

Open Recreational Problems: Simplex Chains

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Founder, Public Invention, an educational non-profit.

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Contributions and suggestions from the reader are encouraged!

2 Introduction

Note: This is a work in progress as a preparation for the Public Invention Mathathon planend for Nov. 30th-Dec. 2nd 2018. Please read it as work designed to give a flavor of the problem area rather than a tehcnical paper being prepared for publication.

The term *simplex* means the simplest regular polyhedron. In two dimensions, a simplex is an equilateral triangle. In three dimensions, a simplex is a regular tetrahedron. In four dimensions, it is just called a simplex. Presumably one can define a simplex in higher-dimensioned Euclidean spaces as well.

Define a *simplex chain* to be a figure of many simplices adjoined face-to-face by a particular rule. The dimensionality of a face is always one less than the dimensionality of the space. In two dimensions, a simplex chain is a series of adjoined equilateral triangles joined edge-to-edge.

Let us number the simplices in a chain starting from 0. Then we can define a simplex chain via a rule that says which edge of the n th simplex to attach the $n+1$ th simplex to. In two dimensions, we can label the edge of the last triangle in simplex chain as *anti-clockwise* or *clockwise*, or just *left* and *right*.

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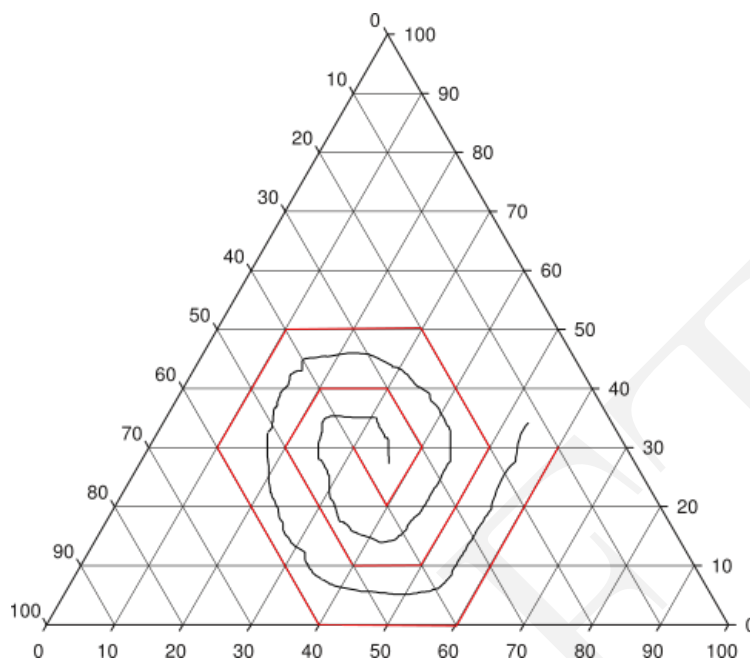


Figure 1: A Space Filling Spiral

It is clear that the rules “always go left” “always go right” produced a pretty hexagon, but then starts putting the triangles right where one already is.

The rule “go left if n is even, right if it is odd” is a bit more interesting; it produces a “ladder” that goes off in a straight line. That is, the whole figure is contained within two parallel lines separated by the height of the triangle.

Exercise: There is probably a simplex chain rule which tiles the entire plane with equilateral triangles. Can you define it?

2.1 The nature of this work

The problems posed here to make the Public Invention Mathathon interesting exist at the intersection of mathematics and computer science. The work is motivated by mechanical engineering and robotics.

2.2 A website to serve as a workbench

With the help of David Jeschke, we have written a website <https://pubinv.github.io/Mathathon-2018-Simplex-Chains/platforms/index.html> designed to allow graphic and interactive play of these concepts. This website will likely evolve faster than this document!

2.3 The mechanically interesting shapes

Although simplex chains are interesting in their own right purely theoretically and for recreation, they are of practical applied value as well. Mechanical and structural engineers are interested in simplex chains because when constructed out of real, physical members in the shape of the simplex chain, they may form rigid and structural strong shapes. For example, many trusses, used to hold up bridges and the roofs of buildings, are in fact simplex chains.

In two dimensions, there are a number of shapes which are interesting not just to mathematicians, but to engineers and architects. Often, these shapes are made of *trusses* which approximate geometric forms. In two dimensions the mechanically interesting shapes are:

- The line
- The triangle
- The circle
- The spiral
- The sine wave

The line is used to make columns and beams. The triangle is the basis of all trusses. Fractions of circles are arcs used to make bridges. Spirals are used to make clock springs. The sine wave is similar to the barrel vault and is used to make corrugated roofs and cardboard and baffles. Note that the triangle and the circle are closed shapes, the others are not.

General structures in three dimensions made by structural and mechanical engineers are called *space frames*. However, many space frames also follow geometric patterns. In three dimensions the mechanically interesting shapes are:

- The beam
- The triangle
- The tetrahedron
- The ring
- The helix
- The spiral

The beam, the triangle, the ring and the spiral can exist in a narrow space, like inside a wall, so they are similar to their 2-d analogs, and used for the same purposes.

The helix is used to make staircases and springs and heat exchangers.

Note that the ring and the triangle are a closed shape, the others are not. The helix does not really exist in two space, although the others do.

The only four-dimensional shape I can conceptualize is that of the beam: that is, a region close to a line embedded in 4-space.

3 On “Open Problems”

This research effort is focused on real *unsolved open problems*. Usually when we say “open problem” in mathematics we mean “unsolved, probably has a clear objective solution, is valuable, and is not obvious.” Open problems you hear about are mostly ones that have been studied a lot and remain unsolved, and are therefore probably difficult problems. However, in this research-a-thon, many of the open problems are probably easy—they are open and unsolved because nobody has yet cared to look very closely at them. However, they are still valuable. Hopefully the problems in this research-a-thon are “real”, even if they not as “hard” as those professional researchers talk about.

We will give opinions about problems on a scale¹ of 1 – 10, with 1 being quite easy and 10 being very difficult. Problems of difficulty 1, 2, 3 are generally “easy”, those of “8,9,10” hard. A problem of difficulty 10 is assumed to be publishable in a non-very-prestigious forum, though possibly easier problems will also be of interest.

4 Some problems of Regular 2D Simplex Chains

4.1 Generators

Define a *regular simplex chaining generator* to be a function takes natural numvbers and returns a symbol from the set $\mathbb{N} \mapsto \{L, R, S\}$, for *left*, *right*, and *stop*. The generator generates a simplex chain by defining to which edge a new triangle should be attached. That is, measured from the direction of a previous edge, should *n*th addition of a triangle be made by adding it to the left or right edge?

Here is an example of a generator in JavaScript:

```
function hex_generator(n) {
  return ((n < 6) ? "L" : "S");
}
```

It produces a simple hexagon. We will assume that regular triangles of unit side length.

¹This scale is inspired by Prof. Norman J Wildberger, who has ranked true open problem using a similar (but much more difficult!) gamut.

Although a 2D simplex chain exists in \mathbb{R}^2 , a generator is a much simpler object. It is somewhat easier to ask questions about generators.

- (Easy 1) R2D-01: Can you create a generator which creates a beam?
- (Easy 2) R2D-02: Can you create a generator which fills the plane?
- (Easy 3) R2D-03: Can you create a generator which completely covers a circle of radius r with no unnecessary triangles?
- (Medium 4) R2D-04: Can you create a generator which creates golden spiral?
- (Medium 4) R2D-05: Can you create a generator which completely and minimally covers a sinusoidal wave?
- (Medium 5) R2D-06: Can you create a generator which defines a logarithmic spiral which does not rely on mapping first to the Cartesian plane?

Problem R2D-06 raises an interesting issue. It is of course possible to create an algorithm which solves a problem in Cartesian space, and then map this back to the turnings of the generator. However, it would be more elegant to create an algorithm which never in fact uses real numbers.

4.2 Triangle Space Numbering

This is because any Regular 2D Simplex Chain, although it may be thought of as points in \mathbb{R}^2 , may also be thought of as simply selecting a set of triangles in an abstract space, which we call the *Triangle Space*, which is \mathbb{Z}^2 . By convention, we begin numbering the Triangle Space at an equilateral triangle point upward, whose apex is at the origin. Then any triangle can be named by (x, y) , where $y = 0$ on is the horizontal row of triangles below the x -axis. A triangle at position x has an apex pointing either upward or downward at position $x/2$.

²

4.3 On the reachability of points

Although too simple for a professional mathematician in two dimensions, let us consider the problem of *reaching* certain regions or points with a simplex chain from a starting point.

Let us for the time being consider *regular* simplex chains. In the Cartesian plane, it is pretty clear that any point can be *reached*, that is, intersected, by a simplex chain of finite

²Amit proposes a different numbering scheme for regular triangles: <http://www-cs-students.stanford.edu/~amitp/game-programming/grids/>

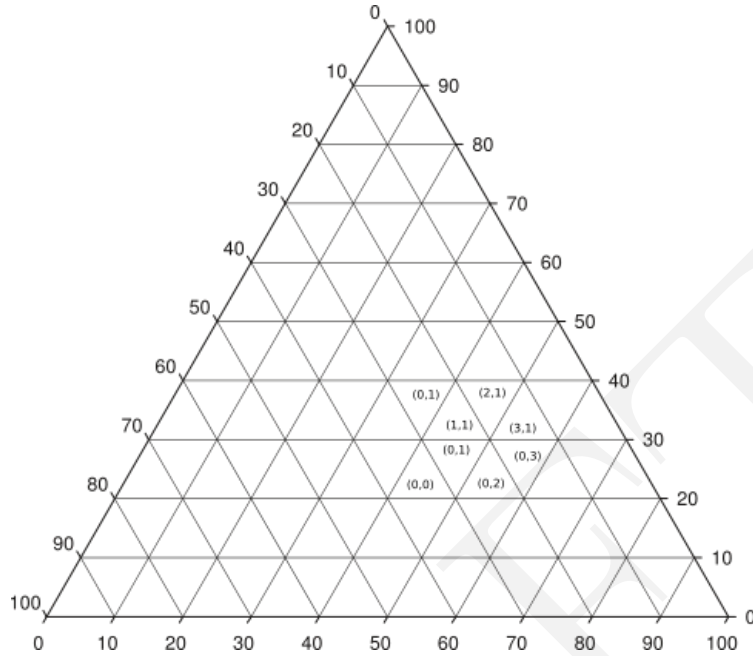


Figure 2: Triangle Space Numbering

length. Developing an algorithm to produce such a chain from the coordinates of a point is an easy problem. We could then pose questions about the shortest such chain, or the number of such chains.

Now consider the problem of reaching a point in 3D with a simplex chain formed only of regular tetrahedra. This is a much harder problem. It seems likely that any point can be intersected, except perhaps points very close to the starting point. Producing an algorithm to do so is more of a challenge. This is a very interesting problem to mechanical engineers, robotocists, and possibly molecular scientists.

Now if instead of merely *intersecting* a point with the body of simplex we instead seek to move the *apex* of a tetrahedra close to the point, we have a real challenge: What simplex chain of regular tetrahedra gets us closest to a point, and of what length? Can we come arbitrarily close to a point with arbitrary length chains?

An interestig, if somewhat wild idea, is to build a coordinate system around points based on reaching them with simplex chains. In this model we do not start with the Cartesian plane; rather we “name” points by the simplex chain we reaches them. What is the nature of a space of such points?

5 Relaxations of Regularity

5.1 Boundedness

If we relax the rule that all the lengths of simplex are exactly the same, then it becomes possible to create new figures which are very close to regular. In particular we can define a simplex chain, or any structure, be *x-bounded* if the ratio of the longest edge in the figure to the shortest edge is x .

We can then ask questions (in 3D) such as:

What is the smallest number of tetrahedra needed to define an $b+c/b$ torus as a function of b and c ?

Or:

What is the lowest x -boundedness of a figure that defines a torus?

A bound on this number can be given based on my previous work; however, there is almost certainly a better solution that “twists” about the torus.

One way to investigate this is simply to place points on the surface of a torus and draw edges between them so as to create an irregular tetrahelix.

5.2 Rational Regularity

Define a *k-regular tetrahelix* to be tetrahelix have at most k distinct edge lengths. Define a *k-regular rational tetrahelix* to be a k -regular tetrahelix in which all lengths are rational.

We can then ask what is the smallest figure with a hole in each of these dimensions?

Note: There exists a 3-regular rational tetrahelix that is a straight line, or a beam.

6 Two dimensions

Two-dimensional shapes are much easier to draw and think about than three-dimensional shapes, but there are still some interesting open problems.

- (Easy 3 R2D-07) Can we produce more interesting shapes by creating chain rules? (This is largely a recreational problem.)
- (Medium 4 R2D-08) Assuming that all lengths in a regular simplex are of size 1 and we seek to approximate a circle of radius r with a simplex chain, what rule best does that? What is the best definition of “best”?
- (Easy 3 R2D-09) If we consider irregular simplices, how do we best approximate a circle?
- (Medium 4 R2D-10) What is the best way to define “approximating” an arbitrary curve with a simplex chain? Is it the sum of the distances of the centroid to the curve? Or some notion based on completely covering the curve and not self-colliding?

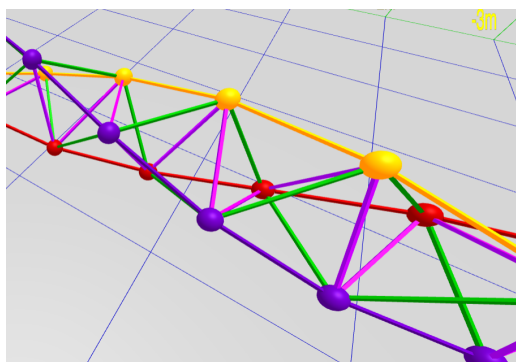


Figure 3: BC Helix Close-up (partly along axis)

- (Easy 3 R2D-11) How do we best approximate an Archimedian spiral? A logarithmic spiral? A Fibonacci spiral?
- (Hard 7 R2D-08) Can we define a notional of “translational periodicity” of a regular simplex chain? That is, can we prove that a given rule for producing a simplex chain produces of series of translations of a smaller simplex chain? For example, if we have a simplex chain which approximates a sine wave or sawtooth wave, can prove that with a periodicity of k it is self-similar?

7 Three dimensions

The problem of simplex chains in three dimensions is much more interesting. In particular, there is a structure known as the tetrahelix which is analogous to 2D ladder: it fits inside a cylinder. That is, it is a “straight”, uncurved stack of tetrahedra extending infinitely.

In order to produce a tetrahelix as a simplex chain, we must have a labeling scheme for the faces of a tetrahedron. Although symmetric, the first tetrahedron has 4 faces against which we can place the second tetrahelix. Having placed the second tetrahelix, there are 3 faces against which we can place the third, if we seek not to collide with the first. These 3 are symmetric; there is no good way to distinguish them, so in a sense it doesn’t matter where we place the third.

But when we come to place the fourth, the figure changes. The three tetrahelices have a clear “spine”. If we place the fourth on the face that does not touch the spine, we start to create a shape somewhat like a spindle or a top. In fact if we carry on this way we can place 6 such tetrahedral, with a 10% gap remaining. If we carry on, we just keep going around, creating a solid of revolution.

But what if we place the fourth to the left or right of the spine, and adopt this as a rule? Then we create clockwise or counterclockwise tetrahelix, respectively, and the we can go on forever, always staying inside a cylinder. It is curious that the rotation about

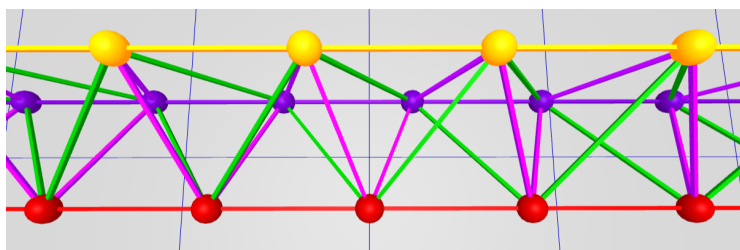


Figure 4: Equitetrabeam, an irregular tetrahelix

the center of the cylinder is an irrational fraction of circle, so we will never come back to precisely the same point.

- (Hard 8 R3D-01) Can we produce more interesting shapes by adopting more complex rules?
- (Hard 9 I3D-02) Can we find a rule that fills all space with irregular tetrahedra?
- (Hard 8 R3D-03) If we forbid collisions, can we find a rule that eventually comes near to all points?
- (Hard 7 R3D-04) What is the maximum fraction of the total volume of 3-Space can we fill (without collisions)?
- (Hard 7 R3D-05) Can we define a rule that fits optimally inside a torus?
- (Hard 7 R3D-06) Can we define a rule that follows a helix (that is, that stays close to a helix, as if we took a cylinder and bent it into a helical shape, like a thick wire of steel coiled into a spring)?
- (Hard 9 R3D-07) Can we define ANY rule using only regular tetrahedra that forms a closed loop with no gaps? If not, can we prove that we cannot?

Note: We use the term “equitetrabeam” to refer to an irregular tetrahelix that has uncurved “rails”, as in the figure belows:

- (Medium 6 I3D-07) What change length is required to make a finger ring?
- (Hard 8 I3D-08) Clearly Equitetrabeams are stackable, in fact you can make a hexagon of them. Can we make a twisted tetrabeam with 3 or 6 columns that is even more efficient (in terms of the ratios of edge lengths)?
- (Medium 7 I3D-09) What is the minimum change length to make a torus?

- (Medium 6 I3D-10) What is the minimum change length of a tetrobot that can touch its toe?
- (Hard 8 I3D-11) What is the densest way to pack a tetrobot into a given volume? (Bounds acceptable)
- Can we develop easy math for a tetrahelical “disc touching”?
- (Hard 8 R3D-12) Can we develop math for “regular variations” (no kinks)?
- (Hard 8 R3D-13) Can we develop math for “one kink” regular variations?
- (Hard 10 I3D-14) Can we make a torus that turns itself inside out via continuous changes of edge lengths, creating circular motion of a point around cross-section of torus? (In a sense, this would be a nano-technology motor.)³
- (Hard 8 I3D-15) Is there a formula for producing a helix of tetrahelices?
- (Hard 8 I3D-16) Is there a way to “knot” Tetrahelices into a slab?⁴
- (Hard 8 I3D-17) Is there a way to “wind” tetrahelices into column?
- (Hard 8 XXX) Is there a way to build a tensegrity tetrahedron?⁵
- If there is a tensegrity tetrahedron, how do we mathematically relate to edge lengths?
- (Hard 8 I3D-03) How fast can the tip of a tetrahelix move given a rate of change of member length? Can we define a this derivative with a closed form expression, rather than using matrix-based arbitrary Jacobians?
- (Hard 8 I3D-18) Can we improve a software toolkit that let’s us play with composing tetrahelices.
- (Hard 9 I3D-19) If we mate tetrahelices face to face, how regular a “tetrahelical tetrahedron” can we make?
- (Hard 8 R3D-20) Is it possible to make a perfectly regular structure that has a “hole”?
- (Medium 7 R3D-21) If it is not possible, what is most regular system that an make a “hole”?

Define *point reachability* to mean place the apex of a simplex chain near a point.

³I might attack this problem by attempting to disprove it, and then writing code to make it happen if I could not disprove it/

⁴This is poorly defined.

⁵This problem has been well-researched, it should start with a literature search in robotics.

- (Medium 7 R2D-03) In 2D, give an algorithm that produces the shortest regular simplex chain coming as close as possible to a point.
- (Hard 7 R3D-22) In 3D, define spaces which are not reachable by a simplex chain of regular tetrahedra of any length, assuming the first tetrahedra is fixed at the origin.
- (Hard 7 R3D-22) Given a limit to the number of simplices, what is the reachable space to with distance x ?
- (Hard 9 I3D-23) What is the shape of reachable points for N simplices?
- (Hard 9 I3D-24) In 3D, given an algorithm that produces the shortest simplex chain that comes with x of a target point.

8 Is there a rule that produces “straight” 4D simplex chains?

If we imagine moving into a four dimensional space, does it make sense to imagine a simplex chain, even though this will have not solid physical reality as we know it.

The “face” of such a figure is in face a 3-dimensional shape, can “adjoining” means that two 4D simplices have faces in the same 3D volume.

Possibly we could write a computer program that would allow us to render projections of such figures. I cannot imagine such shapes, and in a way they become less interesting and less practical than 3D simplex chains. However, one question seems interesting:

Is there a rule that produces a simplex chain in 4D which would produce a “straight” figure? That is, a figure which always remained within in a fixed distance of a line embedded in 4-space?

Idea: The problem doesn’t limit the computational complexity of the chain rule producing the simplex. However, if we evaluate the simplest rules first, we could write a computer program to simply try a large number of them. A “simple” rule could be considered a function only of the number “ n ” modulo some small number “ m ”.

If you consider only simple rules, how does a 4D simplex chain behave? Does it spiral out of control, or remain within a finite region of space?

9 Goals

For each of the “mechanically interesting” shapes, we can ask does there exist a simplex chain, and if so, what is the smallest, that matches that shape that has the properties of regularity, k -regularity, k -rational regularity, and x -boundedness.

10 Hunting techniques

In hunting for these open problems, it would be nice to define a toolkit of techniques that we can use.

One technique is to start with a shape you are trying to make, and then place nodes on the surface of the shape, connecting them to form simplex chains as you go. Then from the shape you have made, consider deriving a simplex chain rule, if that is possible (it might not be.) In this technique you don't start with a simplex chain rule, but attempt to find a simplex chain rule.

On the other hand, it may be more elegant to create a simplex chain rule, see what shape is produced, and then attempt to modify the simplex chain rule to match a shape you are seeking.

11 G-Brain's relation to other mathematics

In comments in a Reddit post, https://www.reddit.com/r/math/comments/9l4qmw/a_mathathon_a_cooperative_free_virtual_hackathon/, G-Brain point out this a simplex chain can be related to typical mathematical language:

I would say a simplex chain is a (countable) pure simplicial d -complex in R^d such that its facet graph (vertices are facets, edges are between facets which meet in a codimension 1 face) admits a Hamiltonian path (visits each vertex exactly once, without repetition of vertices or edges).

We have not emphasized this language because it seems more user-friendly to approach some of the problems from the position of Computer Science than from pure mathematics.

12 Rules of the Mathathon

12.1 Rules

The intention of the Mathathon is to perform all the work in the light and to encourage math as a social activity. It is our hope that all members will add their written work to a GitHub repo as quickly as they can—for example, every 30 minutes or more often would not be unreasonable. To be eligible for an award on the basis of content, all content must be published in a GitHub repo whose entire contents are licensed under Create Commons By Attribution(0), or a the participant must permit their work to be so published in writing. These repos must remain publicly accessible for one month after the date of the hackathon.

It is preferable to name one file in each repo “FinalSubmission.pdf”, with the expectation that this will be the first file read by the judges, and that this file may be included in proceedings, if the organizers choose to create a proceedings publication.

However, to be considered for an award all material must be placed in public GitHub repo, or emailed in a pdf before a deadline. The deadline for email submissions may be early than the deadline for GitHub repo-based submissions.

12.2 Awards

Awards at present do not have monetary prizes. If a sponsor provides monetary support, my current intention is to split the award money evenly between each category below. Participants may receive more than one award.

If sufficient people volunteer, judging will be by a panel of judges.

Awards will be given for:

- Crowd favorite (to be determined by email vote)
- Best contribution by someone without a post-baccalaureate degree
- Best contribution by someone without a college degree
- Best contribution by someone without a High School degree
- Best contribution by a Senior Citizen (at least the age of 65)
- Most creative technique
- Best contribution by someone not a US Citizen
- Most worthy of publication
- Second most worthy of publication
- Best contribution by a child under 16 (may be accomplished in conjunction with a coach, so long as the child is fully engaged)
- Best contribution of a new problem statements
- Most helpful to other participants (not on same team), to be awarded based on written testimonials
- Second most helpful to other participants (not on same team), to be awarded based on written testimonials
- Most useful open repo during the mathathon
- Best mathematical presentation and writing
- Best code contribution
- Best contribution of graphic art, figure, or diagram

13 Key Personnel

Robert L. Read, PhD

13.1 Potential Key Personnel

- Ethan Russo
- David Jeschke

14 Acknowledgement

Many thanks to David Jeschke for assisting in writing the test software.

Thanks to G-Brain of Reddit for relating this to typical mathematical language.