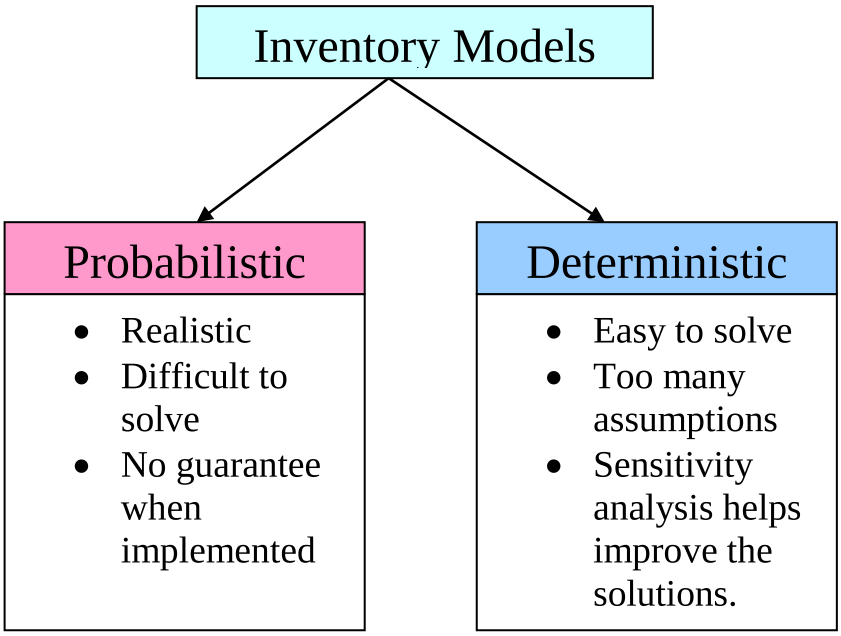
Inventory Control



Costs involved in Inventory Models:

1. Ordering/setup costs (paperwork, billing, fixed costs …).
2. Unit purchasing cost.
3. Holding/carrying cost (storage costs, insurance, cooling, spoilage, taxes, depreciation…).
4. Stock out/shortage costs (loss of sales, loss of goodwill/reputation, higher costs in future, purchasing costs, overtime…).

Deterministic EOQ Models

EOQ – Economic Order Quantity

Assumptions:

1. Repetitive ordering.
2. Constant demand over time.
3. Constant lead times.
4. Continuous ordering – an order can be placed at any time.

Basic EOQ Model

1. Single item problems.
2. Demand is deterministic and at a constant rate.
3. A fixed setup cost is incurred for every order.
4. The lead time for each order is 0.
5. No shortages are allowed.
6. The cost of holding inventory is fixed.

Denote:

q – Number of units per order.

D – demand per year (units/year)

K – fixed ordering cost (independent of q).

p – purchasing cost of 1 unit.

h – annual holding cost per unit.

I(t) – inventory level at time t.

L – lead time. L = time the order arrives – time the order was placed. *In the basic model* L = 0.

Total inventory cost:

TC(q) = annual cost of placing orders + annual purchasing cost + annual holding cost.

A. *annual cost of placing orders*

D – demand per year (units/year)

q – Number of units per order.

→number of orders per year is $\dfrac{D}{q}$

→cost of placing orders per year is $\dfrac{KD}{q}$

B. *annual purchasing cost*

D – demand per year (units/year)

p – unit purchasing cost.

→cost of purchasing per year is pD

C. *annual holding cost*

I(t) – inventory level at time t.

$\bar{I}(t)$ - Average inventory level at the interval $[0,t]$.

→cost of holding per year is $h \cdot 1 \cdot \bar{I}(1\,year)$

$$\bar{I}(T) = \int_0^T \frac{I(t)}{T}\,dt$$

Inventory levels:

Cycle – the length of time between two consecutive orders.

D – demand per year (units/year)

q – Number of units per order.

→number of cycles per year is $\frac{D}{q}$

→length of 1 cycle is: $\frac{1}{D/q} = \frac{q}{D}\,years$

Inventory level:

- At the beginning of a cycle: q.
- At the end of a cycle: 0.

The demand is linear in time → the *average* inventory level is $\frac{q}{2}$.

Therefore, the holding cost per cycle is

$h \cdot \frac{q}{D} \cdot \frac{q}{2} = \frac{hq^2}{2D}$ (cost per unit X length of cycle X average inventory level)

The annual holding cost is: Holding cost per cycle X number of cycles per year - $\frac{hq^2}{2D} \cdot \frac{D}{q} = \frac{hq}{2}$

Total inventory cost:

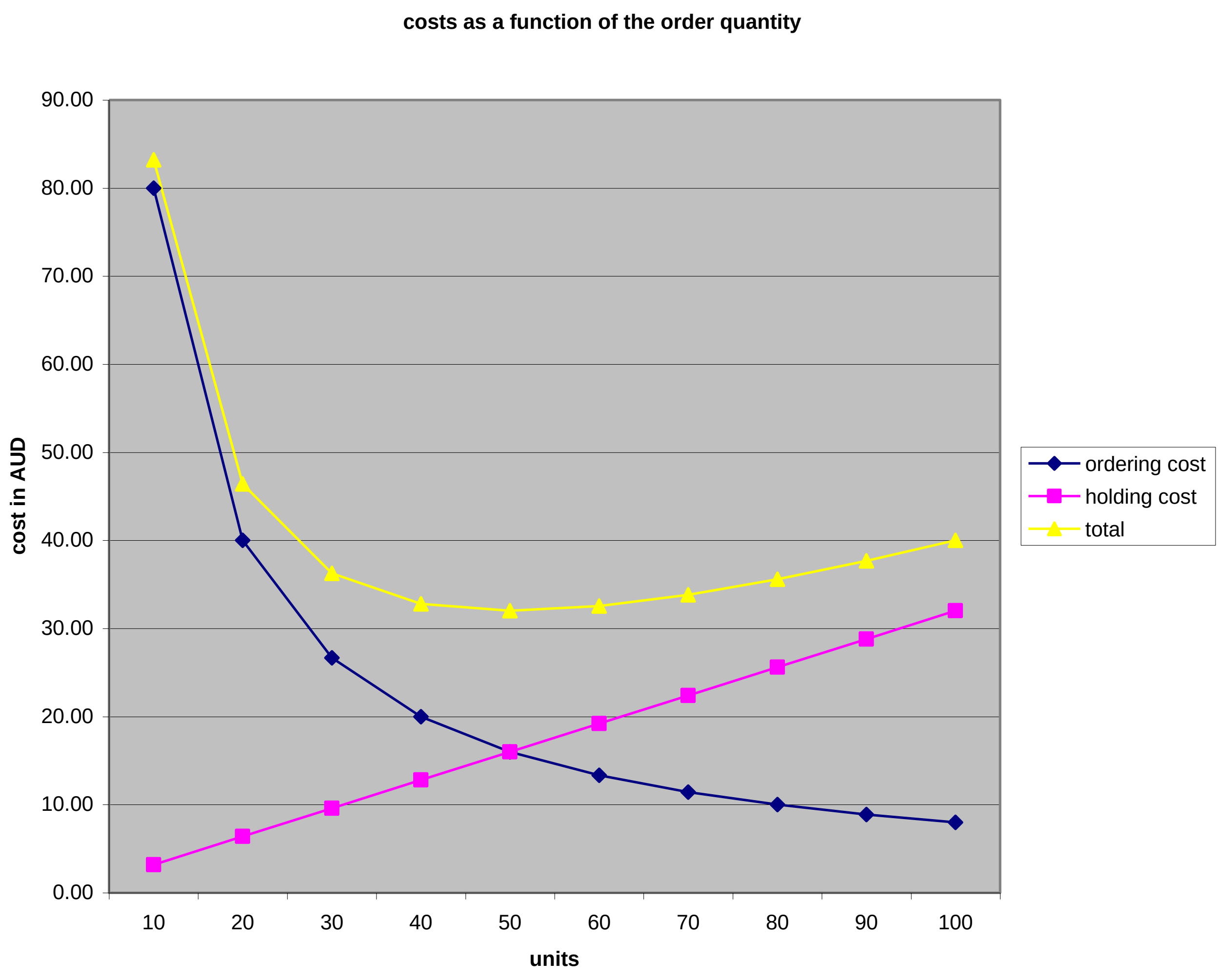
$$TC(q) = \frac{KD}{q} + pD + \frac{hq}{2}$$

$$\frac{\partial TC(q)}{\partial q} = -\frac{KD}{q^2} + \frac{h}{2} = 0 \quad \Rightarrow \quad \frac{KD}{q^2} = \frac{h}{2} \quad \Rightarrow \quad \frac{2KD}{h} = q^2$$

$$\Rightarrow \quad q^* = \sqrt{\frac{2KD}{h}}$$

Note that:

$$\frac{\partial^2 TC(q)}{\partial q^2} = \frac{2KD}{q^3} > 0 \quad for\ all\ q > 0 \qquad \Rightarrow$$ this is a ***minimum*** point.



Example:

The university bookstore would like to order an OR textbook. The following details are known:

Annual demand $D = 100$

Cost of holding stock $h = 0.64\,AUD$

Cost of placing an order $K = 8\,AUD$

How many books should be ordered each time? How frequently should an order be placed?

$$q^* = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2 \cdot 8 \cdot 100}{0.64}} = 50$$

The bookstore should order books in batches of 50.

$$\frac{q^*}{D} = \frac{50}{100} = \frac{1}{2}$$

An order should be placed every half year (two times a year).

Sensitivity analysis:

	q	ordering cost	holding cost	total
D	10	80.00	3.20	83.20
100	20	40.00	6.40	46.40
h	30	26.67	9.60	36.27
0.64	40	20.00	12.80	32.80
k	50	**16.00**	**16.00**	**32.00**
8	60	13.33	19.20	32.53
	70	11.43	22.40	33.83
	80	10.00	25.60	35.60
	90	8.89	28.80	37.69
	100	8.00	32.00	40.00

Try to explain graphically the level of inventory as a function on time.

Note that the graph looks like a saw.

- What happens when the lead time is not zero?
- There exists a reorder point: a time when the order must be placed to ensure that there is no stockout.

$L \neq 0$

- The order should be placed when LD units are left in stock.
- Get your units right!
- If LD exceeds EOQ(q*), then the order should be placed some cycles in advance.
- Divide LD by the EOQ and calculate the remainder. Each time your stock reaches this level, place an order for q units.

Multi-Object EOQ Model

In many cases customers order more than a single item from the producer/supplier. It is difficult to synchronize the arrival/order of items.

Power of two ordering policies:

1. Find EOQ for all items: $q_1^*, q_2^*, \cdots$.
2. Calculate cycles for all items:

$$t_i^* = \frac{q_i^*}{D_i} \quad for\ all\ i$$

3. for each item find m_i : $2^{m_i} \le t_i^* \le 2^{m_i+1}$
4. If $t_i^* \le \sqrt{2} \cdot 2^{m_i}$ reorder every 2^{m_i} time units.

   If $t_i^* > \sqrt{2} \cdot 2^{m_i}$ reorder every 2^{m_i+1} time units.

We ensure no stockouts, and concentrate orders saving money on delivery.

Roundy (1985) - provided proof that this policy increases cost by no more than 6% in comparison to regular EOQ.

Example

A Company orders 4 products:

A – Every 3.9 days.

B – Every 6.8 days.

C – Every 12.1 days.

D – Every 17.6 days.

What is the best power of two policy for this company?

PRODUCT	CYCLE	INTERVAL	m_i	$\sqrt{2}\cdot 2^{m_i}$	ORDERING INTERVAL
A	3.9	$2 \le 3.9 \le 2^2 = 4$	1	2.83	4
B	6.8	$2^2 \le 6.8 \le 2^3 = 8$	2	5.66	8
C	12.1	$2^3 \le 12.1 \le 2^4 = 16$	3	11.31	16
D	17.6	$2^4 \le 17.6 \le 2^5 = 32$	4	22.63	16

Ordering Schedule:

T=0	T=4	T=8	T=12	T=16, 0
A,B,C,D	A	A,B	A	A,B,C,D

EOQ with price breaks/quantity discounts

Larger quantities are usually associated with lower prices per unit:

Price per unit	Quantity
p_1	$q < b_1$
p_2	$b_1 \le q < b_2$
…	…
p_{k-1}	$b_{k-2} \le q < b_{k-1}$
p_k	$b_{k-1} \le q < b_k = \infty$

Assumption: $p_k < p_{k-1} < \ldots < p_2 < p_1$.

How do we solve the problem?

1. For each price find the optimal ordering quantity EOQ_i.
2. If $EOQ_i \ge b_i$ then $q_i^* = EOQ_i$.

   If $EOQ_i < b_i$ then $q_i^* = b_{i-1}$.
3. Calculate TC for all q_i^* values, $i = 1,2,\ldots,k$.
4. Choose the lowest cost as your ordering policy.

Example

Problem 1 pg. 864 in Winston.

Purchasing of computer paper

Ordering cost: $K = \$20$

Holding cost: 20% of the dollar value per year.

Demand: $D = 80$ boxes per month.

Prices:

Price per unit	Quantity
\$10/box	$q < 300$
\$9.8/box	$300 \le q < 500$
\$9.7/box	$500 \le q$

For $q < 300$:

$$EOQ_1 = \sqrt{2 \cdot 20 \cdot 960 / 10 \cdot 0.2} = 138.56$$

$$\Rightarrow q_1^* = 138.56$$

$$\Rightarrow TC = 960 \cdot 10 + \frac{20 \cdot 960}{138.56} + 0.2 \cdot 10 \cdot \frac{138.56}{2} = 9877.12$$

For $300 \le q < 500$:

$$EOQ_2 = \sqrt{2 \cdot 20 \cdot 960 / 9.8 \cdot 0.2} = 139.97$$

$$\Rightarrow q_2^* = 300$$

$$\Rightarrow TC = 960 \cdot 9.8 + \frac{20 \cdot 960}{300} + 0.2 \cdot 9.8 \cdot \frac{300}{2} = 9766$$

For $500 \le q$:

$$EOQ_3 = \sqrt{2 \cdot 20 \cdot 960 / 9.7 \cdot 0.2} = 140.69$$

$$\Rightarrow q_3^* = 500$$

$$\Rightarrow TC = 960 \cdot 9.8 + \frac{20 \cdot 960}{500} + 0.2 \cdot 9.8 \cdot \frac{500}{2} = 9835.4$$

The optimal ordering policy is, therefore: 300 boxes each time incurring a total cost of $9766 per year.

The continuous rate EOQ model

Used when order arrive continuously, e.g., when products are internally produced.

The rate of production is *r* units/year.

Note:

- If $r < D$ - the problem is infeasible.
- If $r = D$ - the problem is solved by continuous production and no stock.
- If $r > D$ - the problem is solved by the continuous EOQ model.

Denote:

q – Number of units produced during each production run.

D – Demand per year (units/year)

K – Setup cost of starting a run.

h – Annual holding cost per unit.

Try and represent the inventory level as a function of *r* and *D* graphically.

Annual Cost = Annual Holding Cost + Annual Setup Cost + Annual Production Cost

Note

- The annual production cost is independent of the number of products produced each run: pD.

- Annual Holding cost = h*(annual average inventory)

Denote:

T – The total time of each production run.

T` – The total time between the end of a production run and the beginning of the next production run.

$I_{\max}$ - The maximum inventory level (at time T)

It follows that:

$$T = \frac{I_{\max}}{r - D} \quad and \quad T' = \frac{I_{\max}}{D}$$

The length of 1 cycle is therefore:

$$Cycle = T + T' = \frac{I_{\max}}{r-D} + \frac{I_{\max}}{D} = \frac{I_{\max}(D+r-D)}{(r-D)D} = \frac{I_{\max} r}{(r-D)D}$$

Next we'll try and link the number of products per run to $I_{\max}$:

q is the number of products produced at a rate of r over a time period of T. Therefore: $q = rT$.

Using the fact that $T = \frac{I_{\max}}{r-D}$, we obtain that

$$I_{\max} = T(r-d) = \frac{q}{r}(r-D)$$

And

Annual average inventory: $\bar{I} = \frac{q}{2r}(r-D)$.

Inserting the value of $I_{\max}$ in the equation for the cycle length yields:

$$Cycle = \frac{I_{\max} r}{(r-D)D} = \frac{T(r-D)r}{(r-D)D} = \frac{\frac{q}{r}(r-D)r}{(r-D)D} = \frac{q}{D}$$

Therefore, the total ***Annual*** holding cost is:

Holding Cost = $h \cdot \dfrac{q(r-D)}{2r}$

The number of cycles per year is:

Cycles per year = $\dfrac{1}{q/D} = \dfrac{D}{q}$

The total ***Annual Cost*** is therefore:

$$h \cdot \frac{q}{2r}(r-D) + \frac{KD}{q} + pD$$

Note that there is no need to optimize again as the cost function is similar to the basic EOQ model:

$$q^* = \sqrt{\frac{2KD}{h(r-D)/r}} = \sqrt{\frac{2KDr}{h(r-D)}}$$

Note that:

$$q^* = \sqrt{\frac{2KDr}{h(r-D)}} = \sqrt{\frac{2KD}{h}} \cdot \sqrt{\frac{r}{r-D}} = Basic\ EOQ \cdot \sqrt{\frac{r}{r-D}}$$

Example:

Ford Australia sells 20,000 Falcons per year. The engines are produced by a single plant based in Sydney. The production rate is 75,000 engines per year.

The setup cost incurred whenever a new batch of engines is produced is \$50,000. The holding cost of an engine is \$350 per year.

How many engines should be produced in every run?

How many times a year should the plant manufacture?

Solution:

$$q^* = \sqrt{\frac{2KDr}{h(r-D)}} = \sqrt{\frac{2 \cdot 50,000 \cdot 20,000}{350} \cdot \frac{75,000}{75,000-20,000}} = 2791.45$$

The number of cycles per year:

$$\frac{D}{q^*} = \frac{20,000}{2791.45} = 7.165\ cycles\ .$$

Special cases:

1. If $r = D$: continuous production.

$$q^* = Basic\ \ EOQ \cdot \sqrt{\frac{r}{r-D}} = \infty$$

2. Setup costs are very high:

$$\lim_{K \to \infty} q^* = \infty$$

production never stops. The rate of production is modified in order to keep a reasonable level of inventory. Examples: oil refineries, electrical plants, nuclear generators, semi-conductor industry.

3. Holding costs are very high:

$$\lim_{h \to \infty} q^* = 0$$

in this case D setups are required. JIT ideology, custom made products: Ships, industrial machines, Ferrari cars.

EOQ Model with Backorders

When demand is not met:

- Loss of goodwill.
- Loss of sales.
- Customer compensation.
- Fines.
- High production costs, overtime, outsourcing.

In most cases it is difficult to assess the cost of shortage. This cost may vary on the market and the business involved.

Model:

q – Order quantity.

M – Maximum shortage that occurs.

s – Shortage cost per unit per year.

Try to represent these values graphically (Inventory level as a function of time).

Every time an order is placed:

- q products arrive.
- M products are immediately dispatched to waiting customers.
- $q-M$ units are placed in stock.

- After $\frac{q-M}{D}$ time units the inventory level is 0 and backorders are accumulated.
- After $\frac{M}{D}$ additional time units the inventory level is at its lowest $-M$ and a new order is placed.

  Two decisions must be made:

1. q - the optimal ordering quantity.
2. M - the maximum shortage allowed (or alternatively, $q-M$ the maximum inventory level).

Annual Cost = Annual Holding Cost + Annual Ordering Cost + Annual Production Cost + ***Annual shortage Cost****.*

Annual Holding cost:

Over 1 cycle: Inventory goes from $q-M$ to 0. At time 0 the inventory level is $q-M$. After $\frac{q-M}{D}$ time units the inventory goes down to 0. During the next $\frac{M}{D}$ time units the inventory is 0, as orders are not met and backorders are accumulated. Therefore, the inventory level over a cycle is: $\frac{q-M}{2}\cdot\frac{q-M}{D}+0\cdot\frac{M}{D}=\frac{(q-M)^2}{2D}$.

Look out this is very tricky!!!

Annual Holding Cost:

$$h\cdot\frac{Average\,inventory}{cycle}\cdot\frac{cycles}{year}=h\frac{(q-M)^2}{2D}\cdot\frac{D}{q}=\frac{h(q-M)^2}{2q}$$

Annual shortage cost:

Similarly, we calculate the annual shortage cost. Over 1 cycle: the shortage level is 0 in the time interval between 0 and $\frac{q-M}{D}$. At time $\frac{q-M}{D}$ the inventory level is 0, and orders begin accumulating. After $\frac{M}{D}$ time units the shortage level reaches M and a new order is placed. Therefore, the shortage level over a cycle is:

$$0 \cdot \frac{q-M}{D} + \frac{M}{2} \cdot \frac{M}{D} = \frac{M^2}{2D}.$$

Look out this is very tricky!!!

Annual Shortage Cost:

$$s \cdot \frac{Average\ shortage}{cycle} \cdot \frac{cycles}{year} = s\frac{M^2}{2D} \cdot \frac{D}{q} = \frac{sM^2}{2q}$$

Annual ordering cost:

As in previous cases: $\frac{KD}{q}$.

Annual purchasing cost:

As in previous cases: pD.

Total Annual cost:

$$TC(q,M) = pD + \frac{KD}{q} + \frac{h(q-M)^2}{2q} + \frac{sM^2}{2q}$$

Finding the minimum for this function:

$$\frac{\partial TC(q,M)}{\partial q} = -\frac{KD}{q^2} + \frac{h(q-M)}{q} - \frac{h(q-M)^2}{2q^2} - \frac{sM^2}{2q^2}$$

$$\frac{\partial TC(q,M)}{\partial M} = -\frac{h(q-M)}{q} + \frac{sM}{q}$$

Equating both expressions to 0 and combining them yields:

$$q^* = \sqrt{\frac{2KD(h+s)}{hs}} = BASIC\ EOQ \cdot \sqrt{\frac{h+s}{s}}$$

$$(q-M)^* = \sqrt{\frac{2KDs}{h(h+s)}} = BASIC\ EOQ \cdot \sqrt{\frac{s}{h+s}}$$

Note that:

$$\frac{\partial^2 TC(q,M)}{\partial q^2} > 0 \quad and \quad \frac{\partial^2 TC(q,M)}{\partial M^2} > 0$$

$$for \ all \, values \ of \ \ q, M > 0$$

$\Rightarrow$ This is a ***minimum*** point.

Example

A car dealer must pay $20,000 for each car purchased. The annual holding cost is estimated to be 25% of the dollar value of inventory. The dealer sells an average of 500 cars per year. He believes that demand is backlogged but estimates that if he is short one car for one year, he will lose $20,000 worth of future profits. Each time the dealer places an order for cars, ordering cost amounts to $10,000. Determine the car dealer's optimal ordering policy. What is the maximum shortage level that will occur?

Solution:

$$q^* = \sqrt{\frac{2KD(h+s)}{hs}}$$
$$= \sqrt{\frac{2 \cdot 10{,}000 \cdot 500(0.25 \cdot 20{,}000 + 20{,}000)}{0.25 \cdot 20{,}000 \cdot 20{,}000}}$$
$$= 50$$

$$(q-M)^* = \sqrt{\frac{2KDs}{h(h+s)}}$$
$$= \sqrt{\frac{2 \cdot 10{,}000 \cdot 500 \cdot 20{,}000}{0.25 \cdot 20{,}000 \cdot (0.25 \cdot 20{,}000 + 20{,}000)}}$$
$$= 40$$

The dealer's optimal ordering policy is:

- Placing an order when the shortage level is 10 ($M^* = 10$).
- Ordering 50 cars at a time.
- Ordering 10 times a year.
- Total annual cost:

$$TC(q,M) = pD + \frac{KD}{q} + \frac{hM^2}{2q} + \frac{s(q-M)^2}{2q} =$$
$$20{,}000 \cdot 500 + \frac{10{,}000 \cdot 500}{50} + \frac{0.25 \cdot 20{,}000 \cdot 40^2}{2 \cdot 50} + \frac{20{,}000(50-40)^2}{2 \cdot 50}$$
$$= 10.2M$$

Extensions:

Try looking at different combinations of these models:

- Continuous Rate EOQ when shortage is allowed. In this case production begins when the max shortage level is reached. Production commences at a rate r while demand keeps arriving.
- Power of 2 ordering policies when shortage is allowed.
- Basic EOQ models when safety inventory is required – an order is placed when the inventory level reaches a certain level $a>0$. Note that for this case the basic EOQ remains the same. Only the Total costs increase.
- EOQ with price breaks (quantity discounts) with shortage.
- Think of some special cases that may occur!