Linear Algebra

GA DAT2
Misrab M. Faizullah-Khan

- We use the notation a_{ij} (or A_{ij}, $A_{i,j}$, etc) to denote the entry of A in the ith row and jth column:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

- We denote the jth column of A by a_j or $A_{:,j}$:

$$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}.$$

- We denote the ith row of A by a_i^T or $A_{i,:}$:

$$A = \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix}.$$

Matrix Multiplication Review

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & 10 \end{bmatrix}$$

Column combination

$$y = Ax = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \\ a_1 \\ \\ \end{bmatrix} x_1 + \begin{bmatrix} \\ a_2 \\ \\ \end{bmatrix} x_2 + \ldots + \begin{bmatrix} \\ a_n \\ \\ \end{bmatrix} x_n$$

Row combination

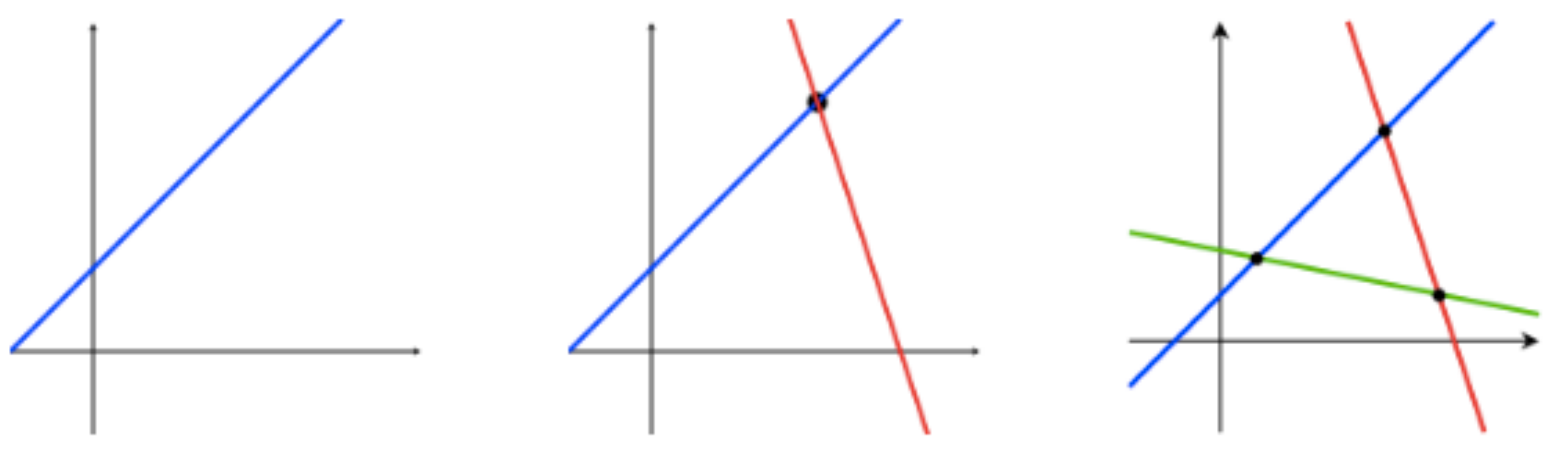
$$\begin{aligned} y^T &= x^T A \\ &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} — & a_1^T & — \\ — & a_2^T & — \\ & \vdots & \\ — & a_m^T & — \end{bmatrix} \\ &= x_1 \begin{bmatrix} — & a_1^T & — \end{bmatrix} + x_2 \begin{bmatrix} — & a_2^T & — \end{bmatrix} + ... + x_n \begin{bmatrix} — & a_n^T & — \end{bmatrix} \end{aligned}$$

Simultaneous System

convert the following into matrix notation

$$2x + 2y = 1$$
$$4x + 5y = 3$$

Simultaneous System



Simultaneous System - Underdetermined $(n < d)$

"more variables than equations"

$$2x + 2y + \mathbf{4z} = 1$$
$$4x + 5y + \mathbf{3z} = 3$$

Simultaneous System - Underdetermined (n < d)

$$2x + 2y + \mathbf{4z} = 1 \quad (1)$$
$$4x + 5y + \mathbf{3z} = 3 \quad (2)$$

(2)-2*(1): $y = 1 + 5z$

sub into (2):

$4x + 5 + 25z + 3z = 3$

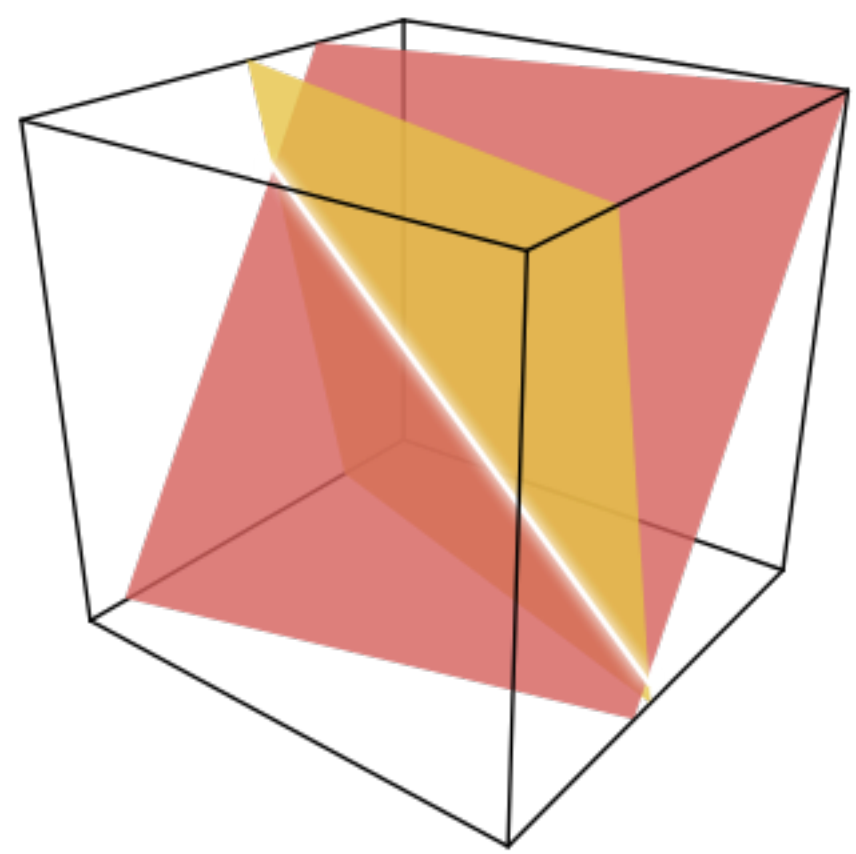
$\Rightarrow x = (-2 - 28z) / 4$

“free variable”

Simultaneous System - Underdetermined (n < d)

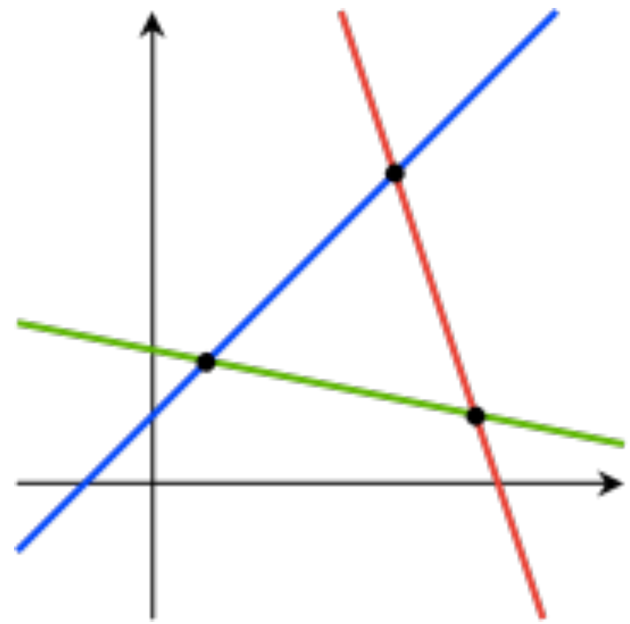
$$y = 1 + 5z$$

$$x = (-2 - 28z) / 4$$



Simultaneous System - Overdetermined (n > d)

"More equations (constraints) than variables"



Simultaneous System - Overdetermined (n > d)

"More equations (constraints) than variables"

Usually the case!! No solution?

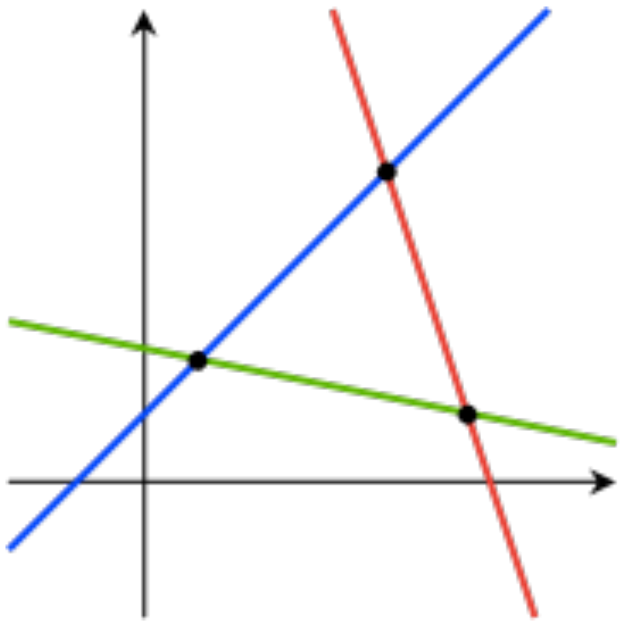
150 observations *(n = 150)*

Fisher's *Iris* Data

Sepal length	Sepal width	Petal length	Petal width	Species
5.1	3.5	1.4	0.2	*I. setosa*
4.9	3.0	1.4	0.2	*I. setosa*
4.7	3.2	1.3	0.2	*I. setosa*
4.6	3.1	1.5	0.2	*I. setosa*
5.0	3.6	1.4	0.2	*I. setosa*
5.4	3.9	1.7	0.4	*I. setosa*
4.6	3.4	1.4	0.3	*I. setosa*
5.0	3.4	1.5	0.2	*I. setosa*

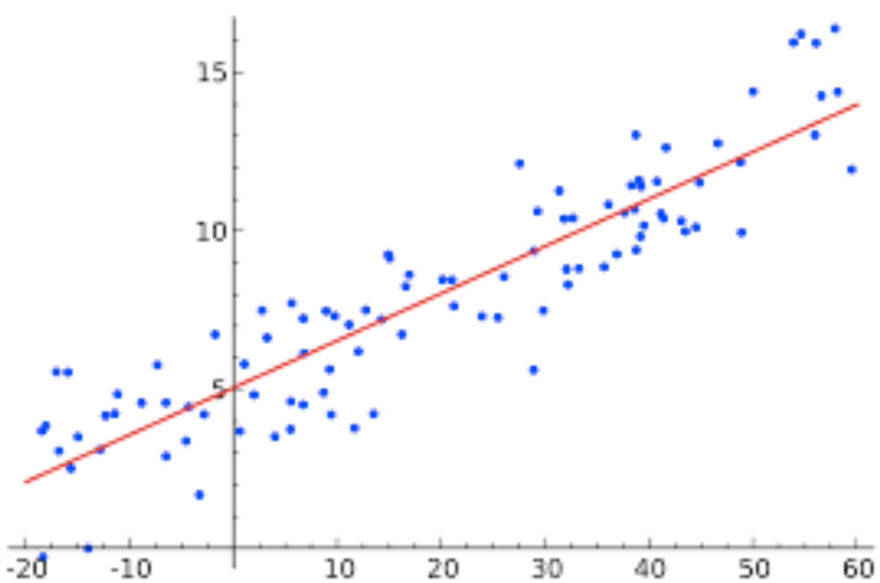
4 features (p = 4)

response

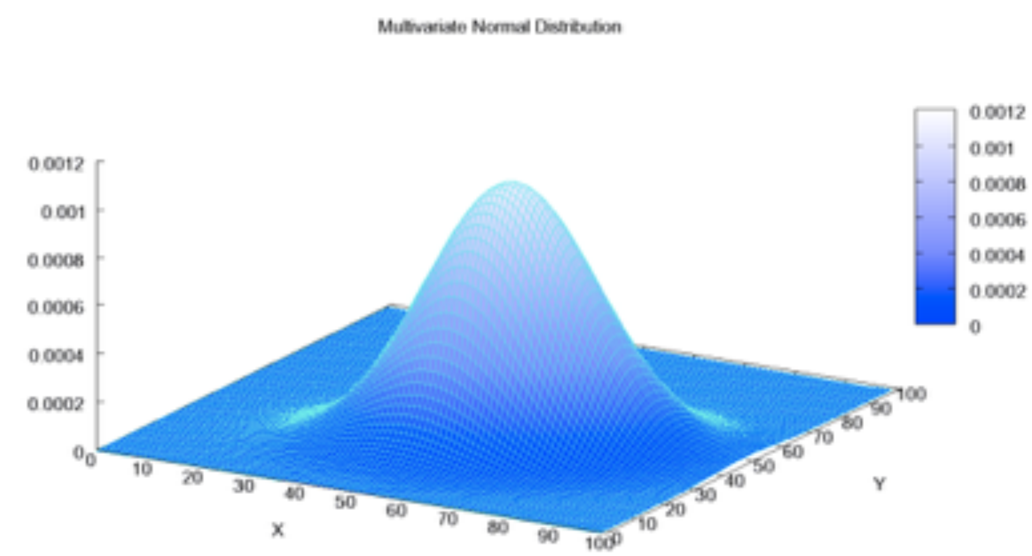


Finding Inverse

Who cares?



least squares



$$f_{\mathbf{x}}(x_1, \ldots, x_k) = \frac{1}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right)$$

multivariate normal

Finding Inverse

https://www.khanacademy.org/math/algebra-home/alg-matrices/alg-determinants-and-inverses-of-large-matrices/v/inverting-matrices-part-3

Finding Pseudo-Inverse

‘best fit’ (least squares) when no solution

If the linear system

$$Ax = b$$

has any solutions, they are all given by

$$x = A^{+}b + [I - A^{+}A]w$$

LU Decomposition

If A is a **square** matrix

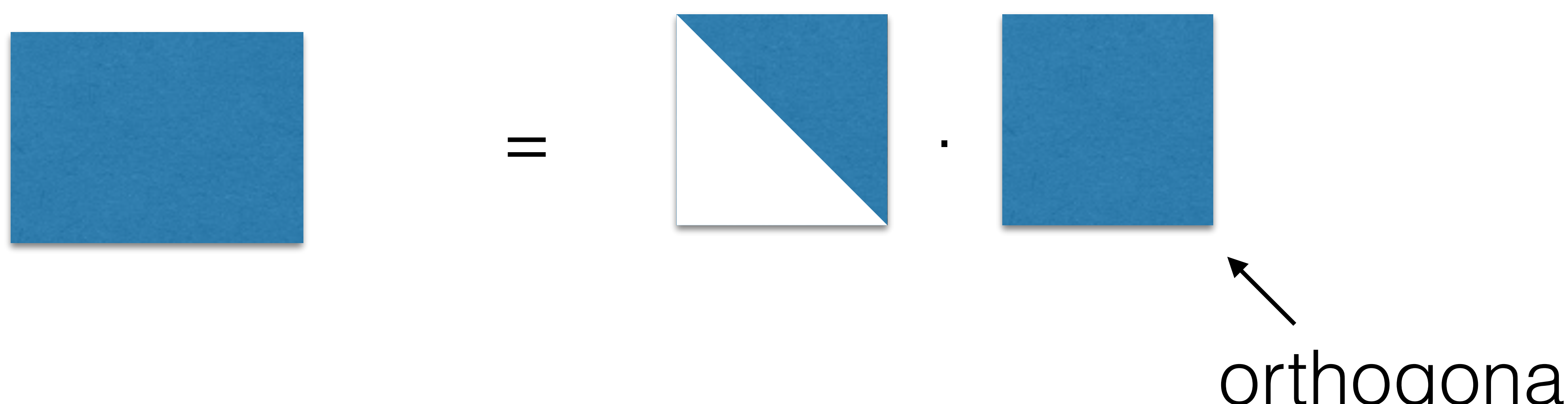
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

A = LU

QR Decomposition

Used in many efficient methods

If A is a (real) **square** matrix



A = QR

QQ.T = I

R upper triangular

SVDecomposition

Norms

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_\infty = \max_i |x_i|.$$

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}.$$

Who wants to draw?

Inverses

$$A^{-1}A = I = AA^{-1}.$$

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^{-1})^T = (A^T)^{-1}$. For this reason this matrix is often denoted A^{-T}.