statop hw2_3

Let

$$X \in \mathbb{R}^{n \times m}$$

Calculate

$$\partial \|X\|_2$$

where $\|X\|_2$ is the spectral norm of X.

Give the explicit subdifferential for when $X = ($

Notes, background needed to answer question:

Let

$$x_0 \in \mathbb{R}^m$$
$$x \in \mathbb{R}^m$$

f be a real-valued convex functions

$\partial f(x_0) \equiv$ the subdifferential of $f(x_0)$, the set of all subgradients of f at x_0.

$g(x_0)$ is a subgradient of a real-valued convex function f at x_0, if:

$$f(x) - f(x_0) \geq \langle g(x_0), x - x_0 \rangle \quad \forall x$$

Let

$$X_0^* \in \mathbb{R}^{m \times n}$$

$$X \in \mathbb{R}^{m \times n}$$

$$G \in \mathbb{R}^{m, p}$$

Then

G is a subgradient of f at X_0^* if

$$f(X) - f(X_0^*) \geq \langle G, X - X_0^* \rangle \quad \forall X$$

By the definition of the inner product of matrices:

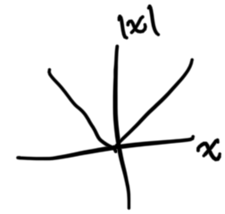
$$\langle G, X - X_0^* \rangle := \sum_{i,j} g_{ij} (x_{ij} - x_{ij}^*)$$

$$= \text{trace}(G^T (X - X^*))$$

Example in the univariate case.

if

$$f(x) = |x|$$
$$x_0 = 0$$
$$x = [-1, 1]$$



then

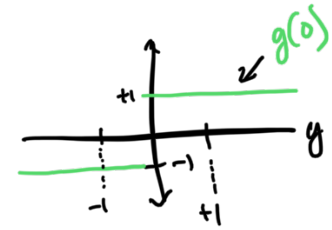
$$f(x) - f(x_0) \geq (g(x_0))^T (x - x_0)$$

$$f(x) - f(0) \geq g(0)(x - 0)$$

$$f(x) \geq g(0)\, x$$

$$|x| \geq g(0)\, x$$

$$\Rightarrow \left\{ \begin{array}{lll} g(0) \geq \frac{|x|}{x} & , & x < 0 \\ g(0) \leq \frac{|x|}{x} & , & x > 0 \\ g(0) = +1 & , & \lim_{x \to 0^+} \\ g(0) = -1 & , & \lim_{x \to 0^-} \end{array} \right\} \Leftrightarrow$$



$$\Rightarrow g(0) \in \{-1, 1\}$$

since this is a univariate case, we can let

$$g(x_0) = c$$

thus the subgradient $g(x_0)$ of $|x_0|$ in $\{-1,1\}$ is

any constant c in $\{-1,1\}$. The set of
is the subdifferential. So in this example

$$\partial|x_0| = [-1,1]$$

Solution attempt:

X is real so

$$\|X\|_2 = \sup_{\|u\|=1,\ \|v\|=1} u^T X v$$

From page 16 of topic 3 we have:

$$d(u^T X_0 v) = \langle u^T v,\ dX_0 \rangle$$

$$= v^T u\, dX_0$$

? $\longrightarrow$ $dX_0 = ?$

$$\Rightarrow \partial\|X_0\| = \text{closure}\Big(\text{conv} \bigcup \partial u^T X_0 v \Big|_{u^T X_0 v = \|X_0\|_2}\Big)$$

$$= \text{conv}\left\{ v^T u\, dX_0 : u^T X_0 v = \|X_0\|_2,\ \|u\|=1,\ \|v\|=1 \right\}$$

$$= \text{conv}\{$$

1.