.**

Search Strategy	Evaluation function $f(n)$
BFS (Breadth-first search)	$g(n)$ (=uniform path cost)
Uniform-cost Search	$g(n)$ (=path cost)
DFS/IDS (see note below)	$-g(n)$
Greedy Best-first Search	$h(n)$
(weighted) A* Search	$g(n) + W \times h(n)$

- **Note:** DFS/IDS is typically implemented differently to achieve the lower space complexity.

Summary: Uninformed search strategies

Algorithm	Complete?	Optimal?	Time complexity	Space complexity
BFS (Breadth-first search)	Yes	If all step costs are equal	$O(b^d)$	$O(b^d)$
Uniform-cost Search	Yes	Yes	Number of nodes with $g(n) \leq C^*$	
DFS	In finite spaces (cycle checking)	No	$O(b^m)$	$O(bm)$
IDS	Yes	If all step costs are equal	$O(b^d)$	$O(bd)$

b: maximum branching factor of the search tree
d: depth of the optimal solution
m: maximum length of any path in the state space
C*: cost of optimal solution

Summary: All search strategies

Algorithm	Complete?	Optimal?	Time complexity	Space complexity
BFS (Breadth-first search)	Yes	If all step costs are equal	$O(b^d)$	$O(b^d)$
Uniform-cost Search	Yes	Yes	Number of nodes with $g(n) \leq C^*$	
DFS	In finite spaces (cycles checking)	No	$O(b^m)$	$O(bm)$
IDS	Yes	If all step costs are equal	$O(b^d)$	$O(bd)$
Greedy best-first Search	In finite spaces (cycles checking)	No	Depends on heuristic	Worst case: $O(b^m)$ Best case: $O(bd)$
A* Search	Yes	Yes	Number of nodes with $g(n) + h(n) \leq C^*$	

Conclusion

- Tree search can be used for planning actions for **goal-based agents** in known, fully observable and deterministic environments.

- Issues are:
  - The large search space typically does not fit into memory. We use a description using a compact **transition model.**
  - The search tree is built on the fly, and we have to deal with **cycles and redundant paths**.

- IDS is a memory efficient methods used in AI often for **uninformed search**.
- **Informed search** uses heuristics based on knowledge or percepts to improve search (i.e., A* to expand fewer nodes).